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Kinks and Tearing Modes in Simple Configurations

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A plane slab of current-carrying plasma is generally shown to be stable against ideal magnetohydrodynamical modes. It is shown that some of the stability criteria appearing in the literature and contradicting this fact can be brought to bear on the stability of the tearing mode. In this connection, simple stability diagrams are shown for the tearing mode in plane and cylindrically symmetric force-free magnetic fields.

I. INTRODUCTION

In this paper several topics related to the stability of a semi-infinite plane slab of plasma are discussed. The reason for the study of such a simple configuration is the appearance of conflicting statements in the literature about this or related configurations. The most important magnetohydrodynamical (MHD) mode is the kink instability, which is usually treated in nonplanar geometry. If resistivity is introduced in the theory, new modes are possible, and one of these, the tearing mode, is most closely related to the kink mode. Because the tearing mode is commonly treated in simple plane geometry, one is led to try the same approach for the kink mode. Here, however, the "kink" mode in one dimension has been shown to be completely stable, not only by means of a study of the equation of motion, but also through a Newcomb-type[1] stability analysis. Next, the relevant literature is discussed, and simple stability criteria are derived for the tearing mode in plane and cylindrically symmetric force-free magnetic fields. Finally, the physical nature of the two types of instability is clearly described and discussed.

II. MODEL AND BASIC EQUATIONS

We consider the geometry of a plane semi-infinite slab of plasma (Fig. 1). The equilibrium is described by

$$\vec{B}_0 = B_{0y}(x)\,\vec{e}_y + B_{0z}(x)\,\vec{e}_z\,, \qquad (1)$$

$$\nabla p_0 = \vec{J}_0 \times \vec{B}_0 = (1/4\pi)(\nabla \times \vec{B}_0) \times \vec{B}_0\,, \qquad (2)$$

where p_0 is the zeroth-order equilibrium pressure, $\vec{J}_0$ the equilibrium current, $\vec{B}_0$ the equilibrium magnetic field, and $\vec{e}_y$, $\vec{e}_z$ are unit vectors in the y and z directions, respectively. We assume that the pressure is isotropic and varies only in the x direction; then Eq. (2) is written as

$$p_0' + \frac{B_{0y} B_{0y}'}{4\pi} + \frac{B_{0z} B_{0z}'}{4\pi} = 0\,, \qquad (3)$$

where the prime denotes differentiation in the x direction.

Outside the slab, $\vec{B}_0$ is constant, and across the boundaries $x = \pm\frac{1}{2}a$, $\vec{B}_0$ can be either discontinuous or continuous, depending on whether surface currents are present or not.

If the above-described equilibrium is perturbed so that a given plasma element moves at a speed $\vec{v} \equiv \partial\vec{\xi}/\partial t$, where $\vec{\xi}$ is a Lagrangian displacement vector, we have the following equations:

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p + \frac{1}{4\pi}\left[(\nabla \times \vec{B}_0)\times \vec{Q} + (\nabla \times \vec{Q}) \times \vec{B}_0\right], \qquad (4)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0\,, \qquad (5)$$

$$\frac{\partial p}{\partial t} + \gamma p_0 \nabla \cdot \vec{v} + \vec{v} \cdot \nabla p_0 = 0\,, \qquad (6)$$

$$\frac{\partial \vec{Q}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}_0)\,, \qquad (7)$$

where the symbols without subscripts denote perturbed quantities; $\vec{Q}$ is the perturbed magnetic field and γ is the adiabaticity index.

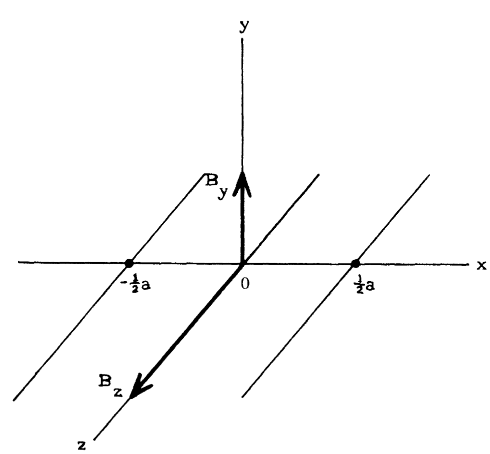

FIG. 1. Plane plasma slab.

