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Exact Screened Calculations of Atomic-Field Pair Production and Low-Energy Screening*

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A discussion of relativistic-pair-production cross sections is given for incident-photon energies in the range 1.073–2.615 MeV. Cross sections are obtained numerically for a screened potential, using an exact partial-wave formulation. Comparisons are made with previous authors' results. For low photon energies the effect of atomic-electron screening on pari-production cross sections is significant. The screening increases the cross section, since the atomic electrons decrease the nuclear Coulomb repulsion of the positron. Comparisons of screened with unscreened results show that angular distributions are essentially unaffected in shape by the screening; cross sections are simply renormalized by replacing $|N_+N_-|^2$ in the point-Coulomb field by $|N_+N_-|^2$ in the screened field, where $N_\pm$ are the normalizations of the positron and electron wave functions at the origin.

I. INTRODUCTION

Recently Øverbø, Mork, and Olsen (ØMO)[1] have reported extensive exact calculations of relativistic atomic-field pair production in a point-Coulomb-potential model (i.e., neglecting screening) for photon energies from threshold to a few MeV, using a method similar to that of Jaeger and Hulme.[2] Large deviations from the Born approximation are found. The theoretical situation has been summarized by Motz, Olsen, and Koch[3] in their review article on this process. However recent experiments of Yamazaki and Hollander[4] disagreed with the deviation from the Bethe-Heitler[5] formula predicted by Jaeger and Hulme. This discrepancy also exists between the experimental data of Yamazaki and Hollander and the point-Coulomb results of ØMO.

We believe that the exact calculation of ØMO is good[6] in so far as the point-Coulomb-potential model is good. The basic question now is to see if the effects of atomic-electron screening on pair-production cross sections in this energy region are actually negligible and if not whether they can be estimated in any simple fashion. Until now such estimates have been made with a form-factor approach based on the Born approximation.[7] Such an approach predicts that the screening effect is small in this low-energy region. For high energies, since the most important contributions to the cross section come from the large-r region, the screening effect becomes important.[8]

Following the methods of our atomic-field bremsstrahlung calculations,[9] we wish here to present a discussion of the calculation of relativistic-pair-production cross sections for incident-photon energies in the range 1.073–2.615 MeV, and to examine the validity of the Born-approximation screening theory.[10] In Sec. II we give a brief survey of pair-production theory and our numerical methods. Comparisons with previous results are presented in Sec. III. We also use there our results to examine screening effects and give a prescription for the low-energy pair-production screening correction. We have verified that the only significant effect of atomic-electron screening comes from the change in normalizations of the continuum-positron-state and the continuum-electron-state wave functions, a result which is reminiscent of the normalization screening theory of the atomic-field photoeffect.[11]

II. THEORY

We use the same formalism we used for the atomic-field bremsstrahlung calculation,[9] namely, Furry's extension of the usual Feynman-Dyson formulations of quantum electrodynamics,[12] including the interaction of the electrons and the positrons with the atomic field in the unperturbed Hamiltonian. For pair production the photon is absorbed instead of emitted, and the initial electron state is replaced by one of a negative energy. The calculation neglects radiative corrections, which are quite small in the energy region we consider here.[13]

The (incoming and outgoing) electrons in the atomic field and the photons are described by the Dirac operator $\Psi(x)$ and the Maxwell operator $A^{\text{rad}}(x)$, respectively. The cross sections in this approximation are obtained from the matrix elements of the S matrix between the initial and final states,

$$\begin{aligned} S_{fi} &= \langle p_- p_+ | S | k \rangle \\ &\cong \langle p_- p_+ | 1 - ie \int d^4x : \overline{\Psi}(x) A\!\!\!/^{\text{rad}}(x) \Psi(x) : | k \rangle \\ &= ie(2\pi/k)^{1/2} \int d^4x\, \psi_2^\dagger(\vec{p}_-, \vec{r}, \vec{\zeta}_-)\vec{\alpha} \\ &\qquad \cdot \vec{\epsilon}\psi_1(-\vec{p}_+, \vec{r}, \vec{\zeta}\,') e^{i\vec{k}\cdot\vec{r}} e^{-i(k-E_+-E_-)t} \\ &= -i2\pi M_{fi}\,\delta(k - E_+ - E_-)\ . \end{aligned}$$

Here

$$M_{fi} = (2\pi\alpha/k)^{1/2} \int d^3r\, \psi_2^\dagger(\vec{p}_-, \vec{r}, \vec{\zeta}_-)\vec{\alpha} \cdot \vec{\epsilon}\psi_1(-\vec{p}_+, \vec{r}, \vec{\zeta}\,') e^{i\vec{k}\cdot\vec{r}}\ ; \quad (2.1)$$

$\psi_2(\vec{p}_-, \vec{r}, \vec{\zeta}_-)$ is the electron wave function asymptotically normalized to a unit amplitude modified plane of four-momentum $(E_-, \vec{p}_-)$ and four-polarization $(0, \vec{\zeta}_-)$ in its rest frame plus an incoming spherical wave; and the positron wave function contains asymptotically spherical incoming waves, as the substitutions $E_1 \to -E_+$, $\vec{p}_1 \to -\vec{p}_+$ (but $|\vec{p}_1| \to |\vec{p}_+|$), and $\vec{\zeta}_1 \to \vec{\zeta}\,' = \vec{\zeta}_+ - 2\hat{p}_+(\hat{p}_+ \cdot \vec{\zeta}_+)$ change outgoing into incoming spherical waves, namely,

