PHYSICAL REVIEW A VOLUME 4, NUMBER 5 NOVEMBER 1971

COMMENTS AND ADDENDA

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Concept of Group Velocity in Resonant Pulse Propagation

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(Received 27 May 1971)

It is shown that the prediction of Garrett and McCumber that a light pulse's velocity may exceed the speed of light in a resonantly absorbing medium is due to asymmetric absorption of energy from the light pulse. More energy is absorbed from the trailing half of the pulse than from the front half, causing the center of gravity of the pulse to move at a velocity greater than the phase velocity of light. It is also shown that in certain cases a pulse's maximum will propagate at the group velocity even when the pulse as a whole is distorted by dispersion.

Textbooks introduce the concept of the group velocity of an electromagnetic pulse as a parameter useful in describing the propagation of a wave packet that is constructed from a narrow band of frequencies.[1] If dispersion is not too large, such a pulse will propagate at the group velocity without significant distortion. However, if a medium is too dispersive, the pulse will become highly distorted and the concept of group velocity is no longer meaningful.

When a pulse's frequency falls in a region of anomalous dispersion, it is possible for the group velocity to be greater than the velocity of light in a vacuum. During the first few years following Einstein's publication of the special theory of relativity, this behavior was difficult to reconcile with the postulate that required that no signal propagate at a velocity faster than the speed of light in a vacuum, c. The apparent contradiction was removed when Sommerfeld[2] showed that no signal could propagate faster than c and, in fact, the wave front progressed with a velocity equal to c in all media. Sommerfeld considered the problem of a light beam that has zero amplitude until it is turned on and thereafter has a constant amplitude. A further study of the transient details of the propagation of such a step-function light pulse was carried out by Brillouin.[3]

Garrett and McCumber[4] have pointed out that the pulses one obtains from some mode-locked lasers have envelopes that are approximately Gaussian in shape. They then proceeded to study analytically and numerically the propagation of a Gaussian-shaped pulse in a resonantly absorbing medium. It was found that under certain conditions the pulses do propagate with velocities greater than c without changing shape. They explained their results by recalling that a light pulse that has an envelope of the form $\mathcal{E}=\mathcal{E}_0 e^{-4t^2/\tau^2}$ extends infinitely in either direction along the time axis and hence has no true beginning or end. It was then argued that the observation of the Gaussian pulse at a depth z in the medium and at a time $t<z/c$ after the pulse entered the absorber is a result of "the action of the dispersive medium on the weak early components of the envelope."[5] Recently, Faxvog *et al.*[6] have measured the velocity of He-Ne mode-locked laser pulses that pass through a resonant Ne absorption cell. The measured velocities were greater than c by a few parts in 10^4, confirming the calculations of Garrett and McCumber. Experimental verification of such a prediction calls for a careful analysis of the dynamics of pulse propagation in order to understand how the results of Garrett and McCumber can be reconciled with the Sommerfeld-Brillouin analysis.

Reference 7 contains an independent prediction of the speeding up of a light pulse in a resonant attenuator and slowing down in a resonant amplifier. The current paper will take that result as a starting point and examine the dynamics of propagation of the superluminal and subluminal light pulses in a way that makes it perfectly obvious that they do not imply energy or signal propagation faster than the phase velocity in the medium. First of all it will be shown that the fact that the pulse has infinitely long leading and trailing edges is not important. This result is reassuring because the pulses from a mode-locked laser do have both a beginning and an end. The conclusion reached in this paper is that the apparent speeding up of a light pulse in an absorber is attributable to the finite response time of the resonant atoms to an applied field. For an attenuating medium, the inertia of the resonant dipoles results in an asymmetric absorption of energy from the light pulse. More energy is absorbed from the trailing half of the pulse than from the leading half and this results in a motion of the pulse maximum at a velocity greater than the phase velocity. Under certain conditions this asymmetric absorption of energy occurs in such a way that the transmitted pulse has the same shape as the incident pulse. The analysis carried out from this point of view has the advantage of making it obvious that, although the center of gravity of the attenuated pulse is moving faster than the speed of light, no signal or energy will propagate faster than the phase velocity c/η of light in the medium.

