x obeys an equation formally written in the Liouville form. It is not necessary that the evolution described by the equation be a time evolution. It may be a formal equation featuring, for example, the variable $1/kT$ instead of t. Similarly, J can have varied forms. In this paper it is R, the operator corresponding to the response. However, an equation in the exact form of Eq. (11a) can be derived,[5] for example, for the reduced density matrix $\mathrm{Tr}\,a_m^\dagger a_m \rho(t)$, with $J = a_m^\dagger a_m$. These other uses of our method will be reported elsewhere.

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Statistical Mechanics of the *XY* Model. IV. Time-Dependent Spin-Correlation Functions*

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We compute the correlation functions $\langle S_0^x(t) S_R^x(0)\rangle$ and $\langle S_0^y(t) S_R^y(0)\rangle$ at $T=0$ for the one-dimensional XY model in the presence of a magnetic field.

I. INTRODUCTION

Dynamical properties of many-particle systems which are very near thermal equilibrium are often studied in terms of time-dependent correlation functions $\langle A(r_1, t_1) B(r_2, t_2)\rangle$. Here $\langle \cdots \rangle$ denotes a thermal average in the canonical ensemble. Contact with macroscopic measurements is made by means of the Kubo formulas[1] and the approximation of linear response theory.

In view of the importance of these time-dependent correlation functions, it would be quite useful to have some nontrivial interacting systems for which the correlation functions can be exactly computed. Until recently no such exactly soluble problems were known. However, in 1967 Niemeijer[2] succeeded in computing exactly the correlation function at all temperatures,

$$\rho_{zz}(R, t) = \langle S_1^z(t) S_{R+1}^z(0)\rangle = \langle e^{iHt} S_1^z e^{-iHt} S_{R+1}^z\rangle , \tag{1.1}$$

for the XY model defined by

$$H = -\sum_{i=1}^{N} \left[(1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y + h S_i^z\right] , \tag{1.2}$$

where S_i^v are $\frac{1}{2}$ the Pauli spin matrices. Niemeijer found that $\rho_{zz}(R, t)$ has the same form as a density-density correlation function of noninteracting fermions which have the dispersion relation

$$\epsilon_k = [(\cos k - h)^2 + \gamma^2 \sin^2 k]^{1/2} . \tag{1.3}$$

In particular, if $\gamma \neq 0$ and $h \neq 1$, ϵ_k can never vanish for real values of k, and for fixed R, $\rho_{zz}(R, t)$ approaches its $t \to \infty$ limit of M_z^2 as t^{-1}.

The purpose of the present paper is to extend Niemeijer's work to the transverse ground-state correlation function

$$\rho_{vv}(R, t) = \langle e^{iHt} S_1^v e^{-iHt} S_{R+1}^v\rangle , \tag{1.4}$$

where $v = X$ or Y. In contrast to ρ_{zz} these correlation functions are *not* expressible as a correlation function of a finite number of field operators of a noninteracting Fermi system. Instead we find that if $h>1$, ρ_{xx} and ρ_{yy} couple to all states of the corresponding free Fermi system (i.e., to all 1, 2, 3, ..., particle states). However, if $h<1$, ρ_{xx} and ρ_{yy} couple only to states with an even number of particles (at least at $T=0$).

In Sec. II we express $\rho_{vv}(R,t)$ in terms of block Toeplitz determinants. This formulation is valid for all T. However, because of the block nature of the determinants, we are only able to extract explicit information when $T=0$. For this special case, we study $\rho_{xx}(R,t)$ for R large and t unrestricted in Sec. III. In Sec. IV we do the same for ρ_{yy}. We conclude in Sec. V with a discussion and summary of the results obtained.

II. FORMULATION

To make precise the system defined by Eq. (1.2), we must specify boundary conditions. We choose cyclic boundary conditions so that

$$S_1^v \equiv S_{N+1}^v , \quad v=x,\ y,\ z . \tag{2.1}$$

In previous computations of the free energy, [3] the correlation functions at $t=0$, [4,5] or $\rho_{zz}(R,t)$, [2] it was possible to make certain particular modifications of the boundary condition (2.1) without altering the behavior of the quantity computed in the $N\to\infty$ limit. However, for $\rho_{xx}(R,t)$ and $\rho_{yy}(R,t)$ we cannot be so cavalier about boundary conditions. The point of difficulty was recognized by Lieb, Schultz, and Mattis (LSM)[3] in their original paper and also by Katsura[3] and amplified in a subsequent article by Schultz, Mattis, and Lieb on the Ising model. [6] For completeness we briefly summarize this work on diagonalizing H before we turn to ρ_{vv}.

LSM approached the problem of diagonalizing (1.2) by first introducing the raising and lowering operators

$$b_j^\dagger = S_j^x + iS_j^y , \tag{2.2a}$$

$$b_j = S_j^x - iS_j^y , \tag{2.2b}$$

with

$$S_j^x = \tfrac{1}{2}(b_j^\dagger + b_j) , \quad S_j^y = (b_j^\dagger - b_j)/2i$$

and

$$S_j^z = b_j^\dagger b_j - \tfrac{1}{2} . \tag{2.3}$$

The operators b_i satisfy the mixed set of commutation relations

$$[b_i^\dagger, b_j] = [b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0 \quad \text{for} \quad i\neq j \tag{2.4a}$$

and anticommutation relations

$$\{b_i, b_i^\dagger\} = 1 , \quad b_i^2 = (b_i^\dagger)^2 = 0 . \tag{2.4b}$$

LSM then perform a Jordan-Wigner transformation to the Fermi operator c_j defined from

$$b_j = \exp\left(+\pi i \sum_{k=1}^{j-1} c_k c_k^\dagger\right) c_j$$

$$= (-1)^{j-1} \exp\left(-\pi i \sum_{k=1}^{j-1} c_k^\dagger c_k\right) c_j , \tag{2.5a}$$

$$b_j^\dagger = c_j^\dagger \exp\left(-\pi i \sum_{k=1}^{j-1} c_k c_k^\dagger\right)$$

$$= (-1)^{j-1} c_j^\dagger \exp\left(\pi i \sum_{k=1}^{j-1} c_k^\dagger c_k\right) \tag{2.5b}$$

In terms of these Fermi operators the Hamiltonian (1.2) becomes

$$H = H^+P^+ + H^-P^- , \tag{2.6}$$

where (we assume N is even for convenience)

$$H^- = \frac{1}{2}\left(\sum_{i=1}^{N-1} [c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + \gamma(c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i)] - 2h\sum_{j=1}^{N} (c_j^\dagger c_j - \tfrac{1}{2}) + c_N^\dagger c_1 + c_1^\dagger c_N + \gamma(c_N^\dagger c_1^\dagger + c_1 c_N)\right) , \tag{2.7a}$$

$$H^+ = \frac{1}{2}\left(\sum_{i=1}^{N-1} [c_i^\dagger c_{i+1}^\dagger + c_{i+1}^\dagger c_i + \gamma(c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i)] - 2h\sum_{j=1}^{N} (c_j^\dagger c_j - \tfrac{1}{2}) - [c_N^\dagger c_1 + c_1^\dagger c_N + \gamma(c_N^\dagger c_1 + c_1 c_N)]\right) , \tag{2.7b}$$

and

$$P_\pm = \frac{1}{2}\left[1 \pm \exp\left(i\pi \sum_{j=1}^{N} c_j^\dagger c_j\right)\right] . \tag{2.8}$$

The operator P_+ (P_-) is a projection operator for states with an even (odd) number of c_j excitations. The decomposition (2.6) is expected because the Hamiltonian (1.2) with or without the boundary condition (2.1) commutes with the "parity" operator

$$\exp\left(i\pi \sum_{j=1}^{N} c_j^\dagger c_j\right)$$

because (1.2) can only change the number of c_j excitations by an even number. Therefore, when acting on a state with an even (odd) number of c_j excitations, H may be replaced by H^+ (H^-). To diagonalize H, we may diagonalize H^+ and H^- separately and then throw out the eigenvectors of H^+ with odd parity and the eigenvectors of H^- with even parity. The difficulty in computing ρ_{vv} lies in the fact that H^+ and H^- do not commute. Therefore the transformation that diagonalizes H^+ will in general leave H^- a mess.