$$i\gamma^2 \psi_+^{\text{in}*}(P_+, \vec{r}, \vec{\zeta}_+) = \psi_1^{\text{out}}(-P_+, \vec{r}, \vec{\zeta}\,')\ ,$$

where ψ_+ is the positron wave function and P is the four-momentum. That is,[14]

$$\left\{ \begin{matrix} \psi_1(\vec{p}, \vec{r}, \vec{\zeta}) \\ \psi_2(\vec{p}, \vec{r}, \vec{\zeta}) \end{matrix} \right\} = 4\pi \sum_{\kappa m} [\Phi_{\kappa m}^\dagger(\hat{p})\chi(\vec{\zeta})]\, i^{l-1} e^{(\pm) i\delta_\kappa} \psi_{\kappa m}(\vec{r})\ , \quad (2.2)$$

where

$$\psi_{\kappa m}(\vec{r}) = r^{-1} \begin{pmatrix} ig_\kappa(r)\Phi_{\kappa m}(\hat{r}) \\ -f_\kappa(r)\Phi_{-\kappa m}(\hat{r}) \end{pmatrix}\ ,$$
$$\Phi_{\kappa m}(r) = \textstyle\sum_s C(l\tfrac{1}{2}j;\, m-s, s)\, Y_{l,m-s}(\hat{r})\chi^s\ . \quad (2.3)$$

Here we use the split representation. The radial wave functions g_κ and f_κ satisfy the equations

$$\frac{dg}{dr} = (p_0 + 1 - V)f - \frac{\kappa g}{r}\ , \quad \frac{df}{dr} = -(p_0 - 1 - V)g + \frac{\kappa f}{r}\ , \quad (2.4)$$

with $p_0 = -E_+$ for ψ_1 and $p_0 = E_-$ for ψ_2, where V is the screened central potential.

The transition probability per unit time between the initial and the final states is then

$$W_{fi} = 2\pi |M_{fi}|^2 \delta(k - E_+ - E_-)\ ,$$

from which we can obtain the cross section by summing over the energies of the final state and dividing by the flux of incoming particles (in this case the velocity of the incident photon $c = 1$ relative to the atom). Since the density of final states is given by

$$\rho_F = \rho(\vec{p}_+)\rho(\vec{p}_-)\, dE_+ dE_-\ ,$$

where

$$\rho(\vec{p}_\pm) = p_\pm E_\pm\, d\Omega_\pm/(2\pi)^3\ ,$$

we obtain the cross section, after integration over energies E_- of the outgoing electron,

$$d\sigma = (2\pi)^{-5} p_- E_- p_+ E_+ |M_{fi}|^2 dE_+ d\Omega_+ d\Omega_-\ . \quad (2.5)$$

Choosing a coordinate system centered at the atomic nucleus with the z axis along $\vec{k}$, $\hat{y}$ along $\vec{k} \times \vec{p}_+$, and $\hat{x}$ in the $(\vec{k}, \vec{p}_+)$ plane, and inserting Eqs. (2.2) into Eq. (2.1), we obtain

$$M_{fi} = 16\pi^2 \left(\frac{2\pi\alpha}{k}\right)^{1/2} \sum_{\kappa_1 m_1 \kappa_2 m_2} [\Phi_2^\dagger(\hat{p}_-)\chi_2]^\dagger [\Phi_1^\dagger(-\hat{p}_+)\chi_1] \times (-)^{l_1} e^{i(\delta_{\kappa 1} + \delta_{\kappa 2})} [\epsilon_- R_+(m_2) + \epsilon_+ R_-(m_2)]\ ,$$

where $R_\pm(m_2)$ are defined by Eqs. (2.9) of Ref. 9 but

$$P_1^\pm(m) = (-)^{(l_2' + l_1 - l)/2} T(l_2', l_1, l;\, m \mp \tfrac{1}{2})\ ,$$
$$P_2^\pm(m) = (-)^{(l_2 + l_1' - l)/2} T(l_2, l_1', l;\, m \mp \tfrac{1}{2})\ . \quad (2.6)$$

Thus, all corresponding formulas for pair production are obtained from the bremsstrahlung formulas by the following substitutions[15]:

$$E_1 \to -E_+,\quad \vec{p}_1 \to -\vec{p}_+(|\vec{p}_1| = |\vec{p}_+|)\ ,$$
$$\lambda_0 \to [(3.86144)^2 \times 10^5](32\alpha/Z^2k)\, p_- E_- p_+ E_+\ ,$$

TABLE I. Pair-production cross sections $\sigma(E_+)$ as computed in this work for $k=2.10\ m_ec^2$. Symbols B, C, HFS refer to the Born approximations, point-Coulomb, modified Hartree-Fock-Slater potentials; and n_{κ_1} and n_{κ_2} are the numbers of κ_1's and of κ_2's used in this work, respectively. σ^B, σ^C, and σ^{HFS} are in units of μb/m_ec^2.