The electric field of a coherent light pulse which is linearly polarized in the x direction and propagates in the positive z direction can be written as

$$\vec{E}(z,t)=\hat{e}_x\,\mathcal{E}(z,t)\cos[\omega(t-\eta z/c)-\phi(z,t)]. \quad (1)$$

The resonant atoms are assumed to be imbedded in a host medium which is characterized by a constant index of refraction η. The pulse's electric field excites each of the resonant atoms into a superposition of its states producing a time varying expectation of its dipole moment operator. Because all the dipole moments are induced by the same coherent light pulse, they have a definite phase relationship and can add up to produce a macroscopic polarization that can be written as

$$P(z,t)=\mathrm{Re}[\mathcal{P}(z,t)e^{-i\omega(t-\eta z/c)}]. \quad (2)$$

When the amplitude $\mathcal{E}$ and phase ϕ of the pulse do not vary significantly over a distance of a wavelength or in a time equal to an optical period, the propagation of the pulse will be described to a good approximation by the reduced wave equation[8,9]

$$\left[\frac{\partial}{\partial z}+\frac{\eta}{c}\frac{\partial}{\partial t}\right]\mathcal{E}(z,t)\,e^{i\phi(z,t)}=\frac{2\pi i\omega}{\eta c}\mathcal{P}(z,t)\ . \quad (3)$$

The pulse propagation will be linear for time t such that

$$\left|\int_{-\infty}^{t}\mu\,\mathcal{E}(z,t)\,e^{i\phi(z,t)}\frac{dt}{\hbar}\right|^2\ll 1 \quad (4)$$

is satisfied.[10] When this condition holds, the atomic populations are not significantly altered and pulse propagation is described by a dispersion theory of the reduced wave equation. This dispersion theory is developed in Ref. 7 and only the important equations will be reproduced here. The amplitude $\mathcal{E}$ and phase ϕ of the pulse envelope will evolve according to

$$\mathcal{E}(z,t)\,e^{i\phi(z,t)}=(1/2\pi)\int_{-\infty}^{\infty}\epsilon(0,\nu)\times\exp\{-[i\nu(t-\eta z/c)+A(\nu)z]\}d\nu, \quad (5)$$

in the linear regime. The Fourier transform of the amplitude and phase of the pulse at the medium boundary $z=0$ is equal to

$$\epsilon(0,\nu)=\int_{-\infty}^{\infty}\mathcal{E}(0,t)\,e^{i[\phi(0,t)+\nu t]}\,dt. \quad (6)$$

The quantity $A(\nu)$ is defined in terms of the homogeneous relation time T_2' and the inhomogeneous line-shape function $g(\Delta)$ according to

$$A(\nu)=\alpha_0\int_{-\infty}^{\infty}\frac{g(\Delta)d\Delta}{1/T_2'+i(\Delta-\nu)}\ . \quad (7)$$

The constant $\alpha_0=-2\pi N\mu^2\omega Z(0)/\hbar\eta c$ is positive when the resonant atoms are initially in their ground state (an attenuating medium) and negative when the atoms are initially in their excited state (an amplifier). In the work that follows it will be assumed that the applied frequency ω is tuned to the center of a symmetric absorption line so that $g(\Delta)=g(-\Delta)$ and that the light pulse has no phase modulation ($\phi=0$).

The group-velocity approximation will be valid for pulses having a spectral width narrow compared with the total linewidth $1/T_2$. If $\epsilon(0,\nu)$ in Eq. (5) is negligible for frequencies $|\nu|>1/\tau$, then the major contribution to the integral comes from the region $|\nu|<1/\tau$. Now if τ is large compared with the reciprocal of the linewidth T_2, then the quantity $A(\nu)$ may be expanded about $\nu=0$ and only the first two terms kept, i.e.,

$$A(\nu)\simeq A(0)+\nu A'(0)\ . \quad (8)$$

Substituting this into Eq. (5) one obtains

$$\mathcal{E}(z,t)=e^{-A(0)z}\mathcal{E}(0,t-z/v_g)\ , \quad (9)$$

where the group velocity is given by

$$v_g=[\eta/c-iA'(0)]^{-1}\ . \quad (10)$$

The Beer's law absorption coefficient α can be identified from Eq. (9) to be equal to $2A(0)$. The derivation of Eq. (9) does *not* require that the pulse have a finite amplitude for all finite time.

It is useful to consider the specific case of a Lorentzian inhomogeneous line shape

$$g(\Delta) = (T_2^* / \pi)\,[1 + (\Delta T_2^*)^2]^{-1} \quad . \tag{11}$$

It was shown in Ref. 7 that for this line shape $A(0) = \alpha_0 T_2$ and $-iA'(0) = \alpha_0 T_2^2$, where the total linewidth is $1/T_2 = 1/T_2' + 1/T_2^*$.