III. EQUATION OF MOTION AND EIGENSOLUTIONS

Because of the planar symmetry of the problem, we can write

$$\vec{\xi}(x, y, z) = \vec{\xi}(x)e^{i(k_y y + k_z z - \omega t)} , \qquad (8)$$

where k_y, k_z are the y, z components of the propagation vector $\vec{k}$. Taking incompressible displacements, $\nabla\cdot\vec{\xi} = 0$, and writing $F = \vec{k}\cdot\vec{B}_0$, we get from Eqs. (4)–(8)

$$(-\omega^2\rho_0 + F^2/4\pi)\vec{\xi} = -\nabla\bar{p} , \qquad (9)$$

where

$$\bar{p} = p + \vec{Q}\cdot\vec{B}_0/4\pi, \qquad \vec{Q} = iF\vec{\xi} - \vec{\xi}\cdot\nabla\vec{B} . \qquad (10)$$

In the following we shall need only the x component of the latter equation:

$$Q_x = iF\xi_x . \qquad (11)$$

Applying the operator $\vec{e}_x\cdot\nabla\times\nabla\times$ on Eq. (9) we obtain the equation of motion in a convenient form:

$$\xi_x'' + (\ln\varphi)'\xi_x' - k^2\xi_x = 0 , \qquad (12)$$

where $\varphi \equiv -\omega^2\rho_0 + F^2/4\pi$.

A quadratic form can be obtained by multiplying Eq. (12) by ξ_x^* and integrating over x, in which case we have

$$\omega^2\int_{-\infty}^{\infty}\rho_0(|\xi_x'|^2 + k^2|\xi_x|^2)\,dx = (1/4\pi)\int_{-\infty}^{\infty}F^2(|\xi_x'|^2 + k^2|\xi_x|^2)\,dx , \qquad (13)$$

where ξ_x and ξ_x' should vanish at $\pm\infty$. The integration over x is taken from $-\infty$ to $+\infty$ because we can extend the definition of $\vec{\xi}$ in the region outside the plasma by writing the vector potential $\vec{A} = \vec{\xi}_x\vec{B}$ and $\vec{Q} = \nabla\times\vec{A} = \nabla\times(\vec{\xi}\times\vec{B})$.[1] Then we can consider the slab configuration of Fig. 1 as the limiting case of a diffuse configuration so that the plasma pressure, density, and magnetic field continuously change into the vacuum configuration with vanishing pressure and density and constant $\vec{B}_0$.

Being interested in the possibility of instabilities, we can restrict the discussion to $\omega^2 < 0$. In that case φ is positive definite and the equation of motion (12) is not singular. The quadratic form (13) now contains no singular solutions and it is evident that for $\omega^2 < 0$ the left-hand side of Eq. (13) is negative definite and the right-hand side is positive definite so that solutions for $\omega^2 < 0$ are impossible and the plasma slab is stable.

On the other hand, considering the slab of Fig. 1 with sharp boundaries, one can solve the eigenvalue problem of Eq. (12) together with the boundary conditions that $\bar{p}$ and ξ_x are continuous across the boundary and that $\xi_x = 0$ in $x = \pm\infty$. Outside the plasma Eq. (12) reduces to

$$\xi_x'' - k^2\xi_x = 0 , \qquad (14)$$

with the solution $\xi_x = Ae^{-k|x|}$, A being a constant. Inside the plasma, if the wavelength of the perturbation is sufficiently long ($k \to 0$), we can write

$$\xi_x'' + (\ln\varphi)'\xi_x' = 0 , \qquad (15)$$

which has the solution

$$\xi_x = B\int\frac{dx}{\varphi(x)} + C , \qquad (16)$$

where B and C are constants. When the wavelength is short ($k \to \infty$) Eq. (12) assumes the form of Eq. (14) which inside the plasma has hyperbolic sines and cosines as solutions. The solutions for the limiting cases of long and small wavelength are particularly interesting because for both cases instabilities have been claimed in the literature. Now, because we restrict ourselves to $\omega^2 < 0$, it is seen that none of these solutions is singular and it is evident from the preceding discussion that it is hopeless to try to match the boundary conditions; the slab is stable. Also it has been argued by Laval, Mercier, and Pellat[2] that for $\omega^2 > 0$ no normal modes exist because of the singularities for $\varphi = 0$. Apparently the plane plasma slab in ideal MHD exhibits neither instabilities nor modes of real frequency and it would appear to be a particularly uninteresting situation.