		$Z=29$		$Z=32$	$Z=92$			
$y=(E_+-1)/(k-2)$	$\sigma^B(E_+)$	$\sigma^C(E_+)$	$\sigma^{\text{HFS}}(E_+)$	$\sigma^{\text{HFS}}(E_+)$	$\sigma^C(E_+)$	$\sigma^{\text{HFS}}(E_+)$	n_{κ_1}	n_{κ_2}
0.95	0.7350	···	4.890	5.248	···	4.370	12	8
0.9	1.012	···	4.660	4.968	···	3.725	12	8
0.8	1.351	···	4.145	4.354	···	2.575	12	10
0.7	1.549	···	3.565	3.679	···	1.657	12	10
0.6	1.656	···	2.933	2.960	0.1637	0.9715	10	10
0.5	1.691	1.872	2.285	2.244	···	0.4999	10	10
0.4	1.656	···	1.575	1.497	···	···	10	10
0.3	1.549	···	0.9273	0.8427	···	0.06613	10	12
0.2	1.351	···	0.3842	0.3269	···	···	10	12
0.1	1.012	0.01335	0.06826	0.05215	···	0.0005056	8	12

$P_n^{\pm}(m)$ are defined by Eqs. (2.6):

$$kZ^{-2}\frac{d\sigma}{dk\,d\Omega_k}\equiv\sigma(k,\theta)\to Z^{-2}\frac{d\sigma}{dE_+\,d\Omega_+}\equiv\sigma(E_+,\theta_+)$$

$$\text{(in mb/sr}-m_ec^2)\ , \quad (2.7)$$

$$kZ^{-2}\frac{d\sigma}{dk}\equiv\sigma(k)\to Z^{-2}\frac{d\sigma}{dE_+}\equiv\sigma(E_+)\quad(\text{in mb}/m_ec^2)\ .$$

The total cross sections are obtained by integration of the energy spectra

$$\sigma=\int_1^{k-1}\frac{d\sigma}{dE_+}\,dE_+\ . \quad (2.8)$$

Pair production and bremsstrahlung are closely related processes, and the method suitable for calculating the latter are also suitable for the former. Since the positron is considered as a negative-energy electron, all the formulas in Sec. III of Ref. 9 are used unchanged except that the factor $[(E+1)/2E]^{1/2}$ is generalized as $[(p_0+1)/2p_0]^{1/2}$ and the factor $[(E-1)/2E]^{1/2}$ is generalized as $-\eta_{p_0}[(p_0-1)/2p_0]^{1/2}$, where $\eta_{p_0}=-p_0/|p_0|$.

The errors of the numerical calculation are estimated to $O(1\%)$ for $\sigma(E_+,\theta_+)$, $O(0.5\%)$ for $\sigma(E_+)$, and $O(1\%)$ for σ results.[16]

TABLE II. Pair-production cross sections $\sigma(E_+)$ as computed in this work for $k=2.60\ m_ec^2$. σ^B, σ^C, and σ^{HFS} are in units of μb/m_ec^2.

		$Z=13$		$Z=79$			
y	$\sigma^B(E_+)$	$\sigma^C(E_+)$	$\sigma^{\text{HFS}}(E_+)$	$\sigma^C(E_+)$	$\sigma^{\text{HFS}}(E_+)$	n_{κ_1}	n_{κ_2}
0.95	14.73	···	23.55	109.4	109.1	20	10
0.9	20.33	26.58	26.44	108.2	108.8	20	10
0.8	27.20	···	30.95	102.5	104.9	18	12
0.7	31.24	34.04	33.98	92.56	96.79	18	14
0.6	33.44	···	35.69	78.38	84.55	16	14
0.5	34.15	35.90	35.94	60.86	68.07	16	16
0.4	33.44	···	34.55	41.65	49.08	14	16
0.3	31.24	30.90	31.06	21.55	28.70	14	18
0.2	27.20	···	24.98	6.840	11.66	12	18
0.1	20.33	14.27	14.68	0.4079	1.572	10	20

III. RESULTS AND DISCUSSION

We list in Tables I–III all cases [specified by k, Z, and $y\equiv(E_+-1)/(k-2)$] we have computed. The tables also give the pair-production cross section $\sigma(E_+)$ integrated over angles which we obtained with the point-Coulomb (C) and the modified Hartree-Fock-Slater (HFS) potential models[17]; these values correspond to single points in the energy spectrum of the pair-production cross section, as shown in Figs. 1 and 2. The total pair-

TABLE III. Pair-production cross sections $\sigma(E_+)$ as computed in this work for the cases other than those shown in Tables I and II.

k (m_ec^2)	Z	Y	$\sigma^B(E_+)$ (μb/m_ec^2)	$\sigma^C(E_+)$ (μb/m_ec^2)	$\sigma^{\text{HFS}}(E_+)$ (μb/m_ec^2)	n_{κ_1}	n_{κ_2}
2.10	1	0.9	1.012	1.102	···	12	8
	1	0.1	1.012	0.9284	···	8	12
	6	0.5	1.691	···	1.767	10	10
	13	0.5	1.691	···	1.993	10	10
	53	0.5	1.691	0.7439	1.537	10	10
	82	0.5	1.691	0.1156	0.6845	10	10
2.20	6	0.5	5.958	···	6.118	12	12
	13	0.5	5.958	···	6.641	12	12
	29	0.5	5.958	7.738	8.212	12	12
	53	0.5	5.958	···	8.491	12	12
	82	0.5	5.958	···	6.259	12	12
	92	0.5	5.958	···	5.333	12	12
2.50	13	0.5	26.35	···	27.98	16	16
	29	0.5	26.35	···	33.51	16	16
	53	0.5	26.35	···	43.95	16	16
	82	0.5	26.35	···	50.93	16	16
	92	0.5	26.35	···	50.76	16	16
2.60	18	0.5	34.15	···	37.55	16	16
	29	0.5	34.15	···	42.51	16	16
	44	0.5	34.15	···	51.18	16	16
	68	0.5	34.15	···	63.83	16	16
	82	0.5	34.15	···	68.63	16	16
	92	0.5	34.15	···	69.67	16	16
3.50	82	0.7	88.42	87.7	···	30	18
5.117	79	0.7	141.2	94.3	192.2	40	22
(2.615 MeV)	79	0.5	152.8	96.2	195.6	30	30