Both H^+ and H^- may be diagonalized by the procedure of Fourier transformation followed by Bogoliubov transformation. [6] For H^+ define operators $a_j^{(+)}$ from

$$c_j^\dagger = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_{\phi_n^{(+)}} \exp(ij\phi_n^{(+)}) a_{\phi^{(+)}}^{(+)\dagger} , \tag{2.9a}$$

$$c_j = \frac{e^{i\pi/4}}{\sqrt{N}} \sum_{\phi_n^{(+)}} \exp(-ij\phi_n^{(+)}) a^{(+)}_{\phi^{(+)}} \quad , \tag{2.9b}$$

where (for convenience N is assumed even) $\phi_n^{(+)}$ takes the values

$$\phi_n^{(+)} = \pm 2\pi(n+\tfrac{1}{2})/N \ , \tag{2.10}$$

with

$$n = 0, 1, 2, \ldots, N/2-1 \ .$$

Similarly for H^- we introduce $a_j^{(-)}$ by

$$c_j^\dagger = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_{\phi_n^{(-)}} \exp(ij\phi_n^{(-)}) a^{(-)\dagger}_{\phi_n^{(-)}} \tag{2.11a}$$

and

$$c_j = \frac{e^{i\pi/4}}{\sqrt{N}} \sum_{\phi_n^{(-)}} \exp(-ij\phi_n^{(-)}) a^{(-)}_{\phi_n^{(-)}} \quad , \tag{2.11b}$$

where $\phi_n^{(-)}$ takes on the values

$$\phi_n^{(-)} = 0, \pm 2\pi n/N, \pi \ , \tag{2.12}$$

with

$$n = 1, 2, \ldots, N/2-1.$$

Then we have

$$H^- = (1-h) a_0^{(-)\dagger} a_0^{(-)} - (1+h) a_\pi^{(-)\dagger} a_\pi^{(-)} + \frac{1}{2} \sum_{n=1}^{N/2-1} \{ 2(\cos\phi_n^{(-)} - h) [a^{(-)\dagger}_{\phi_n^{(-)}} a^{(-)}_{\phi_n^{(-)}} + a^{(-)\dagger}_{-\phi_n^{(-)}} a^{(-)}_{-\phi_n^{(-)}}] - 2\gamma \sin\phi_n^{(-)} [a^{(-)\dagger}_{\phi_n^{(-)}} a^{(-)\dagger}_{-\phi_n^{(-)}} + a^{(-)}_{\phi_n^{(-)}} a^{(-)}_{-\phi_n^{(-)}}] \} + \tfrac{1}{2} Nh \tag{2.13a}$$

and

$$H^+ = +\frac{1}{2} \sum_{n=0}^{N/2-1} \{ 2(\cos\phi_n^{(+)} - h) [a^{(+)\dagger}_{\phi_n^{(+)}} a^{(+)}_{\phi_n^{(+)}} + a^{(+)\dagger}_{-\phi_n^{(+)}} a^{(+)}_{-\phi_n^{(+)}}] - 2\gamma \sin\phi_n^{(+)} [a^{(+)\dagger}_{\phi_n^{(+)}} a^{(+)\dagger}_{-\phi_n^{(+)}} - a^{(+)}_{\phi_n^{(+)}} a^{(+)}_{-\phi_n^{(+)}}] \} + \tfrac{1}{2} Nh \ . \tag{2.13b}$$

To complete the diagonalization we make the Bogoliubov transformation

$$a_\phi^{(p)} = \cos\Theta_\phi \, \eta_\phi^{(p)} + \sin\Theta_\phi \, \eta_{-\phi}^{(p)\dagger} \ , \tag{2.14a}$$

$$a_{-\phi}^{(p)} = \cos\Theta_\phi \, \eta_{-\phi}^{(p)} - \sin\Theta_\phi \, \eta_\phi^{(p)\dagger} \ , \tag{2.14b}$$

where p is either + or −,

$$\tan 2\Theta_\phi = \gamma \sin\phi / (\cos\phi - h) \ , \tag{2.15}$$

and $0 < \Theta_\phi < \frac{1}{2}\pi$. This transformation is to be used for each and every $\phi_n^{(+)}$ and for each and every $\phi_n^{(-)}$ except 0 and π. To account for the exceptional cases $\phi_n^{(-)} = 0, \ \pi$ define

$$a_0^{(-)} = \eta_0^{(-)\dagger} \text{ and } a_\pi^{(-)} = \eta_\pi^{(-)\dagger} \ . \tag{2.16}$$

Then all $\eta_\phi^{(+)}$ and $\eta_\phi^{(-)}$ satisfy the Fermion anticommutation relations

$$\{\eta^{(+)\dagger}_{\phi_1^{(+)}}, \eta^{(+)}_{\phi_2^{(+)}}\} = \delta_{\phi_1^{(+)}, \phi_2^{(+)}} \tag{2.17a}$$

and

$$\{\eta^{(-)\dagger}_{\phi_1^{(-)}}, \eta^{(-)}_{\phi_2^{(-)}}\} = \delta_{\phi_1^{(-)}, \phi_2^{(-)}} \ , \tag{2.17b}$$

and with the definition that if $\phi > 0$,

$$\Lambda(\phi) = [(\cos\phi - h)^2 + \gamma^2 \sin^2\phi]^{1/2} > 0 \tag{2.18}$$

and

$$\Lambda(0) = \begin{cases} h - 1 = \lim_{\phi \to 0} \Lambda(\phi) & \text{for } h \ge 1 \\ h - 1 = -\lim_{\phi \to 0} \Lambda(\phi) & \text{for } h < 1 \ , \end{cases} \tag{2.19}$$

we obtain

$$H^+ = -\tfrac{1}{2} \sum_{\phi^{(+)}} \Lambda(\phi^{(+)}) + \sum_{\phi^{(+)}} \Lambda(\phi^{(+)}) \eta^{(+)\dagger}_{\phi^{(+)}} \eta^{(+)}_{\phi^{(+)}} \ , \tag{2.20a}$$

$$H^- = -\tfrac{1}{2} \sum_{\phi^{(-)}} \Lambda(\phi^{(-)}) + \sum_{\phi^{(-)}} \Lambda(\phi^{(-)}) \eta^{(-)\dagger}_{\phi^{(-)}} \eta^{(-)}_{\phi^{(-)}} \ . \tag{2.20b}$$

It is now easy to see why the computation of $\rho_{xx}(R, t)$ and $\rho_{yy}(R, t)$ involves difficulties not encountered in previous calculations. Consider, for example,

$$\rho_{xx}(R, 0) = Z^{-1} \textstyle\sum_n e^{-\beta E_n} \langle E_n | S_1^x S_{R+1}^x | E_n \rangle \ , \tag{2.21}$$

where Z is the partition function and $|E_n\rangle$ is the complete set of eigenstates of H. The operator $S_1^x S_{R+1}^x$ connects only states which differ by the creation or destruction of $2b_j$ excitations. Moreover, we may consider evaluating ρ_{xx} by writing $S_1^x S_{R+1}^x$ as

$$S_1^x S_{R+1}^x = \tfrac{1}{4}(-1)^R (c_1^\dagger - c_1)(c_2^\dagger + c_2) \times (c_2^\dagger - c_2) \cdots (c_R^\dagger - c_R)(c_{R+1}^\dagger + c_{R+1}) \ . \tag{2.22}$$

This expression contains sums of products of an even number of operators. In the subsequent reductions carried out by LSM, only matrix elements of products of an even number of operators occur. But such even operators have vanishing matrix elements between an eigenstate of H^+ and an eigenstate of H^-. The remaining matrix elements are between eigenstates of H^+ or H^- alone, and are easily evaluated.[3] Furthermore as $N \to \infty$ if $E_n^+ - E_n^- \to 0$, then $\langle E_n^+ | O | E_n^+ \rangle \to \langle E_n^- | O | E_n^- \rangle$ for any operator O. Hence the more elaborate diagonalization procedure carried out above leads to the same result in the $N \to \infty$ limit as the cruder approximation of replacing H by H^-, which was made in previous work.[2–5]

The difficulty in directly evaluating $\rho_{vv}(R, t)$ $(v = x, y)$ is that $e^{iHt} S_1^v e^{-iHt} S_{R+1}^v$ cannot be evaluated in terms of matrix elements of even operators.

Consider first an expansion in terms of the eigenvectors of H

$$\rho_{vv}(R,t)=(1/Z)\sum_{m,n} e^{-\beta E_m}\langle E_m|S_1^v|E_n\rangle \times e^{it(E_m-E_n)}\langle E_n|S_{R+1}^v|E_m\rangle . \quad (2.23)$$

The operator S_1^x connects eigenstates of H^+ with eigenstates of H^-. But while the operator substitutions which express S_1^x in terms of the operators $\eta_\phi^{(+)}$ or the operators $\eta_\phi^{(-)}$ are easily obtainable, there is no simple way to compute matrix elements of $\eta_\phi^{(+)}$ or $\eta_\phi^{(-)}$ between an eigenstate of H^- and an eigenstate of H^+. Therefore, (2.23) is an exceedingly awkward formula to use to evaluate $\rho_{vv}(R,t)$.