The polarization induced by a pulse of the form of Eq. (9) can be obtained by substituting into Eq. (3)

$$\frac{2\pi i\omega}{\eta c}\, \mathcal{P}(z,t) = \left[-A(0) + iA'(0)\, \frac{\partial}{\partial t} \right] \mathcal{E}(z,t) \quad . \tag{12}$$

For the specific case of a Lorentzian inhomogeneous line shape this expression becomes

$$\mathcal{P}(z,t) = \frac{i\eta c}{2\pi\omega}\, \alpha_0 T_2 \left[\mathcal{E}(z,t) - T_2 \frac{\partial}{\partial t} \mathcal{E}(z,t) \right] . \tag{13}$$

If $\partial\mathcal{E}/\partial t$ were zero, Eq. (12) would give the amplitude of the steady-state polarization induced by a constant amplitude light beam. In the case of an attenuating medium, $A(0)$ is positive and this term would lead to a macroscopic polarization that would radiate 180° out of phase with respect to the pulse that induced it. The field radiated by this polarization would cancel part of the applied field, resulting in a decrease in the amplitude of the transmitted light beam. The term involving $\partial\mathcal{E}/\partial t$ is important for understanding the dynamic response of the polarization to a light pulse. Consider the polarization induced by a pulse which has a roughly bell-shaped envelope. During the front half of the pulse the amplitude is rising, $\partial\mathcal{E}/\partial t$ is positive and the second term in Eq. (13) will result in a polarization with an amplitude less than the steady-state value. During the trailing half of the pulse $\partial\mathcal{E}/\partial t$ is negative and the second term would increase the amplitude of the polarization over the steady-state value. In an attenuator, the macroscopic polarization is responsible for absorption of energy from the pulse so that the fact that the amplitude of the polarization is larger during the trailing half of the pulse than during the leading half implies that more energy will be absorbed from the tail of the pulse than from the front. This asymmetric absorption of energy results in a forward motion of the center of gravity of the pulse so that it appears that the pulse is moving faster than c/η.

The evolution of a pulse of the form

$$\mathcal{E}(0,t) = e^{-(4t^2/\tau^2)} \quad , \tag{14}$$

with a value of τ chosen so that Eq. (9) is valid, is illustrated in Fig. 1. The figure shows the pulse amplitude as a function of the retarded time $t - \eta z/c$ at various depths z in the attenuating medium. If the resonantly absorbing atoms were absent, all the curves would coincide with the $z = 0$ curve. This would indicate that the pulse propagated without attenuation at the phase velocity c/η. The presence of the absorbing atoms results in an asymmetric absorption of energy from the pulse which has the result that each successive pulse envelope is Gaussian in shape, having a center which has advanced to the left indicating that the pulse is moving faster than c/η. In Fig. 1 the pulse amplitude for a given value of $t - \eta z/c$ and at a depth $z > 0$ is less than the corresponding value for $z = 0$. It is shown in Appendix A that this is a general property, i.e.,

$$e^{-A(0)z}\,\mathcal{E}(0, t - z/v_g) \le \mathcal{E}(0, t - (\eta/c)z) \tag{15}$$

when the conditions that lead to the derivation of Eq. (9) are satisfied. The implication of Eq. (15) is that at no depth z or time t will the amplitude of a pulse propagating according to the group-velocity expression of Eq. (9) be greater than the ampli-

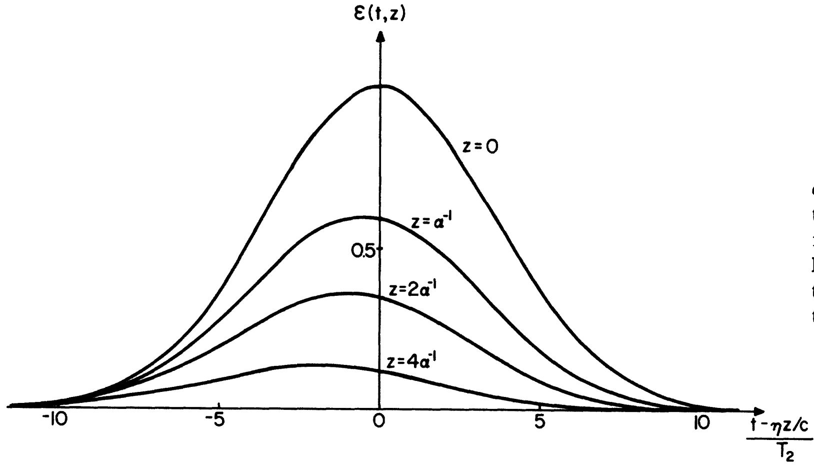

FIG. 1. This figure shows the pulse envelope as a function of the retarded time $t - nz/c$ for various depths in a resonant attenuator. The input pulse has a Gaussian shape and is long enough to satisfy the approximations that lead to Eq. (9).

tude that the pulse would have had if it were just propagating at the phase velocity c/η without loss. This result eliminates any possibility that an energy or a signal velocity might exceed c/η in a resonant attenuator.