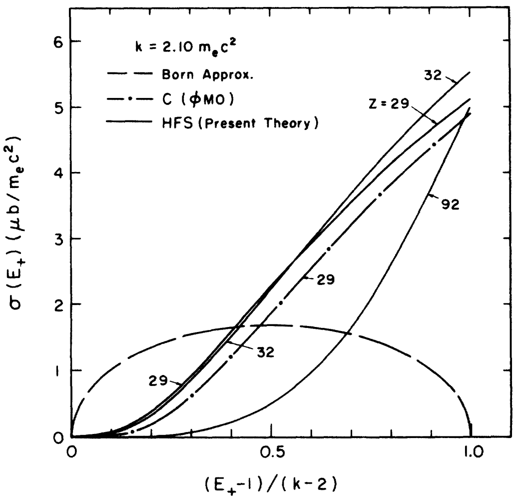

FIG. 1. Pair-production differential cross sections $\sigma(E_+)$ of present theory for HFS field (solid line), compared with the Born-approximation results (broken line), and those of ØMO (dotted-broken line) for point-Coulomb field, $k=2.10\ m_ec^2$. The numbers attached to the curves give the atomic number of the target element.

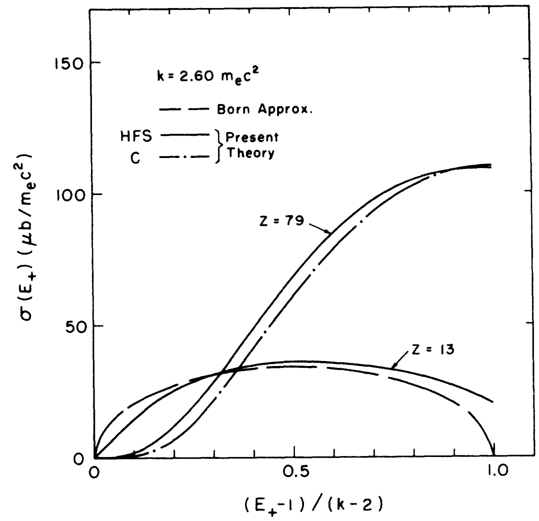

FIG. 2. Comparison of pair-production cross sections $\sigma(E_+)$ for $k=2.60 m_ec^2$.

production cross sections are summarized in Table IV. In Figs. 4–6 we show the corresponding angular distributions $\sigma(E_+, \theta_+)$ in microbarns per steradian per $m_e c^2$ for several sample cases. Our cases provide some coverage of the photon energy range 1.073–2.615 MeV and targets $Z = 1–92$; they were chosen to study screening effects and to permit comparisons with previous work. We begin by comparing point-Coulomb-potential results, which also serve as a check of our calculations. Next we examine the importance of screening effects. After summarizing our theoretical formulation, we compare it with current experimental work.

A. Comparison of Point-Coulomb Results

As for the bremsstrahlung process, the relativistic Born approximation (the Bethe-Heitler formula)[5] is expected to be valid when the Coulomb parameter $2\pi Z\alpha/\beta \ll 1$, where $\beta = \mathrm{Min}(\beta_+, \beta_-)$. This condition is never well satisfied for high-Z elements, or for low-Z elements and low photon energies. Comparisons of the present results with those of the Bethe-Heitler Born approximation are given in Tables I–V, and in Figs. 1, 2, and 4–6.

Because of the Coulomb repulsion of the positrons, the energy distribution is asymmetrical, unlike the symmetrical Bethe-Heitler Born-approximation result. This has consequences for screening which we will discuss below. The asymmetry was realized as early as 1934 by Nishina, Tomonaga, and Sakata[18] and later by Jaeger and Hulme,[2] and examined experimentally in 1935 by Alichanow, Alichanian, and Kosodaew.[19] It can be qualitatively understood if one remembers that the normalization of the positron wave function is proportional to

$$[2\pi\nu_+ e^{-2\pi\nu_+}/(1-e^{-2\pi\nu_+})]^{1/2} :$$

based on either the Schrödinger equation or the Sommerfeld-Maue wave function,[20] where $\nu_+ \equiv Z\alpha E_+/p_+$. For low energies the number of positrons near the nucleus is very strongly suppressed in comparison with the number of electrons since

TABLE IV. Comparison of the total pair-production cross section σ.

k (m_ec^2)	Z	σ^C(mb/atom) ØMO or Øverbø	This work	Born	NST	CNST	σ^{HFS} (mb/atom) This work	NST	CNST	γ_{cal}
2.10	29	0.1705	⋯	0.112	⋯	⋯	0.194	0.200	0.194	1.14
2.10	32	0.2070	⋯	0.136	⋯	⋯	0.242	⋯	⋯	1.17
2.10	92	0.306	⋯	1.12	0.279	0.304	1.03	⋯	⋯	3.37
2.60	13	2.84	2.85	2.72	⋯	⋯	2.86	2.87	2.86	1.00
2.60	79	⋯	213.	100.	⋯	⋯	228.	238.	228.	1.07