Rather than attack the problem of computing matrix elements of odd operators, we reformulate the computation of $\rho_{vv}(R,t)$ so that we only need make use of matrix elements of even operators. To do this, consider instead of the two-spin correlation function $\rho_{vv}(R,t)$ the four-spin correlation

$$C_{vv}(R,t,N)=\langle S_{1+N/2}^v(t)S_{1-R+N}^v(t)S_1^v(0)S_{1-R+N/2}^v(0)\rangle_N , \quad (2.24)$$

where $\langle\cdots\rangle_N$ makes explicit that the thermal average is in a finite lattice of N spins. This four-spin correlation can be evaluated in terms of matrix elements of even operators only. Then $\rho_{vv}(R,t)$ may be recovered by use of the cluster property

$$\lim_{N\to\infty}\langle S_{1+N/2}^v(t)S_{1-R+N}^v(t)S_1^v(0)S_{1-R+N/2}^v(0)\rangle_N$$
$$=\lim_{N\to\infty}\langle S_{1-R+N}^v(t)S_1^v(0)S_{1+N/2}(t)S_{1-R+N/2}(0)\rangle_N$$
$$=\rho_{vv}^2(R,t) . \quad (2.25)$$

The four-spin correlation function is easily reduced to the evaluation of a Pfaffian by the methods of LSM.[3] We find that

$$C_{xx}^2(R,t,N)=4^{-4}\begin{vmatrix} 0 & \tilde{S}^x & \tilde{T}^x & \tilde{U}^x \\ -\tilde{S}^{xT} & 0 & -\tilde{U}^x & \tilde{V}^x \\ -\tilde{T}^x & \tilde{U}^x & 0 & -\tilde{S}^x \\ -\tilde{U}^x & -\tilde{V}^x & \tilde{S}^{xT} & 0 \end{vmatrix} , \quad (2.26)$$

where each of the submatrices is of dimension $N/2-R$ and, with $0\le m\le N/2-R-1$ and $0\le n\le N/2-R-1$,

$$\tilde{S}_{m,n}^x=\frac{1}{N}\sum_\phi e^{-i(m-n)\phi}S(\phi)=\frac{1}{N}\sum_\phi e^{-i(m-n-1)\phi}\Phi(\phi)\tanh\tfrac{1}{2}\beta\Lambda(\Phi) , \quad (2.27a)$$

$$\tilde{T}_{m,n}^x=\frac{1}{N}\sum_\phi e^{-i(m+n+R)\phi}T(\phi)=-\frac{1}{N}\sum_\phi e^{-i(m+n+R)\phi}\Phi(\phi)\left(e^{-i\Lambda(\phi)t}-\frac{e^{i\Lambda(\phi)t}+e^{-i\Lambda(\phi)t}}{e^{\beta\Lambda(\phi)}+1}\right) , \quad (2.27b)$$

$$\tilde{U}_{m,n}^x=\frac{1}{N}\sum_\phi e^{i(m+n+R+1)\phi}\left(e^{-i\Lambda(\phi)t}-\frac{e^{-i\Lambda(\phi)t}-e^{i\Lambda(\phi)t}}{e^{\beta\Lambda(\phi)}+1}\right) , \quad (2.27c)$$

$$\tilde{V}_{m,n}^x=-\frac{1}{N}\sum_\phi e^{i(n+m+R+2)\phi}T(\phi)=-\frac{1}{N}\sum_\phi e^{i(n+m+R+2)\phi}\Phi(\phi)\left(e^{-i\Lambda(\phi)t}-\frac{e^{i\Lambda(\phi)t}+e^{-i\Lambda(\phi)t}}{e^{\beta\Lambda(\phi)}+1}\right) , \quad (2.27d)$$

where

$$\Phi(\phi)=e^{-i\phi}\left(\frac{(1-\lambda_1^{-1}e^{i\phi})(1-\lambda_2^{-1}e^{i\phi})}{(1-\lambda_1^{-1}e^{-i\phi})(1-\lambda_2^{-1}e^{-i\phi})}\right)^{1/2} \quad (2.28)$$

and

$$\lambda_1=\frac{h+[h^2-(1-\gamma^2)]^{1/2}}{1-\gamma} , \quad (2.29)$$

$$\lambda_2=\frac{h-[h^2-(1-\gamma^2)]^{1/2}}{1-\gamma} , \quad (2.30)$$

and the square root is defined to be positive at $\phi=\pi$. Similarly,

$$C_{yy}^2(R,t,N)=4^{-4}\begin{vmatrix} 0 & \tilde{S}^y & \tilde{T}^y & \tilde{U}^y \\ -\tilde{S}^{yT} & 0 & -\tilde{U}^y & \tilde{V}^y \\ -\tilde{T}^y & \tilde{U}^y & 0 & -\tilde{S}^y \\ -\tilde{U}^y & -\tilde{V}^y & \tilde{S}^{yT} & 0 \end{vmatrix} , \quad (2.31)$$

where

$$\tilde{S}_{m,n}^y=\frac{1}{N}\sum_\phi e^{-i(m-n-2)\phi}S(-\phi) , \quad (2.32a)$$

$$\tilde{T}_{m,n}^y=-\frac{1}{N}\sum_\phi e^{-i(m+n+R)\phi}T(-\phi) , \quad (2.32b)$$

$$\tilde{U}_{m,n}^y=\tilde{U}_{m,n}^x , \quad (2.32c)$$

$$\tilde{V}_{m,n}^y=\tfrac{1}{4}\sum_\phi e^{i(m+n+R+2)\phi}T(-\phi) . \quad (2.32d)$$

III. $\rho_{xx}(R,t)$

We now specialize our consideration to the case $T=0$. Then, as formulated in Sec. II, the evaluation of $\rho_{yy}(R,t)$ is similar to the evaluation of the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ of the two-dimensional Ising model carried out by Cheng and Wu.[7] The separation of spins in the vertical direction M of the Ising-model problem is analogous to the time separation t of the XY-model problem.

As in the case $t=0$ there are three distinct cases: (A) $h<1$, (B) $h>1$ and (C) $h=1$. In this paper we treat case (A) and (B) by modifying the procedure of CW. Furthermore, we consider only the anisotropic case $0<\gamma\leq 1$.

A. $h<1$

In this case, $|\lambda_1|>1$ and $|\lambda_2|>1$ and we may follow the $T<T_c$ procedure of CW.

Consider the ratio

$$\tilde{f}^4(R,t,N)=\frac{C^2_{xx}(R+1,t,N)}{C^2_{xx}(R,t,N)}\quad, \tag{3.1}$$

where $C^2_{xx}(R+1,t,N)$ is to be obtained from (2.26) by omitting the first of the $N/2-R$ rows (columns) in the third and fourth rows (columns) shown explicitly in (2.30) and by omitting the last of the $N/2-R$ rows (columns) from the first and second rows (columns) shown explicitly in (2.30). Define the 4×4 matrix of $N/2-R$ component vectors $\tilde{X}^a_b$ by the linear equations

$$\begin{bmatrix} 0 & \tilde{S}^x & \tilde{T}^x & \tilde{U}^x \\ -\tilde{S}^{xT} & 0 & -\tilde{U}^x & \tilde{V}^x \\ -\tilde{T}^x & \tilde{U}^x & 0 & -\tilde{S}^x \\ -\tilde{U}^x & -\tilde{V}^x & \tilde{S}^{xT} & 0\end{bmatrix}\begin{bmatrix}\tilde{X}^1_1 & \tilde{X}^2_1 & \tilde{X}^3_1 & \tilde{X}^4_1 \\ \tilde{X}^1_2 & \tilde{X}^2_2 & \tilde{X}^3_2 & \tilde{X}^4_2 \\ \tilde{X}^1_3 & \tilde{X}^2_3 & \tilde{X}^3_3 & \tilde{X}^4_3 \\ \tilde{X}^1_4 & \tilde{X}^2_4 & \tilde{X}^3_4 & \tilde{X}^4_4\end{bmatrix} = \begin{bmatrix}\tilde{\delta}_D & 0 & 0 & 0 \\ 0 & \tilde{\delta}_D & 0 & 0 \\ 0 & 0 & \tilde{\delta}_U & 0 \\ 0 & 0 & 0 & \tilde{\delta}_U\end{bmatrix}, \tag{3.2}$$

where the $N/2-R$ component vectors $\tilde{\delta}_i$ are

$$\tilde{\delta}_D=\begin{bmatrix}0\\ \vdots \\ 0 \\ 1\end{bmatrix},\qquad \tilde{\delta}_U=\begin{bmatrix}1\\ 0\\ \vdots \\ 0\end{bmatrix}. \tag{3.3}$$