For an amplifier, $A(0)$ is negative and an argument similar to the one presented above leads to the conclusion that the macroscopic polarization radiates more energy into the trailing half of the pulse than it does into the leading half. This asymmetric amplification of the pulse in an amplifier results in an apparent slowing down of the pulse. This effect has been observed experimentally.[11]

If the conditions that were required for the derivation of Eq. (9) are not satisfied because the pulse spectrum is too broad, then pulse distortion becomes important. In the case of an attenuator the pulse distortion can take the form of oscillations of the pulse envelope between positive and negative values (corresponding to changes of phase of the amount π). For an amplifier, the pulse can show an asymmetric broadening. Behavior of this type was extensively studied in Ref. 7.

A result which is relevant to the experiments of Faxvog *et al.* and Casperson and Yariv is proven in Appendix B. In the case of a pulse having an envelope which is initially symmetric about $t=0$ [that is, if $\mathcal{E}(0,t)=\mathcal{E}(0,-t)$], the pulse maximum will travel at the group velocity even when the pulse's spectrum is so broad that the next highest term in the expansion of Eq. (8) must be retained and the pulse exhibits distortion. Thus an experiment that determines the velocity of a pulse by observing the arrival of its maximum may confirm the group-velocity expression even when the pulse is being distorted.

APPENDIX A

This appendix will establish conditions under which

$$e^{-A(0)z}\mathcal{E}(0,q-iA'(0)z)\le\mathcal{E}(0,q) \tag{A1}$$

will be satisfied. When $q=t-\eta z/c$, the left-hand side of the inequality is the group-velocity expression for the propagation of the pulse. The right-hand side is the amplitude of the pulse that would have been observed at position z and time t if no absorbing atoms were present. The above expression may be rewritten as

$$\frac{\ln\mathcal{E}(0,q-iA'(0)z)-\ln\mathcal{E}(0,q)}{-iA'(0)z}\le\frac{A(0)}{-iA'(0)} \quad . \tag{A2}$$

Considering the limit of Eq. (A2) for small z, one finds that a necessary condition for Eq. (A1) to be satisfied is

$$\frac{1}{\mathcal{E}(0,t)}\,\frac{\partial\mathcal{E}(0,t)}{\partial t}\le\frac{A(0)}{-iA'(0)} \quad . \tag{A3}$$

Written for the case of a Lorentzian line shape of the form shown in Eq. (11), the condition becomes

$$\frac{T_2}{\mathcal{E}(0,t)}\,\frac{\partial\mathcal{E}(0,t)}{\partial t}\le 1 \quad . \tag{A4}$$

Equation (A4) will be satisfied when the pulse amplitude does not change a great deal in a time T_2. The left-hand side of Eq. (A3) will be significantly less than the right-hand side for a pulse that satisfies the conditions leading to Eq. (9).

The argument that leads to Eq. (A3) can be reversed if one considers a medium of thickness z to be composed of many thin slices of thickness Δz. This means that Eqs. (A1) and (A3) are equivalent.

APPENDIX B

Consider a pulse which propagates according to Eq. (5) with $\phi=0$. Assume the pulse envelope has a single maximum at $t=0$ and is symmetric about that point. It is necessary for

$$\frac{\partial\mathcal{E}(z,q)}{\partial q}=0=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\nu\epsilon(0,\nu)e^{-[i\nu q+A(\nu)z]}d\nu \tag{B1}$$

[where q is equal to $t-(\eta/c)z$] to be satisfied in order for the pulse to have a maximum. The symmetry of the pulse envelope ($\mathcal{E}(0,t)=\mathcal{E}(0,-t)$) implies that $\epsilon(0,\nu)=\epsilon(0,-\nu)$, and using this result shows that $q=0$ and $z=0$ does satisfy Eq. (B1). When $A(\nu)$ is expanded up to and including the quadratic term, Eq. (B1) becomes

$$(-i/2\pi)\int_{-\infty}^{\infty}\nu\epsilon(0,\nu)\exp\{-i\nu[q+iA'(0)z]-A''(0)\nu^2z\}\times d\nu\, e^{-A(0)z}=0 \quad . \tag{B2}$$

The fact that $A''(0)\nu^2$ is also symmetric about $\nu=0$ implies that Eq. (B2) will be satisfied for $t=[\eta/c-iA'(0)]z$. Thus the pulse maximum moves along with the group velocity defined by Eq. (10) even though the additional term will lead to pulse distortion.