IV. NEWCOMB-TYPE STABILITY ANALYSIS

Without restricting oneself to incompressible perturbations from Eqs. (4)–(8), the following well-known[3] quadratic form δW can be obtained:

$$\delta W = \tfrac{1}{2}\int[\gamma p_0(\nabla\cdot\vec{\xi})^2 + (\vec{\xi}\cdot\nabla p_0)\nabla\cdot\vec{\xi} + (1/4\pi)\vec{Q}^2 - \vec{J}_0\cdot\vec{Q}\times\vec{\xi}\,]\,d\tau . \qquad (17)$$

For definiteness we shall assume here that the plasma slab is diffuse and bounded by two perfectly conducting walls at $x = x_1$ and $x = x_2$. This restriction will not strongly influence the final results and it can be removed easily by taking the limits $x_1 \to -\infty$, $x_2 \to +\infty$. Using real variables, Eq. (17) can be written as

$$\delta W = \frac{1}{2}\int_{x_1}^{x_2}\left[\gamma p_0(\eta + \xi_x')^2 + \frac{k^2}{4\pi}\left(\zeta - \frac{k_z B_{0y} - k_y B_{0z}}{k^2}\,\xi_x'\right)^2 + \frac{F^2}{4\pi k^2}|\xi_x'|^2 + \frac{F^2}{4\pi}|\xi_x|^2\right]dx , \qquad (18)$$

where we have defined

$$\eta \equiv \nabla\cdot\vec{\xi} - \xi_x' = i(k_y\xi_y + k_z\xi_z) ,$$

$$\zeta \equiv i(\vec{\xi}\times\vec{B}_0)_x = i(B_{0z}\xi_y - B_{0y}\xi_z) \ .$$

Minimization with respect to η and ζ is trivial and yields

$$\delta W = (1/8\pi k^2)\int_{x_1}^{x_2} F^2(|\xi_x'|^2 + k^2|\xi_x|^2)\,dx \ . \quad (19)$$

In this form δW is seen to be never negative, so that the configuration is always stable according to ideal magnetohydrodynamics.

Because of the singularities for $F=0$ in the Euler-Lagrange equation belonging to δW the description of the theory in terms of the displacement vector $\vec{\xi}$ is not completely equivalent to the description in terms of the perturbed magnetic field $\vec{Q}$. This has led to a number of errors in the literature. To illustrate this point let us transform Eq. (19) by means of Eq. (11):

$$\delta W = \frac{1}{8\pi k^2}\int_{x_1}^{x_2}\left[|Q_x'|^2 + \left(\frac{F''}{F}+k^2\right)|Q_x|^2\right]dx - \frac{1}{8\pi k^2}\frac{F'}{F}|Q_x|^2\Big|_{x_1}^{x_2} \ , \quad (20)$$

where we have conserved the second term because we need it in a later section. Because of the term F''/F, which can be negative, the integral can be negative. However, now one should remember that the singular points ($F=0$) have a stabilizing effect in ideal magnetohydrodynamics. According to Newcomb's theory,[1] which can be modified so as to apply to this situation, the singular points split the interval (x_1, x_2) into independent subintervals which should be studied separately, as far as stability is concerned. If F''/F is negative it is clear that δW from Eq. (20) could give a negative result only if the independent subintervals were large enough, because, at the boundary of an independent subinterval, Q_x^2 vanishes and the negative contribution to the integral could dominate only if regions of small $Q_x'^2$ were included. Newcomb shows that δW can be negative for an independent subinterval only if there exist solutions of the Euler-Lagrange equation, following from minimization of δW, having more than one zero point in the independent subinterval. Stated differently: The zero points of Q_x should alternate more rapidly than the zero points of F.