Application of Jacobi's theorem then gives

$$\tilde{f}^4(R,t,N)=\det\begin{bmatrix}\tilde{X}^1_{1,N/2-R-1} & \tilde{X}^2_{1,N/2-R-1} & \tilde{X}^3_{1,N/2-R-1} & \tilde{X}^4_{1,N/2-R-1} \\ \tilde{X}^1_{2,N/2-R-1} & \tilde{X}^2_{2,N/2-R-1} & \tilde{X}^3_{2,N/2-R-1} & \tilde{X}^4_{2,N/2-R-1} \\ \tilde{X}^1_{3,0} & \tilde{X}^2_{3,0} & \tilde{X}^3_{3,0} & \tilde{X}^4_{3,0} \\ \tilde{X}^1_{4,0} & \tilde{X}^2_{4,0} & \tilde{X}^3_{4,0} & \tilde{X}^4_{4,0}\end{bmatrix}. \tag{3.4}$$

We must evaluate $\tilde{f}(R,t,N)$ in the $N\to\infty$ limit. To do this we partition the indices of the $N/2-R$ dimensional determinants into the sets

$$(1)\quad 0\leq n\leq N/4 \tag{3.5a}$$

and

$$(2)\quad N/4<n\leq N/2-R-1\ , \tag{3.5b}$$

where the precise value when the separation is made (here taken as $N/4$) is unimportant. In (3.2) the matrix elements connecting the subspace 1 with the subspace 2 are exponentially small in N except for elements of S. However, those elements of S whose matrix elements between 1 and 2 are of order 1 are a distance $N/4$ away from the place when the right-hand side of (3.2) is different from zero. Therefore, as $N\to\infty$ eight elements of the determinant in (3.4) in the lower left and upper right quadrants vanish and we obtain

$$f^4(R,t)=\lim_{N\to\infty}\tilde{f}^4(R,t,N)=\left\{\det\begin{bmatrix}X^3_{3,0} & X^4_{3,0} \\ X^3_{4,0} & X^4_{4,0}\end{bmatrix}\right\}^2, \tag{3.6}$$

where X^a_b is an ∞ vector obtained as the solution of

$$\begin{bmatrix} 0 & S^x & T^x & U^x \\ -S^{xT} & 0 & -U^x & V^x \\ -T^x & U^x & 0 & -S \\ -U^x & -V^x & S^T & 0\end{bmatrix}\begin{bmatrix}X^3_1 & X^4_1 \\ X^3_2 & X^4_2 \\ X^3_3 & X^3_4 \\ X^3_4 & X^4_4\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & 0 \\ \delta & 0 \\ 0 & \delta\end{bmatrix}, \tag{3.7}$$

where

$$\delta=\begin{bmatrix}1\\ 0\\ \vdots\end{bmatrix}$$

and S^x, T^x, U^x, and V^x are obtained from (2.27) by replacing $(1/N)\sum_\phi$ by $(1/2\pi)\int_{-\pi}^{\pi}d\phi$.

Once $f(R,t)$ is known we may obtain $\rho_{xx}(R,t)$ by using the cluster property (2.25) and determine the $\pm$ sign from the known $t=0$ results by continuity to obtain

$$f(R,t)=\frac{\rho_{xx}(R+1,t)}{\rho_{xx}(R,t)}\ . \tag{3.8}$$

Therefore

$$\rho_{xx}(R,t)=\rho_{xx}(\infty,t)\prod_{r=R}^{\infty}f(r,t)^{-1}\ , \tag{3.9}$$

where $\rho_{xx}(\infty,t)$ is independent of t and from Paper

II has the value

$$\rho_{xx}(\infty, t) = [2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4} . \tag{3.10}$$

We proceed to evaluate $f(R, t)$ for arbitrary t when R is large by using the iterative scheme of CW. We find from (3.14) of CW that

$$f^2(R, t) \doteq (\delta^T S^{x-1}\delta)^2\{1 - 2(\delta^T S^{x-1}\delta)^{-1}\delta^T X^{x-1} \times [U^x S^{x-1} U^x + T^x (S^{xT})^{-1} V^x] S^{x-1}\} , \tag{3.11}$$

when $\doteq$ means that as $R \to \infty$ the asymptotic expansion of both sides of the relation are the same.

The required inverse matrix elements may be obtained by solving the Wiener-Hopf equation

$$\sum_{l=0}^{\infty} S^x_{ml}(S^{x-1})_{ln} = \delta_{m,n} , \quad 0 \le m , \quad 0 \le n . \tag{3.12}$$

We find

$$(S^{x-1})_{m,n} = -(2\pi)^{-2}\oint d\xi\, \xi^{-n-1}(1-\lambda_2^{-1}\xi)^{1/2}(1-\lambda_1^{-1}\xi)^{1/2} \times \oint d\xi'\, \xi'^m (\xi'-\xi)^{-1}(1-\lambda_1^{-1}\xi'^{-1})^{-1/2}(1-\lambda_2^{-1}\xi'^{-1})^{-1/2} , \tag{3.13}$$

where the contours of integration are the unit circles except that the one for ξ' is indented outward near $\xi' = \xi$. In particular,

$$\delta^T S^{x-1}\delta = (S^{x-1})_{0,0} = 1 , \tag{3.14a}$$

$$(S^{x-1})_{m,0} = (2\pi i)^{-1}\oint d\xi\, \xi^{m-1}(1-\lambda_1^{-1}\xi^{-1})^{-1/2} \times (1-\lambda_2^{-1}\xi^{-1})^{-1/2} , \tag{3.14b}$$

$$(S^{x-1})_{0,n} = (2\pi i)^{-1}\oint d\xi\, \xi^{-n-1}(1-\lambda_2^{-1}\xi)^{1/2} \times (1-\lambda_1^{-1}\xi)^{1/2} . \tag{3.14c}$$

Substituting (3.14) and (2.27) into (3.11) and then into (3.9), we obtain the desired expansion for large R:

$$\begin{aligned}\rho_{xx}(R, t) &\doteq \rho_{xx}(\infty)\{1 + (2\pi)^{-2}\oint d\xi \oint d\eta\, \xi^R \eta^{-R}(\xi-\eta)^{-2} \times e^{-it[\Lambda(\xi)+\Lambda(\eta)]}[M(\xi,\eta)-1]\} \\ &= \rho_{xx}(\infty)\{1 + (2\pi)^{-2}\oint d\xi \oint d\eta\, \xi^R \eta^{-R}(\xi-\eta)^{-2} \times e^{-it[\Lambda(\xi)+\Lambda(\eta)]}\tfrac{1}{2}[M(\xi,\eta)+M(\eta,\xi)-2]\} ,\end{aligned} \tag{3.15}$$

where

$$M(\xi, \eta) = \left(\frac{(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2^{-1}\xi^{-1})(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)}{(1-\lambda_1^{-1}\eta^{-1})(1-\lambda_2^{-1}\eta^{-1})(1-\lambda_1^{-1}\eta)(1-\lambda_2^{-1}\eta)}\right)^{1/2} , \tag{3.16}$$

where in the first equation of (3.15) the η contour is indented outward at $\eta = \xi$.

Further discussion of (3.15) will be postponed until Sec. V.

B. $h > 1$

In this case $\lambda_1 > 1$ but $\lambda_2 < 1$ and we may follow a procedure closely related to the $T > T_c$ procedure of CW.

The problem which arises when $h > 1$ is that

$$\ln S(2\pi) - \ln S(0) = 2\pi i . \tag{3.17}$$

We want to solve a Wiener-Hopf equation with something like $S(\phi)$ as the Fourier transform of the kernal and hence it is desirable to work with a $S(\phi)$ where the right-hand side of (3.17) would be zero. This will be the case if instead of $S(\phi)$ we can deal with $e^{-i\phi}S(\phi)$. Therefore it is convenient to consider besides the determinant (2.31) the determinant

$$D^2(R, t, N) = \begin{vmatrix} \underline{0}| & |\underline{S}^x \quad \underline{T}^x| & |\underline{U}^x \\ -\bar{S}^{xT}| & |\bar{0} \quad -\bar{U}^x| & |\bar{V}^x \\ -\underline{T}^x| & |\underline{U}^x \quad \underline{0}| & -|\underline{S}^x \\ -\bar{U}^x| & -|\bar{V}^x \quad \bar{S}^{xT}| & |\bar{0} \end{vmatrix} , \tag{3.18}$$

where a horizontal (vertical) bar indicates the addition of a row (column). For example,

$$|\underline{S}^x = \begin{bmatrix} \tilde{S}^x_{0-1} & \tilde{S}^x_{00} & \cdots & \tilde{S}^x_{0,N/2-R-1} \\ \tilde{S}^x_{1-1} & \tilde{S}^x_{10} & \cdots & \tilde{S}^x_{1,N/2-R-1} \\ \vdots & \vdots & & \vdots \\ \tilde{S}^x_{N/2-R,-1} & \tilde{S}^x_{N/2-R,0} & & \tilde{S}^x_{N/2-R,N/2-R-1} \end{bmatrix} . \tag{3.19}$$

All the submatrices displayed in (3.18) are $(N/2-R+1)\times(N/2-R+1)$.