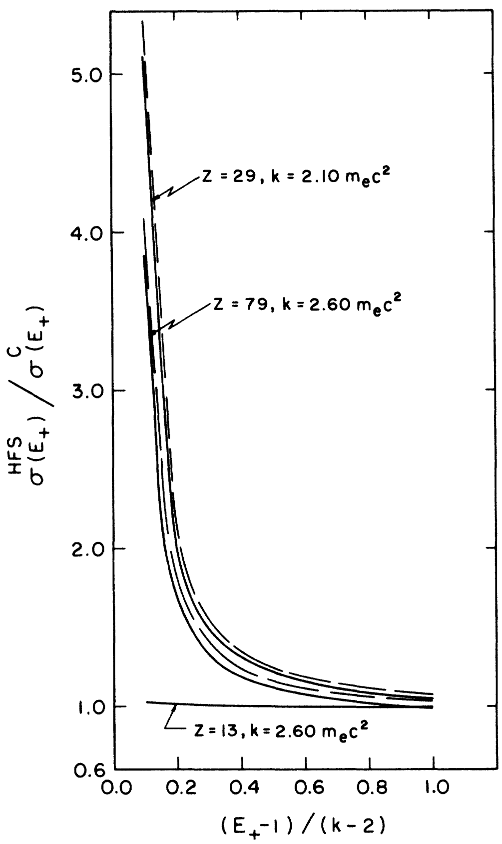

FIG. 3. Comparison of the screening factors $\gamma(E_+)$ for the exact calculation (solid line) and the effective normalization screening theory (broken line).

$$|\psi_+(r=0)|^2/|\psi_-(r=0)|^2 = e^{-2\pi\nu_+} . \qquad (3.1)$$

This conclusion also follows in the limit that $p_+ \to 0$ for the Dirac equations. The same exponential is the dominant factor in nuclear reactions between positively charged nuclei at low bombarding energies—the well-known Gamow factor.[21] When the positron energy $E_+ = m_e c^2$, the cross sections vanish since the positron wave functions vanish, but not when $E_- = m_e c^2$ with $k > 2m_e c^2$. However the Born-approximation theory gives zero cross section for both cases.

TABLE V. Comparison of the pair-production cross section $\sigma(E_+)$ of point-Coulomb-potential model.

k (m_ec^2)	Z	y	$\sigma^C(E_+)$ (μb/m_ec^2) ØMO	This work	Born
2.10	1	0.9	1.10	1.102	1.012
2.10	1	0.1	0.925	0.9284	1.012
2.10	29	0.5	1.87	1.872	1.691
2.20	29	0.5	7.74	7.738	5.958
3.50	82	0.7	189.	187.7	88.42

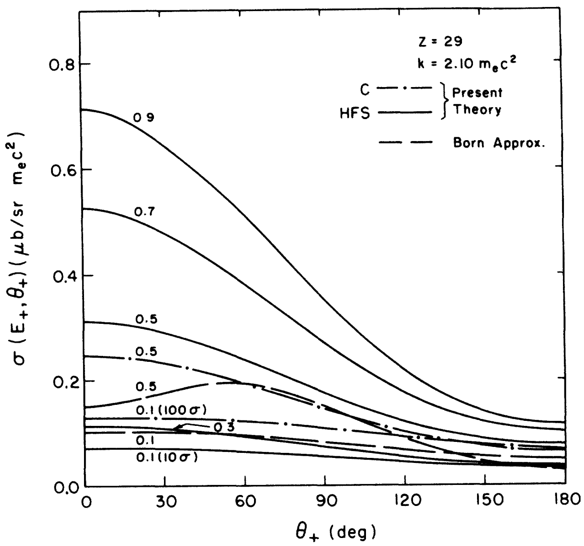

FIG. 4. Pair-production angular distributions $\sigma(E_+, \theta_+)$ for $k = 2.10\ m_ec^2$, $Z = 29$. The numbers attached to the curves give the values of $y \equiv (E_+ - 1)/(k-2)$.

We do not attempt here to give a detailed discussion of the higher Born-approximation corrections (Coulomb effects), which have been studied quite extensively by ØMO.[1] We do show test comparisons of the point-Coulomb results of ØMO with

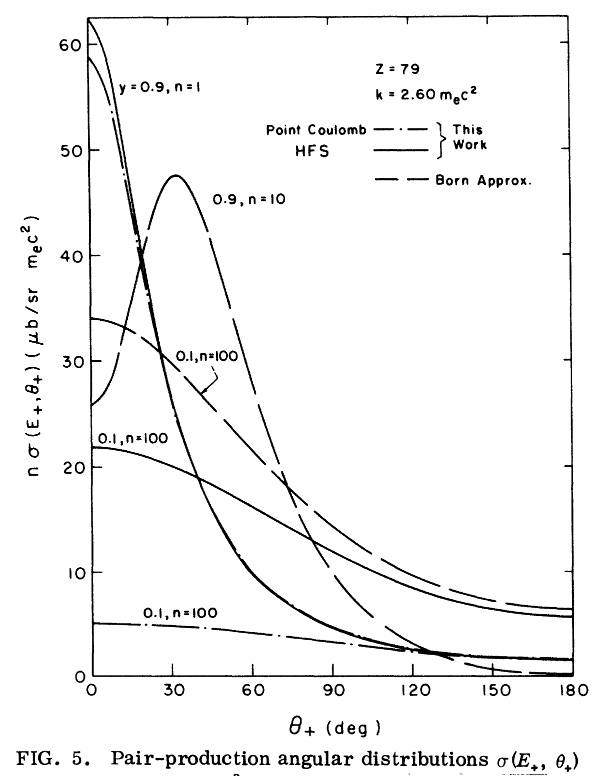

FIG. 5. Pair-production angular distributions $\sigma(E_+, \theta_+)$ for $k = 2.60\ m_ec^2$, $Z = 79$, $y = 0.1$, and $y = 0.9$.