The Euler-Lagrange equation in terms of Q_x can be written in the following form:

$$Q_x''/Q_x = F''/F + k^2 \ . \quad (21)$$

Because Q_x''/Q_x and F''/F are negative if Q_x and F oscillate, it follows immediately from Sturm's fundamental theorem that Q_x oscillates more slowly than F. As a result, a single independent subinterval (bounded by two succeeding zeros of F) cannot contain more than one zero of Q_x and the system is stable. So we finally arrive at the same conclusion which was reached directly from Eq. (19), viz., that the plasma is stable. It seems that in the formulation of the theory in terms of ξ_x one knows already beforehand that the singular points suppress possible instabilities, whereas in the formulation in terms of Q_x one must use the stabilizing influence of these points explicitly.

V. DISCUSSION

In this section we will briefly discuss some results, obtained by various authors, pertinent to the configuration we have been discussing. It turns out that frequently the influence of the destabilizing term F''/F in Eq. (20) and the associated singularities are not properly taken into account.

One of the earliest papers concerned with the stability of a plane plasma layer is that of Loughhead.[4] In that paper it is claimed that such a layer, with a uniform current flowing between a pair of parallel planes, and with vanishing current and magnetic field outside this region, is unstable with respect to ideal MHD modes. The derivation is essentially in terms of Q_x and this conclusion is reached because the solutions for Q_x have more than one zero. However, the influence of the singular points is not taken into account and accordingly the obtained incorrect condition for instability [Eq. (54) of Ref. 4] is the condition which ensures the existence of singular points.

Recently, a similar statement was made in a paper by Hasegawa,[5] where an instability of a plane plasma system is considered as an explanation for the observed instabilities during auroral substorms. This instability (which is called a "kink") should have wavelengths and growth rates corresponding to those observed in the aurora band. The model is similar to the one described above, where the y-z plane coincides with the aurora, with the only difference being that the dimension of the layer in the z direction is not infinite. Currents flowing along the magnetic field in the y direction now produce a magnetic field $B_z(x)$ decaying in the x direction because of the limited extension of the sheet in the z direction. Next, the stability theory of this model is developed, assuming that in the middle of the sheet a local treatment of the instability is possible and taking an $\exp i(k_y y + k_z z - \omega t)$ dependence of the perturbed quantities. Therefore, with respect to stability, Hasegawa's model is essentially the one described above, where the x dependence of the equilibrium magnetic field is caused by a geometrical factor. Hence, we conclude that this model cannot be unstable.

Apparently the local treatment is in disagreement with the presence of a gradient in the magnetic field outside the plasma layer. This term dB_z/dx is the driving force of the instability in Hasegawa's model. However, outside the sheet $J_y = 0$ and hence $dB_z/dx = dB_x/dz$, so that either $dB_z/dx = 0$ and the field is straight so that the plane analysis can be applied but there is no instability, or $dB_z/dx \neq 0$ and the field is curved so that the plane analysis breaks down.

As a further example let us prescribe the magnetic field in the plane layer to be force free with a constant α, where α is the parameter from the defining equation for force-free fields: $\nabla\times\vec{B} = \alpha\vec{B}$. For this case $F''/F = \alpha^2$ and the Euler-Lagrange equation in terms of Q_x reads

$$Q_x'' + (\alpha^2 - k^2)Q_x = 0 \,, \qquad (22)$$

having oscillating solutions if $k^2 < \alpha^2$, viz.,

$$Q_x \sim \sin[(\alpha^2 - k^2)^{1/2} x + \text{const}] \,.$$

The singular points are determined by the zero points of $F \sim \sin(\alpha x + \text{const})$. In accordance with our expectations, Q_x can oscillate at most as rapidly as F, so that stability is ensured.