Consider the ratio

$$r^2(R, t, N) = [D^2(R, t, N)]^{-1} \times \det\begin{bmatrix} 0 & \tilde{S}^x & \tilde{T}^x & \tilde{U}^x \\ -\tilde{S}^{xT} & 0 & -\tilde{U}^x & \tilde{V}^x \\ -\tilde{T}^x & \tilde{U}^x & 0 & -\tilde{S}^x \\ -\tilde{U}^x & -\tilde{V}^x & \tilde{S}^{xT} & 0 \end{bmatrix} \tag{3.20}$$

Define the 4×4 matrix of $N/2-R+1$ component vectors $\tilde{X}^b_a$ by

$$\begin{bmatrix} \underline{0}| & |\underline{S}^x \quad \underline{T}^x| & |\underline{U}^x \\ -\bar{S}^{xT}| & |\bar{0} \quad -\bar{U}^x| & |\bar{V}^x \\ -\underline{T}^x| & |\underline{U}^x \quad \underline{0}| & -|\underline{S}^x \\ -\bar{U}^x| & -|\bar{V}^x \quad \bar{S}^{xT}| & |\bar{0} \end{bmatrix} \begin{bmatrix} \tilde{X}^1_1 & \tilde{X}^2_1 & \tilde{X}^3_1 & \tilde{X}^4_1 \\ \tilde{X}^1_2 & \tilde{X}^2_2 & \tilde{X}^3_2 & \tilde{X}^4_2 \\ \tilde{X}^1_3 & \tilde{X}^2_3 & \tilde{X}^3_3 & \tilde{X}^4_3 \\ \tilde{X}^1_4 & \tilde{X}^2_4 & \tilde{X}^3_4 & \tilde{X}^4_4 \end{bmatrix} = \begin{bmatrix} \tilde{\delta}_D & 0 & 0 & 0 \\ 0 & \tilde{\delta}_U & 0 & 0 \\ 0 & 0 & \tilde{\delta}_D & 0 \\ 0 & 0 & 0 & \tilde{\delta}_U \end{bmatrix} , \tag{3.21}$$

where the $N/2 - R + 1$ component vectors $\bar{\delta}_U$ and $\bar{\delta}_D$ are given by (3.3). From X_a^b we obtain $r(R, t, N)$ by use of Jacobi's theorem

$$r^2(R, t, N) = \det \begin{bmatrix} \tilde{X}^1_{1,N/2-R} & \tilde{X}^2_{1,N/2-R} & \tilde{X}^3_{1,N/2-R} & \tilde{X}^4_{1,N/2-R} \\ \tilde{X}^1_{2,0} & \tilde{X}^2_{2,0} & \tilde{X}^3_{2,0} & \tilde{X}^4_{2,0} \\ \tilde{X}^1_{3,N/2-R} & \tilde{X}^2_{3,N/2-R} & \tilde{X}^3_{3,N/2-R} & \tilde{X}^4_{3,N/2-R} \\ \tilde{X}^1_{4,0} & \tilde{X}^2_{4,0} & \tilde{X}^3_{4,0} & \tilde{X}^4_{4,0} \end{bmatrix}, \tag{3.22}$$

where we have relabeled all the submatrices so that their indices all run from 0 to $N/2 - R$. As we did for $h<1$ we now consider dividing the indices into two groups: (1) $0 \le n \le N/4$ and (2) $N/2 < n \le N/2 - R$. Then applying the same argument as for $h<1$ we take the $N \to \infty$ limit and obtain

$$r^2(R, t) = \lim_{N\to\infty} r^2(R, t, N) = \left\{ \det \begin{bmatrix} X^2_{2,0} & X^4_{2,0} \\ X^2_{4,0} & X^4_{4,0} \end{bmatrix} \right\}^2, \tag{3.23}$$

where the 2×2 matrix of infinite dimensional vectors X_b^a is obtained from

$$\begin{bmatrix} 0 & \hat{S} & \hat{T} & \hat{U} \\ -\hat{S}^T & 0 & -\hat{U} & \hat{V} \\ -\hat{T} & U & 0 & -\hat{S} \\ -\hat{U} & -\hat{V} & \hat{S}^T & 0 \end{bmatrix} \begin{bmatrix} X^2_1 & X^4_1 \\ X^2_2 & X^4_2 \\ X^2_3 & X^4_3 \\ X^2_4 & X^4_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \delta & 0 \\ 0 & 0 \\ 0 & \delta \end{bmatrix}, \tag{3.24}$$

where we have used

$$\hat{S} = \lim_{N\to\infty} |\underline{S}^x, \qquad \hat{T} = \lim_{N\to\infty} \underline{T}^x| = T^x, \qquad \hat{U} = \lim_{N\to\infty} \bar{U}^x|, \qquad \hat{V} = \lim_{N\to\infty} |\bar{V}^x. \tag{3.25}$$

Furthermore, by following a procedure analogous to that of the $h<1$ case, we may show that

$$\lim_{R\to\infty} \lim_{N\to\infty} D(R, t, N) = D \tag{3.26}$$

exists and is independent of t. Therefore we obtain

$$\rho_{xx}(R, t) = \tfrac{1}{4}[Dr(R, t)]^{1/2}. \tag{3.27}$$

To obtain $\Gamma(R, t)$ for R large we expand the matrix on the left-hand side of (3.24) in a perturbation series since the off-diagonal blocks are exponentially small in R. We obtain to lowest order

$$X^2_2 = X^4_4 = 0 \tag{3.28a}$$

and

$$X^2_4 = -X^4_2 = \hat{S}^{-1}\hat{T}(\hat{S}^T)^{-1}\delta. \tag{3.28b}$$

Therefore

$$\rho_{xx}(R, t) = \pm D^{1/2}[\hat{S}^{-1}\hat{T}(S^T)^{-1}]_{0,0}. \tag{3.29}$$

The inverse matrix elements of $\hat{S}$ may be found by solving the Wiener-Hopf equation

$$\sum_{m=0}^{\infty} \hat{S}_{l,m}\hat{S}^{-1}_{m,n} = \delta_{l,m}, \quad 0 \le l, \quad 0 \le n \tag{3.30}$$

and we obtain (in particular)

$$[\hat{S}^{-1}]_{0,m} = \frac{-(\lambda_1\lambda_2)^{-1/2}}{2\pi i} \oint d\xi\, \xi^{-m-1} \times (1-\lambda_1\xi^{-1})^{1/2}(1-\lambda_2^{-1}\xi^{-1})^{-1/2}. \tag{3.31}$$

Therefore, using (3.25) and (2.27b) for T^x we obtain

$$\rho_{xx}(R, t) \doteq \pm\tfrac{1}{4} D^{1/2} \frac{1}{2\pi i} \oint d\xi\, \xi^{R-1} e^{-it\Lambda(\xi)} \times \left[\frac{(1-\lambda_1^{-1}\xi)(1-\lambda_1^{-1}\xi^{-1})}{(1-\lambda_2\xi)(1-\lambda_2\xi^{-1})}\right]^{1/2}. \tag{3.32}$$

Finally we obtain $D^{1/2}$ by comparing (3.32) with $t=0$ with $\rho_{xx}(R, 0)$ as obtained from (4.17) and (4.21) of II and hence find the desired result

$$\rho_{xx}(R, t) \doteq \tfrac{1}{4}[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4} \times \frac{1}{2\pi i} \oint d\xi\, \xi^{R-1} e^{-it\Lambda(\xi)} \times \left[\frac{(1-\lambda_1^{-1}\xi)(1-\lambda_1^{-1}\xi^{-1})}{(1-\lambda_2\xi)(1-\lambda_2\xi^{-1})}\right]^{1/2}, \tag{3.33}$$

where the square root is defined as positive at $\xi = -1$.

IV. $\rho_{yy}(R,t)$

As for ρ_{xx} we again treat two cases: $h>1$ and $h<1$.