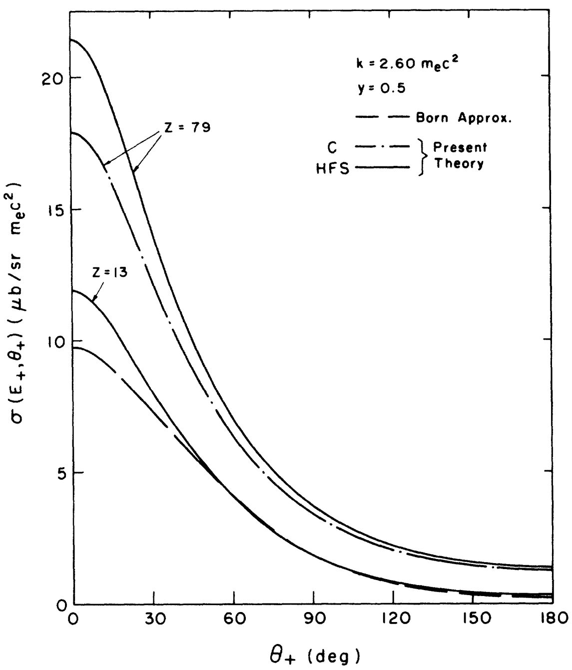

FIG. 6. Pair-production angular distributions $\sigma(E_+, \theta_+)$ for $k = 2.60\ m_ec^2$, $y = 0.5$, $Z = 13$, and $Z = 79$.

those of our present calculations in Tables IV and V. The agreement between these two point-Coulomb calculations is good, increasing our confidence in the numerical techniques of both calculations.

B. Screening Effects

We now turn to a discussion of the effects of atomic-electron screening on the point-Coulomb pair-production cross sections. We wish to understand when screening effects are important and to see whether they can be estimated in any simple fashion. The usual method of estimating screening is the form-factor approach based on Born approximation,[7] so we will examine in some detail the validity of this procedure.

In the Born approximation, the pair-production matrix element is proportional to

$$\int V(r) e^{i\vec{q}\cdot\vec{r}}\, d^3r \equiv M' ,$$

where[22]

TABLE VI. Comparison of the variation of the screening factor at two points of the energy spectrum for fixed Z with incident photon energy.

y	$\sigma^{HFS}(E_+)/\sigma^{C}(E_+)$ $(Z=79)$ $k = 2.60 m_ec^2$	$k = 2.615$ MeV $\simeq 5.12 m_ec^2$
0.5	1.12	0.997
0.7	1.05	0.989

TABLE VII. Contribution of the dominant partial-wave cross section for low photon energies. Values in the third column are in percentages for $y = 0.95$ to 0.1.

k (m_ec^2)	Z	$\sigma^{HFS}(E_+, \kappa_1 = 1, \kappa_2 = -1) / \sigma^{HFS}(E_+)$
2.10	29	41 to 57
2.10	92	49 to 52
2.60	79	25 to 39

$$V(r) = \int [\rho_N(r') - \rho_e(r')] d^3r' / |\vec{r} - \vec{r}'| , \qquad (3.2)$$

ρ_N and ρ_e are the nuclear-charge and electron-charge densities, respectively; and $\vec{q} = \vec{k} - \vec{p}_+ - \vec{p}_-$ is the momentum transferred to the nucleus. From Eq. (3.2) we have[22]

$$M' = (4\pi Z/q^2)[F_N(q) - F_e(q)] , \qquad (3.3)$$

with

$$F_N(q) = Z^{-1} \int \rho_N(r) e^{i\vec{q}\cdot\vec{r}} d^3r ,$$

$$F_e(q) = Z^{-1} \int \rho_e(r) e^{i\vec{q}\cdot\vec{r}} d^3r .$$

In the energy region we considered in this paper, we have[23] $F_N(q) \cong 1$. Thus the unscreened cross section may be corrected for screening effects by including the multiplicative factor $[1 - F_e(q)]^2$.

The standard description of screening effects in pair production is based on the parameters of the Thomas-Fermi (TF) model.[24] Here $F_e(q)$ depends on the quantity qr_{TF}, where $r_{TF} = 137 Z^{-1/3}$ is the radius of the TF atom. Screening effects are classified by the screening parameter ξ_s which is approximately equal to r_{TF}/r_{max}. Here r_{max} is the maximum-impact parameter discussed by Heitler, equal to q_{min}^{-1}, $q_{min} = k - p_+ - p_-$. If r_{max}

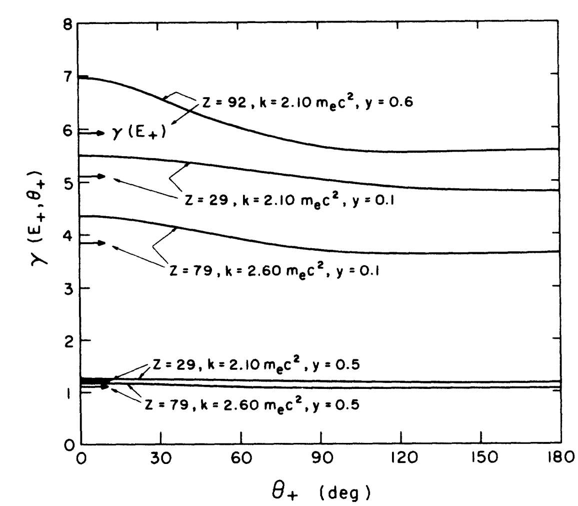

FIG. 7. Comparison of the pair-production screening factors $\gamma(E_+\theta_+)$ with $\gamma(E_+)$. The screening factors $\gamma(E_+)$ are shown by arrows.