Schmidt[6] gives an expression for δW in terms of $\vec{Q}$ for general force-free fields using a special normalization which was found by Voslamber and Callebaut.[7] If one were to determine the stability of the plane force-free field from the expression (5-132) of Ref. 6 for δW, one would find instability if $(\alpha^2 - k^2)^{1/2} a > \pi$, where $x_2 - x_1 = a$. This criterion is plotted in Fig. 2. It turns out to be the condition which ensures that Q_x has more than one zero point in the interval (x_1, x_2). However, we notice again that the singular points are ignored in Schmidt's treatment, so that the stability diagram of Fig. 2 makes no sense in ideal magnetohydrodynamics.

VI. TEARING MODES IN FORCE-FREE MAGNETIC FIELDS

Surprisingly enough, the stability diagram of Fig. 2, which is incorrect within ideal MHD theory, can be given a simple physical interpretation within the resistive theory. In Refs. 8 and 9 it was shown that in the limit of high conductivity the stability of a constant pressure plasma with respect to tearing modes is determined by the same expression for δW as for ideal MHD modes, if the resistive layer (surrounding a point where $\vec{k}\cdot\vec{B} = 0$) were simply replaced by a vacuum. Because ξ_x has no physical meaning in a vacuum one should start from the expression (20) rather than from the expression (19) in order to derive the criterion for the tearing modes from this statement. For a plasma-vacuum-plasma system, where the vacuum is situated in a small region $(x_s - \epsilon,\ x_s + \epsilon)$ containing the singular point, δW becomes

$$\delta W \sim \int_{x_1}^{x_s-\epsilon} [\,|Q_x'|^2 + (F''/F + k^2)\,|Q_x|^2\,]\,dx$$

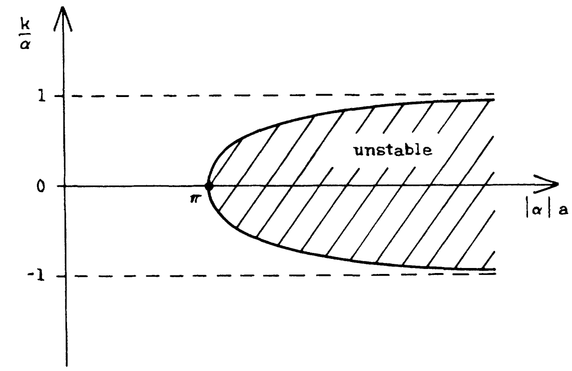

FIG. 2. Stability diagram for the tearing mode in a plane force-free magnetic field.

$$+ \int_{x_s+\epsilon}^{x_2} [\,|Q_x'|^2 + (F''/F + k^2)\,|Q_x|^2\,]\,dx \,, \qquad (23)$$

where the contribution of the vacuum integral vanishes because ϵ is small and the contributions of the second term of Eq. (20) cancel because Q_x can be taken approximately constant in the thin vacuum region. Using Eq. (21) and integrating by parts, we obtain the condition for the stability of the tearing mode:

$$(Q_{x1}'/Q_{x1})_{x_s-\epsilon} > (Q_{x2}'/Q_{x2})_{x_s+\epsilon} \,, \qquad (24)$$

where Q_{x1} satisfies the boundary condition $Q_x = 0$ in $x = x_1$ and Q_{x2} satisfies the boundary condition $Q_x = 0$ in $x = x_2$. This criterion is the same as the criterion $\Delta' < 0$ of Refs. 8 and 9. If we now take the limit $\epsilon \to 0$, corresponding to taking the resistivity $\eta \to 0$, inequality (24) provides the condition that the solution Q_{x1}, being zero in $x = x_1$, should not have a second zero upon continuation in the interval (x_s, x_2). So, criterion (24) is equivalent to the condition that Q_x should not have more than one zero point in the whole interval (x_1, x_2). In this way, working in terms of Q_x, one can simply ignore the singular points and find the criterion for stability of a force-free field of constant α against tearing modes in a simple manner. Here, we explicitly restrict our consideration to force-free fields of constant α because for these fields the equation for Q_x is not singular, as is evident from Eq. (21). Clearly, the singularities which turn up in force-free fields of $\alpha \neq$ const in general prevent us from stating the stability criteria against tearing modes in the same simple manner.