A. $h>1$

In this case we may treat $C^2_{yy}(R, t, N)$ by a procedure analogous to that used for C^2_{xx} for $h>1$. The principle modification is that, instead of (3.18) we consider

$$D_y^2(R, t, N) = \begin{vmatrix} |\bar{0} & \bar{S}^y| & |\bar{T}^y & \bar{U}^y| \\ -|\underline{S}^{yt} & \underline{0}| & -|\underline{U}^y & \underline{V}^y| \\ -|T^y & \bar{U}^y| & |\bar{0} & -S^y| \\ -|\underline{U}^y & -\underline{V}^y| & |\underline{S}^{yt} & \underline{0}| \end{vmatrix}, \tag{4.1}$$

where the bars indicate the position of added rows and columns. The rest of the evaluation is identical and we obtain for large R

$$\rho_{yy}(R,t)=\tfrac{1}{4}\left[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}\right]^{1/4}\times\frac{1}{2\pi i}\oint d\xi\,\xi^{R-1}e^{-it\Lambda(\xi)}\left[(1-\lambda_1^{-1}\xi)\times(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2\xi)(1-\lambda_2\xi^{-1})\right]^{1/2}, \tag{4.2}$$

where the square root is positive for $\xi=-1$.

B. $h<1$

We proceed as in the previous case except that we add two rows and columns whereas previously we added only one. Therefore consider (4.1) with the bars reinterpreted as signifying the addition of two rows (columns) and consider the ratio

$$r_y^2(R,t,N)=[D_y^2(R,t,N)]^{-1}\times\begin{vmatrix} 0 & \bar{S}^y & \bar{T}^y & \bar{U}^y \\ -\bar{S}^{yT} & 0 & -\bar{U}^y & \bar{V}^y \\ -\bar{T}^y & \bar{U}^y & 0 & -\bar{S}^y \\ -\bar{U}^y & -\bar{V}^y & \bar{S}^{yT} & 0 \end{vmatrix}. \tag{4.3}$$

This ratio may be obtained from the 4×8 matrix of $N/2-R+2$ component vectors X_b^a which are the solutions of the linear equations

$$\begin{bmatrix} |\bar{0} & \bar{S}^y| & |\bar{T}^y & \bar{U}^y| \\ -|\underline{S}^{yT} & \underline{0}| & -|\underline{U}^y & \underline{V}^y| \\ -|\bar{T}^y & \bar{U}^y| & |\bar{0} & -\bar{S}^y| \\ -|\underline{U}^y & -\underline{V}^y| & |\underline{S}^{yT} & \underline{0}| \end{bmatrix}\begin{bmatrix}\tilde{X}_1^i \\ \tilde{X}_2^i \\ \tilde{X}_3^i \\ \tilde{X}_4^i\end{bmatrix}=\tilde{\delta}^i, \tag{4.4}$$

where $\tilde{\delta}^i$ are the eight columns of

$$\begin{bmatrix} \tilde{\delta}_U^0 & \tilde{\delta}_U^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\delta}_D^0 & \tilde{\delta}_D^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\delta}_U^0 & \tilde{\delta}_U^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\delta}_D^0 & \tilde{\delta}_D^1 \end{bmatrix}, \tag{4.5}$$

where the $N/2-R+2$ component vectors $\tilde{\delta}_{U,D}^i$ are

$$\tilde{\delta}_U^0=\begin{bmatrix}1\\0\\ \vdots \\ 0\end{bmatrix},\quad \tilde{\delta}_U^1=\begin{bmatrix}0\\1\\ \vdots \\ 0\end{bmatrix},\quad \tilde{\delta}_D^0=\begin{bmatrix}0\\ \vdots \\ 1\\0\end{bmatrix},\quad \tilde{\delta}_D^1=\begin{bmatrix}0\\ \vdots \\ 0\\1\end{bmatrix}. \tag{4.6}$$

The ratio is expressible as an 8×8 determinant of the X_b^a. In the $N\to\infty$ limit we apply the argument used before and obtain

$$r_y(R,t)=\lim_{N\to\infty} r_y(R,t,v)=\pm\det\begin{bmatrix} X_{1,0}^1 & X_{1,0}^2 & X_{1,0}^5 & X_{1,0}^6 \\ X_{1,1}^1 & X_{1,1}^2 & X_{1,1}^5 & X_{1,1}^6 \\ X_{3,0}^1 & X_{3,0}^2 & X_{3,0}^5 & X_{3,0}^6 \\ X_{3,1}^1 & X_{3,1}^2 & X_{3,1}^5 & X_{3,1}^6 \end{bmatrix}, \tag{4.7}$$

where the X_b^a in the expression are ∞ component vectors obtained from the ∞ set of equations corresponding to (4.4). In this set of equations the right-hand side is

$$\begin{bmatrix} \delta^0 & \delta^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^0 & \delta^1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.8}$$

where

$$\delta^0=\begin{bmatrix}1\\0\\ \vdots \\ \vdots\end{bmatrix}\quad\text{and}\quad \delta^1=\begin{bmatrix}0\\1\\0\\ \vdots \\ \vdots\end{bmatrix}. \tag{4.9}$$

This equation may now be solved iteratively for X_b^a. The leading term of the iteration is

$$X_{1i}^1=X_{1i}^2=X_{3i}^5=X_{3i}^6=0,$$
$$X_{3i}^1=-X_{1i}^5=[(\tilde{S}^{yT})^{-1}V^y\hat{S}^{y-1}]_{i,0}, \tag{4.10}$$
$$X_{3i}^2=-X_{1i}^6=[(\hat{S}^{yT})^{-1}V^y\hat{S}^{y-1}]_{i1}\quad\text{for } i=0,1,$$

where for fixed rows and columns

$$\hat{S}^y=\lim_{n\to\infty}\bar{S}^y|=S^{xT}. \tag{4.11}$$

Therefore, substituting (4.10), (4.11), (2.36d), and (3.14) into (4.7), arguing as in Sec. III that

$$\lim_{R\to\infty}\lim_{N\to\infty} D_y(R,t,N)=D$$

is independent of t, and determining D by comparison of the $t=0$ result with (5.6) and (5.12) of Paper II [8] we obtain the final result

$$\rho_{yy}(R,t)\doteq\rho_{xx}(\infty)\begin{vmatrix} Y_R(t) & Y_{R+1}(t) \\ Y_{R-1}(t) & Y_R(t)\end{vmatrix}. \tag{4.12}$$

Here $\rho_{xx}(\infty)$ is given by (3.8) and

$$Y_R\doteq\frac{1}{2\pi i}\oint d\xi\,\xi^{R-1}e^{-it\Lambda(\xi)}\left[(1-\lambda_1^{-1}\xi)(1-\lambda_1^{-1}\xi^{-1})\times(1-\lambda_2\xi)(1-\lambda_2\xi^{-1})\right]^{-1/2}. \tag{4.13}$$

V. SUMMARY AND DISCUSSION

Time-dependent spin correlation functions are more commonly studied as functions of the frequency ω and the wave number k rather than as functions of t and R. Our expansions have all been for R large and therefore without further discussion we cannot compute the Fourier transform with respect to R. However, if R is large our expansions are valid for all times t. Therefore, defining the Fourier transform of $\rho_{vv}(R,t)-\rho_{vv}(\infty,t)$ with respect to t to be

$$\tilde{\rho}_{vv}(R,\omega)=\int_{-\infty}^{\infty} dt\, e^{i\omega t}[\rho_{vv}(R,t)-\rho_{vv}(\infty,t)]\,, \quad (5.1)$$

we find from (3.15), (3.33), (4.2), and (4.12) that for $h>1$, $0\le\gamma\le 1$,

$$\tilde{\rho}_{xx}(R,\omega)\doteq\tfrac{1}{4}[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4} \times\left(\frac{d\Lambda(k)}{dk}\right)^{-1}_{k=k(\omega)}(e^{ik(\omega)R}+e^{-ik(\omega)R}) \times\left(\frac{(1-\lambda_1^{-1}e^{ik(\omega)})(1-\lambda_1^{-1}e^{-ik(\omega)})}{(1-\lambda_2^{-1}e^{ik(\omega)})(1-\lambda_2^{-1}e^{-ik(\omega)})}\right)^{1/2} \quad (5.2a)$$

and

$$\tilde{\rho}_{yy}(R,\omega)=\tfrac{1}{4}[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4} \times\left(\frac{d\Lambda(k)}{dk}\right)^{-1}_{k=k(\omega)}(e^{ik(\omega)R}+e^{-ik(\omega)R})2\lambda_2\omega/(1+\gamma)\,, \quad (5.2b)$$

where $k(\omega)$ satisfies

$$\omega=\Lambda(k)=[(\cos k-h)^2+\gamma^2\sin^2 k]^{1/2}\,, \quad (5.3)$$

with

$$0\le k\le\pi\,. \quad (5.4)$$

From (5.3) we see that (5.2) vanishes unless

$$h-1<\omega<h+1\,. \quad (5.5)$$

Similarly we have for $h<1$, $0<\gamma\le 1$,

$$\tilde{\rho}_{xx}(R,\omega)=-[2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4}\frac{1}{2\pi}\int_{\pi}^{\pi}d\phi_1 \times\int_{-\pi}^{\pi}d\phi_2\,\delta(\Lambda(\phi_1)+\Lambda(\phi_2)-\omega)\,e^{iR(\phi_1+\phi_2)} \times|e^{i(\phi_1+\phi_2)}-1|^{-2}\frac{1}{2}\left[\frac{\Lambda(\phi_1)}{\Lambda(\phi_2)}+\frac{\Lambda(\phi_2)}{\Lambda(\phi_1)}-2\right] \quad (5.2c)$$

and

$$\tilde{\rho}_{yy}(R,\omega)=[2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4} \times\frac{1}{2\pi}\int_{\pi}^{\pi}d\phi_1\int_{\pi}^{\pi}d\phi_2\,\delta(\Lambda(\phi_1)+\Lambda(\phi_2)-\omega)\,e^{iR(\phi_1+\phi_2)} \times\tfrac{1}{8}[e^{-i(\phi_1-\phi_2)}+e^{i(\phi_1-\phi_2)}-2]\Lambda^{-1}(\phi_1)\Lambda^{-1}(\phi_2)\lambda_2^{-2}(1+\gamma)^2\,. \quad (5.2d)$$