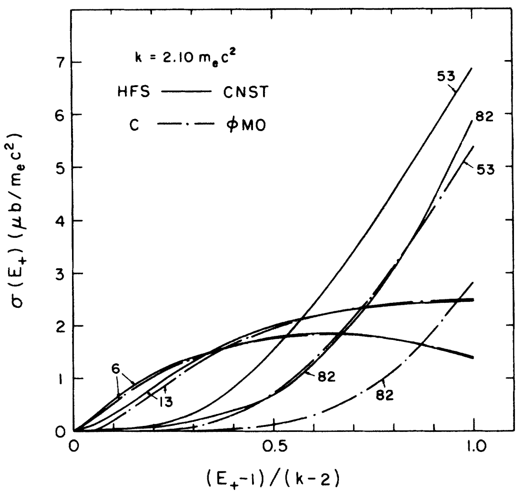

FIG. 8. Comparison of pair-production cross sections $\sigma(E_+)$ for $k=2.10\ m_ec^2$. The numbers attached to the curves give the atomic number of the target element.

is small compared to r_{TF}, then ξ_s is large and $F_e(q)$ is small, which means electron screening is not important. If r_{max} is of the order of r_{TF}, then $\xi_s \sim 1$, F_e is large, and screening is important. For low photon energies, $r_{max}=O(\frac{1}{2})$ since $q_{min}\sim 2$, and thus the form-factor theory of screening suggests very small screening effects.

In Fig. 3 we show in several cases our results for the energy distribution of the screening factor $\gamma(E_+)$—the ratio of screened to point-Coulomb integrated cross sections $\sigma(E_+)$; the corresponding screening effects on the total cross sections are summarized in Table IV. Table VI shows for two

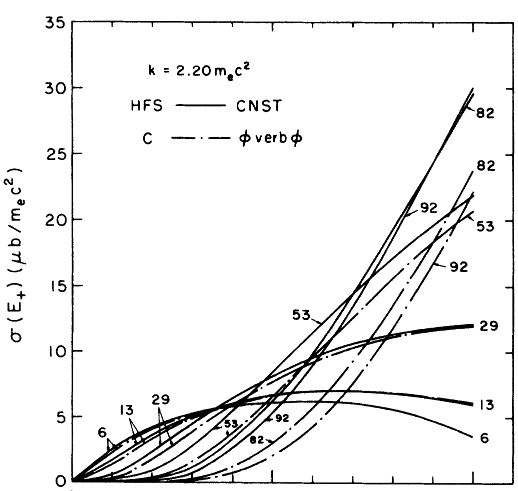

FIG. 9. Same as Fig. 8 except that the photon energy k is 2.20 m_ec^2.

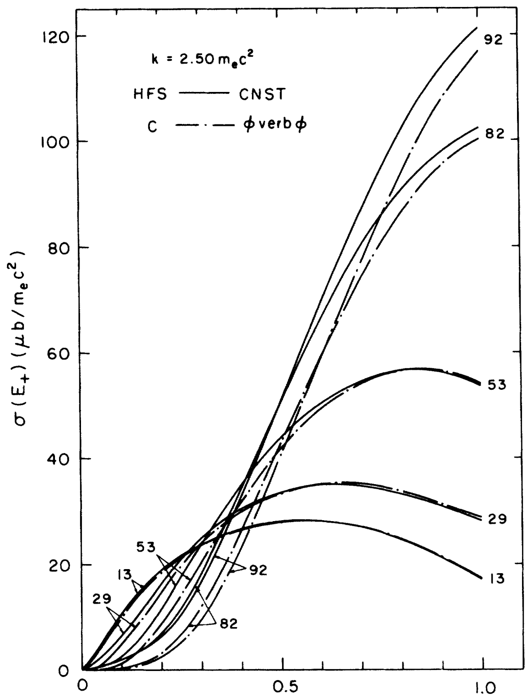

FIG. 10. Same as Fig. 8 except that the photon energy k is 2.50 m_ec^2.

points of the spectrum the result of going to higher incident energy for fixed Z. As we can see from Tables IV and VI and Figs. 1–3 the screening effect is significant for low photon energies, becomes small well above threshold, and then changes sign, decreasing the total cross section as the photon energy becomes high. Note that screening increases the cross section for low photon energies. This is because the atomic electrons decrease the Coulomb repulsion of the positrons (which is responsible for the asymmetric energy distribution).

Since the angular cross sections have received little attention,[3] we provide several sample results in Figs. 4–6. In Fig. 7 we show the corresponding screening factor $\gamma(E_+, \theta_+)$—the ratio of screened to point-Coulomb angular cross sections $\sigma(E_+, \theta_+)$. We see that the shapes of angular distributions are almost independent of screening, namely, $\gamma(E_+, \theta_+)$ can be fairly well represented by $\gamma(E_+)$. This suggests that the atomic-electron screening is primarily a normalization effect.

We conclude that the Born-approximation theory is partly right; namely, the main region of importance for the matrix element when the photon energy is low is the region of space where $r=O(\frac{1}{2})$. In this dominant small-r region the shape of the continuum electron and positron wave functions is not much affected by screening, so the screened cross section is proportional to the point-Coulomb

TABLE VIII. Total pair-production cross sections converted from point-Coulomb results of ØMO or Øverbø with the corrected effective normalization screening theory.

k (m_ec^2)	Z	$\sigma^{C}_{\text{ØMO}}$(mb/atom) or Øverbø	$\sigma^{\text{HFS}}_{\text{CNST}}$(mb/atom)
2.10	6	5.086×10^{-3}	5.17×10^{-3}
	13	2.768×10^{-2}	2.84×10^{-2}
	29	0.1705	0.194
	53	0.4134	0.625
	82	0.3719	1.01
2.20	6	3.475×10^{-2}	3.52×10^{-2}
	13	0.1784	0.180
	29	1.144	1.21
	53	4.437	5.17
	82	8.486	12.6
	92	9.161	15.1
2.50	13	1.840	1.85
	29	10.79	10.9
	53	48.90	50.7
	82	151.2	168.
	92	198.4	228.
2.60	18	5.643	5.66
	29	16.32	16.5
	44	45.14	46.1
	68	142.8	149.
	82	234.9	252.
	92	314.3	351.