The condition for the absence of solutions Q_x having more than one zero point in (x_1, x_2) is $(\alpha^2 - k^2)^{1/2} a > \pi$. So Fig. 2 gives the correct stability diagram for the plane force-free field with constant α with respect to tearing modes. It is seen that in the long-wavelength limit $(k \to 0)$ the plasma is unstable if $|\alpha| a > \pi$, that is for high currents in the layer. It seems attractive to consider these resistive tearing modes as an alternative explanation for the aforementioned aurora phenomena. However, it turns out that the current density of the sheet current is too low to satisfy the criterion $|\alpha| a > \pi$ and the resistivity is many orders of magnitude too low to give a reasonable value for the growth rates of resistive tearing modes in these cases. Apparently, collisionless tearing modes, as have been described by Coppi, Laval, and Pellat,[10] are more appropriate for the description of the aurora phenomena.

We have shown that a plane force-free field can develop tearing-mode instabilities and therefore our treatment is subject to the criticism of Barston,[11] who claims that a plane plasma layer in the absence of gravity is exponentially stable for constant resistivity. The proof of this statement is given in the Appendix of his paper, where it is shown that a plane plasma layer with $\rho = \text{const}$, $\eta = \text{const}$, $F'' = 0$, and $g = 0$ is stable. Indeed, it is clear that also in our case there will be no instability if $F'' = 0$, because in that case $\alpha = 0$ and the source of the instability is simply neglected. Barston's reason for taking $F'' = 0$ is the fact that he considers completely consistent equilibrium situations where $\vec{v} = 0$, $\partial\vec{B}/\partial t = 0$, and also Ohm's law $\eta\vec{j} = \vec{E} + \vec{v}\times\vec{B}$ is satisifed. For constant resistivity this leads directly to $\Delta^2\vec{B} = 0$ and therefore $F'' = 0$. This line of reasoning is not very useful, however, because one can well study the tearing stability in the situation where there is no perfect equilibrium ($\eta = \text{const}$ and $\partial\vec{B}/\partial t \neq 0$ or $\vec{v} \neq 0$) by introducing a time scale for the stability problem which is much shorter than the time scale according to which the equilibrium quantities vary. Physically this makes sense because tearing modes exponentiate in a time $\tau_t \sim \eta^{-3/5}$, whereas the magnetic field decays in a time $\tau_d \sim \eta^{-1}$. So, in the limit of small resistivity $\tau_t \ll \tau_d$ and we have a quasiequilibrium. This means that, although strictly speaking Barston is right, his results imply that the tearing stability problem should be stated in terms of different time scales for the equilibrium and the stability.

A similar interpretation of the stability criteria can be given for force-free fields in cylindrical geometry. The ideal MHD stability of a force-free field of constant α in a cylinder (a so-called Lundquist field) was investigated by Voslamber and Callebaut.[7] They found it to be unstable with respect to kink modes if $|\alpha| r_0 > 3.176$, where r_0 is the radius of the wall surrounding the plasma. The stability with respect to ideal MHD modes of the Lundquist field is governed by the behavior of $r\xi_r = -ir\, Q_r/F$, where

$$rQ_r \sim m J_m([1-(k/\alpha)^2]^{1/2}|\alpha| r) + (k/\alpha)[1-(k/\alpha)^2]^{1/2}|\alpha| rJ'_m \times ([1-(k/\alpha)^2]^{1/2}|\alpha| r), \quad (25)$$

$$F \equiv \frac{mB_\theta}{r} + kB_z \sim \frac{\alpha}{|\alpha|}\frac{m}{r} J_1(|\alpha| r) + kJ_0(|\alpha| r). \quad (26)$$

Whereas for the plane force-free field Q_x could oscillate at most in step with the singular points, for the Lundquist field rQ_r can oscillate more rapidly than F(for $m=1$) in a very thin region of the $k/\alpha - |\alpha| r_0$ plane, as indicated in Fig. 1 of Ref. 7. Only the presence of this very thin region makes the Lundquist field unstable with respect to ideal MHD modes.