Both (5.2c) and (5.2d) vanish unless,

$$2\omega_{\min}<\omega<2(h+1) \quad (5.6)$$

where

$$\omega_{\min}=1-h \quad \text{if} \quad 1-\gamma^2\le h \quad (5.7a)$$

$$=\gamma[1-h^2(1-\gamma^2)^{-1}]^{1/2} \quad \text{if} \quad 0\le h<1-\gamma^2\,. \quad (5.7b)$$

The fact that for $h>1$ and $R\gg 1$, the quantities ρ_{xx} and ρ_{yy} depend on ω only through $k(\omega)$ may be interpreted to mean that in (5.2a) and (5.2b) we are observing the one-elementary-excitation contribution of $\tilde{\rho}_{vv}(R,\omega)$ from elementary excitations with the dispersion relation $\omega=\Lambda(k)$ (see Fig. 1). The fact that for $h<1$ the quantity $\tilde{\rho}_{vv}(R,\omega)$ depends on ω not as $k(\omega)$, but as an integral over k_1 and k_2 satisfying $\omega=\Lambda(k_1)+\Lambda(k_2)$, implies that there are no single-elementary-excitation contributions to $\tilde{\rho}_{vv}(R,\omega)$ and that the first nonvanishing contribution is that from the two-elementary-excitation mode.

Of course, (5.2) is not the complete expression for $\tilde{\rho}_{vv}(R,\omega)$. It is merely the first term in an expansion. This expansion was originally conceived of as an expansion whose orders of magnitude are $(e^{-KR})^n$. However, each factor of e^{-KR} comes from an integral like $\int d\xi\,\xi^R e^{-it\Lambda(\xi)}$. Therefore, an equivalent interpretation of the expansion is that we are expanding $\rho_{vv}(R,t)$ in terms of 1, 2, ..., n, ..., elementary excitations. If we continue the expansion of higher orders we observe that for $h>1$, the quantity $\rho_{vv}(R,t)$ has contributions from 1, 2, 3, ..., elementary excitations. However, if $h<1$ the only contributions are from states with an even number of excitations.

When $h>1$, the vanishing of ρ_{vv} when $\omega<h-1$ is expected from (2.23). If we set $T=0$, this expansion reduces to

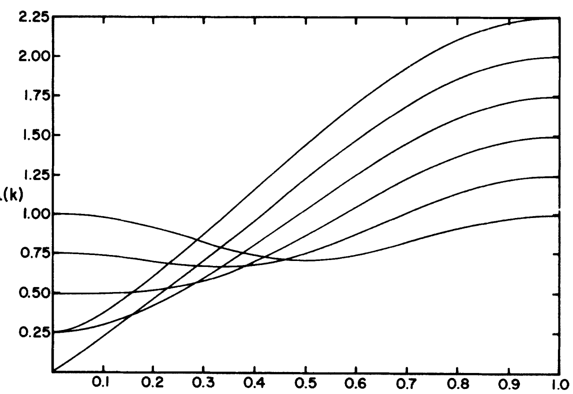

FIG. 1. $\Lambda(k)$ as a function of k for various values of h with $\gamma^2=\frac{1}{2}$. Note that $h=\Lambda(\pi)-1$.

$$\rho_{vv}(R,t)-\rho_{vv}(\infty,t)=\sum_{m,\,E_m\neq E_0}\langle E_0|S_0^v|E_m\rangle e^{it(E_m-E_0)}\times\langle E_m|S_R^v|E_0\rangle\,. \quad (5.8)$$

Since $E_m - E_0$ is made up of a sum of a finite number of $\Lambda(k)$ and since for $h>1$ we have $\Lambda(k)>h-1$, the Fourier transform of (5.8) will have to vanish if $\omega<h-1$. However, the vanishing of ρ_{vv} for $\omega<2\omega_{\min}$ shown in (5.6) for $h<1$ does not follow from these energetic considerations alone.

The vanishing of $\tilde\rho_{vv}(R,\omega)$ when $\omega<h-1$ or $\omega<2\omega_{\min}$ is a property of the correlation function of $T=0$ only. If $T>0$ the expansion (2.23) is still possible, but in this case the dominant contributions to ρ_{vv} come from $E_m=\langle H\rangle$ and, owing to the existence at $T>0$ of hole as well as particle excitations, there will exist states E_n (for example, the one-hole-one-particle state) for which $H-E_n$ is arbitrarily small (and even negative). Therefore, if $T>0$ we is general expect $\tilde\rho_{vv}(R,\omega)$ to be nonvanishing over the entire real ω axis. However, if T is sufficiently small and $h>1$ the contribution from the one elementary excitation is still expected to make the largest contribution.

If we interpret (5.2) as the contribution to $\tilde\rho_{vv}(R,\omega)$ from the one-excitation ($h>1$) or two-excitation ($h<1$) states, we may Fourier transform the results with respect to R and interpret the results not as an expansion in terms of a small parameter but rather as an expansion in the number of excitations. Defining the Fourier transform

$$\hat\rho_{vv}(k,\omega)=\sum_{R=-\infty}^{\infty}e^{ikR}\,\tilde\rho_{vv}(R,\omega)\,, \quad (5.9)$$

we obtain for $h>1$

$$\hat\rho_{xx}(k,\omega)\doteq\tfrac14\left[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)\right]^{1/4}\left[\delta(k-k(\omega))+\delta(k+k(\omega))\right]\left(\frac{d\Lambda(k)}{dk}\right)^{-1}_{k=k(\omega)}\left(\frac{(1-\lambda_1^{-1}e^{ik(\omega)})(1-\lambda_1^{-1}e^{-ik(\omega)})}{(1-\lambda_2e^{ik(\omega)})(1-\lambda_2e^{-ik(\omega)})}\right)^{1/2}$$

$$=\tfrac14\left[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)\right]^{1/4}\delta(\omega-\Lambda(k))\left(\frac{(1-\lambda_1^{-1}e^{ik})(1-\lambda_1^{-1}e^{-ik})}{(1-\lambda_2e^{ik})(1-\lambda_2e^{-ik})}\right)^{1/2} \quad (5.10a)$$

and

$$\hat\rho_{yy}(k,\omega)=\tfrac14\left[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1})\right]^{1/4}\left[\delta(k-k(\omega))+\delta(k+k(\omega))\right]\left(\frac{d\Lambda(k)}{dk}\right)^{-1}_{k=k(\omega)},\quad 2\lambda_2\omega/(1+\gamma)$$

$$=\tfrac14\left[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)\right]^{1/4}\delta(\omega-\Lambda(k))\,2\lambda_2\omega/(1+\gamma) \quad (5.10b)$$

and for $h<1$

$$\hat\rho_{xx}(k,\omega)=[2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4}\tfrac14[1-\cos k]^{-1}\left(\frac{\Lambda(k_1)}{\Lambda(k_2)}+\frac{\Lambda(k_2)}{\Lambda(k_1)}-2\right)\left(\frac{d}{dk_1}\left[\Lambda(k_1)+\Lambda(k-k_1)\right]\right)^{-1}, \quad (5.10c)$$

$$\hat\rho_{yy}(k,\omega)=[2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4}[\cos k-1]^{-1}\Lambda^{-1}(k_1)\Lambda^{-1}(k_2)\lambda_2^{-2}(1+\gamma)^2\left(\frac{d}{dk_1}\left[\Lambda(k_1)+\Lambda(k-k_1)\right]\right)^{-1}, \quad (5.10d)$$