cross section for the same photon energy, the constant of proportionality being the ratio of $|N_+N_-|^2$ screened to unscreened, where $N_\pm$ are the normalizations of the positron and electron wave functions at the origin. Now, for low photon energies the low-κ partial waves dominate the cross section, as shown in Table VII for several sample cases. Thus we may calculate the ratio of the dominant partial-wave normalizations. This we call the effective normalization screening theory (NST); with it we may convert all the point-Coulomb predictions of ØMO. Comparisons of our exact calculations with those calculated with NST are given in Table IV. The corresponding comparisons for $\gamma(E_+)$ are given in Fig. 3. As we can see from Fig. 3, the prediction based on the NST is quite good.

The ratio of the exact calculation to the NST is almost independent of energy E_+. Thus we can do better if we calculate one point of the screened cross section $\sigma(E_+)$ exactly. This we call the corrected effective normalization screening theory (CNST). The results calculated from the CNST are presented in Table IV. The agreement between the exact calculation and the CNST is very good. By using the CNST, we converted all the point-Coulomb results of ØMO[1] for incident-photon energies in the range $2.10-2.60\ m_ec^2$. The results for $\sigma(E_+)$ and σ are given in Figs. 8–11 and in Table VIII, respectively.

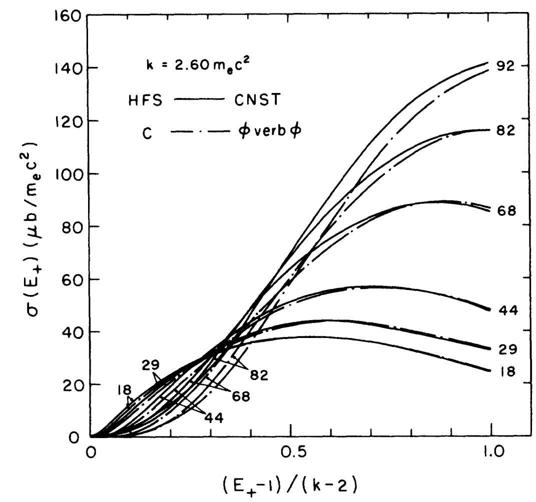

FIG. 11. Same as Fig. 8 except that the photon energy k is 2.60 m_ec^2.

C. Comparisons with Experiments

The status of experimental work on the pair-production cross section to 1970 has been summarized by Motz, Olsen, and Koch,[3] and by Øverbø.[1] We do not attempt here to make detailed comparisons with experiments and wish here only to note some experimental evidence of the low-energy screening effect on pair production.

From the examples we have given in this paper we conclude that atomic-electron screening effects on pair-production cross sections are not negligible in the low-photon-energy region. The screening increases the total cross section. This explains the discrepancy between the experimental data of Yamazaki and Hollander[4] and the point-Coulomb results of ØMO. Recently, Rao, Murty, Premchand, and Parthasaradhi[25] noted significant deviations between the point-Coulomb results of ØMO and the experimental results of Rao *et al.*[26] for $k=1.1192\pm0.0006$ MeV.[27] This provides further qualitative experimental evidence of this low-photon-energy

TABLE IX. Comparisons with experiments. The value of $\sigma_{\text{expt}}/\sigma_B$ for $Z=32$, $k=2.10\ m_ec^2$ is the extrapolated value from the experiments of Yamazaki and Hollander (Ref. 4). The values of $\sigma^{\text{HFS}}/\sigma_B$ for $k=1.1192$ MeV ($\cong 2.190\ m_ec^2$), $Z=29$, and 82 are the interpolated values from the calculated results.

k	Z	Ref.	$\sigma_{\text{expt}}/\sigma_B$	$\sigma^{\text{HFS}}/\sigma_B$	$\sigma_{\text{expt}}/\sigma^{\text{HFS}}$
$2.10 m_ec^2$	32	4	2.05 ± 0.31	1.78	1.15 ± 0.17
1.1192 MeV	29	26	2.07 ± 0.13	1.57	1.32 ± 0.08
($\cong 2.190 m_ec^2$)	82	26	2.70 ± 0.13	1.94	1.39 ± 0.07

screening behavior. Due to the large experimental error (5–10% or more) Rao *et al.* were unable to observe significant difference between their experimental data and the point-Coulomb results from the γ-ray source Co^{60} ($k = 1.17323 \pm 0.00003$, 1.33252 ± 0.00003, 2.158, and 2.505 MeV with absolute photon intensity per 100 decays of Co^{60} being 99.88, 100, 0.0012, and 4×10^{-5}, respectively[28]). This is easy to understand from Table IV since the difference between the screened result and the point-Coulomb result is only 7% for $Z=79$ and $k=2.60\ m_e c^2$ (about 1.329 MeV).

Comparisons of theory and experiment are shown in Table IX. The agreement between the experimental results of Yamazaki and Hollander and our exact result is good for the case $Z=32$, $k=2.10\ m_e c^2$. For the cases $k=1.1192$ MeV ($\cong 2.190\, m_e c^2$), $Z=29$ and 82, there is about 30% discrepancy between the experimental data of Rao *et al*, and the results calculated with the CNST.

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