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[6]F. R. Faxvog, C. N. Y. Chow, T. Bieber, and J. A. Carruthers, Appl. Phys. Letters 17, 192 (1970).

[7]M. D. Crisp, Phys. Rev. A 1, 1604 (1970).

[8]The use of this equation precludes the phenomena of precursors because the index of refraction is constant implying that the mechanisms that contribute to η have an instantaneous response time.

[9]The use of Eq. (3) instead of the second-order wave equation neglects the light that is radiated in the backward direction. This approximation is discussed in the following: M. D. Crisp, Opt. Commun. 1, 54 (1969).

[10]M. D. Crisp (unpublished).

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PHYSICAL REVIEW A VOLUME 4, NUMBER 5 NOVEMBER 1971

Classical Limit of the Binary-Collision Expansion

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(Received 5 March 1971)

It is shown that the classical form of the binary-collision expansion follows naturally from the multiple-scattering expansion for the T matrix. The limits of applicability of this classical expansion are discussed.

The classical limit of the Lee-Yang binary-collision expansion[1] was first discussed by Pais and Uhlenbeck.[2] They showed that the binary expansion can be looked upon as a quantum-mechanical extension of the classical Ursell-Mayer expansion, and that to each classical Ursell graph there corresponds a class of quantum graphs. They gave a prescription for writing down the contribution to the cluster function from these quantum graphs, and showed that in the limit as $\hbar \to 0$, these contributions can be summed to give the correct classical expression corresponding to the Ursell graph.[3]

More recently, Hecht and Lind[4] studied the particular case of hard spheres. They obtained an expression for the classical limit of the binary kernel, and explicitly evaluated the terms appearing in the binary-collision expansions of the first four cluster coefficients. They showed that the correct classical values for these cluster coefficients are obtained provided one sums all the terms in the series. For coefficients greater than the second, these series contain an infinite number of terms.

The purpose of this note is to give an alternative formulation of the classical limit. It is well known[5,6] that there is a close relationship between the binary-collision expansion and the Watson multiple-scattering expansion for the T matrix[7]—in fact, the latter is the "time-independent" counterpart of the former, and they are related by a Laplace transform. We shall show that the passage to the classical limit follows more naturally in the T matrix formalism, and that the character of the resulting expansion is more readily apparent.

We first summarize the multiple-scattering form of the binary-collision expansion for the quantum-mechanical case. The details can be found in Refs. 2, 6, and 7. Consider a system of N particles in a container of volume V with Hamiltonian

$$H_N = H_N^0 + \sum_{i>j} v_{ij} , \qquad (1)$$

where H_N^0 is the kinetic energy of the N particles and v_{ij} is a pair potential. The lth-cluster coefficient b_l is given in terms of the lth-cluster operator U_l by

$$b_l = \lim (1/Vl!)\,\mathrm{Tr}(U_l) \quad \text{as } V \to \infty , \qquad (2)$$

where the trace is to be taken over a complete set of l-particle states. In order to obtain an expression for U_l, we introduce a two-body scattering operator $T_\alpha(z)$ defined by

$$T_\alpha(z) = v_\alpha + v_\alpha G_0(z)\, T_\alpha(z), \qquad (3)$$

where

$$G_0(z) = 1/(z - H_l^0). \qquad (4)$$

α is a pair index and z is a complex variable. We also need the operator

$$\mathcal{L}_\beta^{-1} = \frac{1}{2\pi i} \int_C dz\, e^{-\beta z}, \qquad (5)$$

where C is a contour in the z plane enclosing the spectrum of H_N on the real axis. U_l is then given in terms of the T operator by

$$U_l = \mathcal{L}_\beta^{-1} \sum (G_0\, T_{\alpha_1}\, G_0\, T_{\alpha_2} \cdots G_0\, T_{\alpha_m}\, G_0). \qquad (6)$$

The various terms under the summation sign in (6) are represented by graphs. Each graph consists of l vertical particle lines, and the T operators are represented by horizontal "blocks" connecting two particle lines. The sum in (6) is then over all possible connected l-particle graphs, with the following restrictions: (a) No two adjacent T_α's can have the