where

$$k_1+k_2=k\,, \quad (5.11a)$$

$$\Lambda(k_1)+\Lambda(k_2)=\omega\,. \quad (5.11b)$$

Finally we consider the explicit asymptotic expansion of $\rho_{vv}(R,t)$ when either $R\gg t$ or $t\gg R$. If $R\to\infty$ and t is fixed we obtain[9] for $h>1$

$$\rho_{xx}(R,t)-\rho_{xx}(R,0)\sim-\tfrac{1}{64}(1-\lambda_2^2)^{3/4}(1-\lambda_1^{-2})^{1/4}(1-\lambda_1^{-1}\lambda_2)(1-\lambda_1^{-1}\lambda_2^{-1})^{3/2}\,\pi^{-1/2}(1+\gamma)^2\lambda_2^{R-2}R^{-1/2}(t^2/R) \quad (5.12a)$$

and

$$\rho_{yy}(R,t)-\rho_{yy}(R,0)\sim-\tfrac{3}{128}\,(1-\lambda_2^2)^{7/4}(1-\lambda_1^{-1}\lambda_2)(1-\lambda_1^{-2})^{1/4}(1-\lambda_1^{-1}\lambda_2^{-1})^{3/2}\,\pi^{-1/2}(1+\gamma)^2\lambda_2^{R-2}R^{-3/2}(t^2/R)\,; \quad (5.12b)$$

for $1-\gamma^2<h^2<1$,

$$\rho_{xx}(R,t)-\rho_{xx}(R,0)\sim-\tfrac{1}{64}\pi^{-1}[\gamma^2(1-h^2)]^{1/4}(1+\gamma)(1-\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2)(\lambda_2^2-1)^{-1}\lambda_2^{-2R}R^{-2}(t^2/R)\,, \quad (5.12c)$$

$$\rho_{yy}(R,t)-\rho_{yy}(R,0)\sim\tfrac{3}{32}\pi^{-1}(1+\gamma)[\gamma^2(1-h^2)]^{1/4}\lambda_2^{-2(R+1)}R^{-3}(t^2/R)\,; \quad (5.12d)$$

for $0\le h^2<1-\gamma^2$,

$$\rho_{xx}(R,t)-\rho_{xx}(R,0)\sim\tfrac{1}{32}\pi^{-1}(1+\gamma)[\gamma^2(1-h^2)]^{1/4}\alpha^{2R}R^{-2}(t^2/R)\ \mathrm{Re}\{2e^{2iR\phi}(1-\alpha^2)(1-e^{-2i\phi})(\alpha^{-2}e^{-2i\phi}-1)^{-1}$$

$$+\left|e^{i\psi}-e^{-i\psi}\right|\left|1-\alpha^2e^{2i\psi}\right|(\alpha^{-1}-1)^{-1}[3(\alpha^{-1}e^{-i\psi}-\alpha e^{i\psi})(\alpha^{-1}e^{i\psi}-\alpha e^{-i\psi})^{-1}+1]\}\,, \tag{5.12e}$$

$$\rho_{yy}(R,t)-\rho_{yy}(R,0)\sim\tfrac{1}{16}\pi^{-1}(1+\gamma)[\gamma^2(1-h^2)]^{1/4}(\alpha-\alpha^{-1})^2\alpha^{2R}R^{-1}(t^2/R)\mathrm{Re}\{(\alpha^{-1}e^{i\psi}-\alpha e^{-i\psi})^{1/2}(\alpha e^{i\psi}-\alpha^{-1}e^{-i\psi})^{-1/2}\}\,, \tag{5.12f}$$

where

$$\alpha=|\lambda_2|^{-1}=[(1-\gamma)/(1+\gamma)]^{1/2} \tag{5.13a}$$

and

$$\psi=\cos h/(1-\gamma^2)^{1/2}>0. \tag{5.13b}$$

If R is fixed and $t\to\infty$ we obtain for $h>1$

$$\rho_{xx}(R,t)\sim\tfrac14(2\pi)^{-1/2}[(1-\lambda_2^2)(1-\lambda_1^2)(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4}t^{-1/2}\{e^{\pi i/4}(1-\lambda_1^{-1})(\lambda_2-1)^{-1}(h-1)^{1/2}(h+\gamma^2-1)^{-1/2}e^{-it(h-1)}$$
$$+(-1)^Re^{\pi i/4}(1+\lambda_1^{-1})(1+\lambda_2)^{-1}(h+1)^{1/2}(h+\gamma^2+1)^{-1/2}e^{-it(h+1)}\}\,, \tag{5.14a}$$

$$\rho_{yy}(R,t)\sim\tfrac14(2/\pi)^{1/2}[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4}\lambda_2(1+\gamma)^{-1}t^{-1/2}$$
$$\times\{e^{-\pi i/4}(h-1)^{3/2}(h+\gamma^2-1)^{-1/2}e^{-it(h-1)}+(-1)^Re^{\pi i/4}(h+1)^{3/2}(h+1-\gamma^2)^{-1/2}e^{-it(h+1)}\}; \tag{5.14b}$$

for $1-\gamma^2<h<1$,

$$\rho_{xx}(R,t)\sim[2(1+\gamma)]^{-1}[\gamma^2(1-h_2)]^{1/4}\{1-(4\pi)^{-1}(-1)^Rh^2[(1-h^2)(h^2-(1-\gamma^2)^2)]^{-1/2}e^{-2it}t^{-1}\}, \tag{5.14c}$$

$$\rho_{yy}(R,t)\sim\tfrac14\lambda_2^{-2}(1+\gamma)[\gamma^2(1-h^2)]^{1/4}\pi^{-1}(-1)^R[(1-h^2)(h^2-(1-\gamma^2)^2)]^{-1/2}e^{-2it}t^{-1}; \tag{5.14d}$$

and for $0\le h<1-\gamma^2$,

$$\rho_{xx}(R,t)\sim[2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4}\{1+(4\pi t)^{-1}(-1)^{R+1}e^{\pi i/2}h^2[(1-h^2)(h^2-(1-\gamma^2)^2)]^{-1/2}e^{-2it}$$
$$+\cos(R+1)\theta_0\,\omega_0^{-1/2}(1-\gamma^2)^{3/2}[\omega_0-1+h]^2[(1-h)(1-\gamma^2-h^2)(1-\gamma^2+h)(1-\gamma^2-h)^3]^{-1/2}e^{-it[\omega_0+1-h]}$$
$$+(-1)^{R+1}\cos(R+1)\theta_0\,\omega_0^{-1/2}(1-\gamma^2)^{3/2}[\omega_0-1-h]^2[2(1+h)(1-\gamma^2+h^2)(1-\gamma^2-h)(1-\gamma^2+h)^3]^{1/2}e^{-it[\omega_0+1+h]}, \tag{5.14e}$$

$$\rho_{yy}(R,t)\sim\tfrac14\lambda_2^{-2}(1+\gamma)[\gamma^2(1-h^2)]^{1/4}\pi^{-1}t^{-1}\{(-1)^R[(1-h^2)((1-\gamma^2)^2-h^2)]^{-1/2}e^{\pi i/2}e^{-2it}+\tfrac12e^{-\pi i/2}\omega_0^{-1}(1-\gamma^2)^{-1}e^{-2it\omega_0}$$
$$+\sqrt{2}\cos R\theta_0[(1-h)(1-\gamma^2+h)\omega_0(1-\gamma^2)]^{-1/2}e^{-it(1-h+\omega_0)}$$
$$+\sqrt{2}(-1)^R\cos R\theta_0[(1+h)(1-\gamma^2-h)\omega_0(1-\gamma^2)]^{-1/2}e^{-it(1+h+\omega_0)}, \tag{5.14f}$$

where

$$\cos\theta_0=h/(1-\gamma^2) \tag{5.15a}$$

and

$$\omega_0=\Lambda(\theta_0)=\gamma[1-h^2(1-\gamma^2)^{-1}]^{1/2} \tag{5.15b}$$

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[8]Note that there is a misprint in (5.12) of Paper II.

[9]In the expansion of $\rho_{xx}(R,0)$ for $R\to\infty$ and $0\le h^2<1-\gamma^2$ given by (4.14) of II, a term has been omitted. The complete expression should be

$$\rho_{xx}(R)\sim\tfrac12(-1)^R(1+\gamma)^{-1}[\gamma^2(1-h^2)]^{1/4}[1+\pi^{-1}R^{-2}\alpha^{2R}$$
$$\times\mathrm{Re}(e^{2i\psi R}(\alpha^{-1}e^{-i\psi}-\alpha e^{i\psi})^{-2}\{1+(2R)^{-1}[\alpha^2(1-\alpha^2)^{-1}$$