Adopting the same line of reasoning as for the plane force-free field, we now ignore the singular points ($F=0$) in order to find the stability criterion for the Lundquist field with respect to tearing modes. It can be seen from Eq. (25) that rQ_r has more than one zero point in the interval $(0, r_0)$ for $|k| < |\alpha|$ and for $|\alpha| r_0$ large enough. The stability criterion for the $m=1$ tearing mode turns out to be the most restrictive one. It is depicted in Fig. 3. The critical curve is taken from Fig. 1 of Ref. 7, but now the unstable region in the $k/\alpha - |\alpha| r_0$ plane is more extended, owing to neglect of singularities. It is seen that with respect to tearing modes the plane and the cylindrical force-free fields are much more similar than they are with respect to ideal MHD modes, which are present for $m=1$ in the cylindrical case and absent in the plane case. From Eq. (25) it follows that for $\alpha r_0 > 3.832$ also $m=0$ tearing modes are unstable, as indicated in Fig. 4. In fact the Lundquist field turns out to be unstable with respect to tearing modes for every value of m, only the unstable region for higher values of m is displaced more to the right in the stability diagram.

Finally we mention that the resistive instabilities

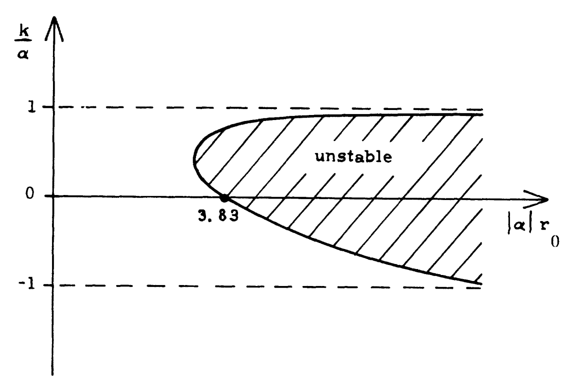

FIG. 3. Stability diagram for the tearing mode in a Lundquist field ($m=1$).

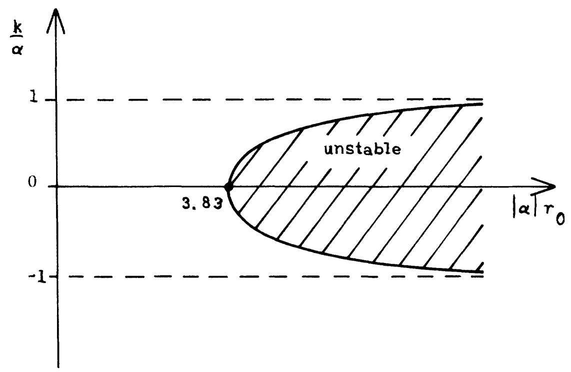

FIG. 4. Stability diagram for the tearing mode in a Lundquist field ($m = 0$).

for force-free fields are especially relevant for constant α, because it has been proved recently[12] that these force-free fields are the only ones which remain force-free in time if the plasma is at rest ($\vec{v} = 0$) and if the resistivity is constant and isotropic. Under these conditions the force-free field decays exponenitally in a time $\tau_a = (4\pi/\alpha^2)\eta^{-1}$, without changing its direction.

VII. CONCLUSIONS

The tearing mode and the kink mode are both current-driven modes. In the plane case the effect of the presence of current in the plasma is shear in the magnetic field which leads to $F''/F \neq 0$. In the cylindrical case, current in general causes both shear (except in a constant-pitch field) and curvature and again $F''/F \neq 0$. Both the kink and the tearing mode have the same source of energy, i.e., the energy that comes from the fact that the magnetic field is not a vacuum field.

We have proved that the kink mode is stable in the plane case, so that for comparison one really has to consider the cylindrical case. Then, for both modes the plasma moves in such a way as to undo the deformation of the magnetic field imposed by the presence of current. In the case of the tearing mode the plasma motion takes place about the resistive layer where the plasma moves into opposite directions. Resistivity makes possible the subsequent decoupling of plasma and field lines, so that the latter appear to collapse towards the resistive layer annihilating this way that part of the magnetic field which gives rise to $F''/F < 0$. In the kink mode the same thing is accomplished by a global distortion of the plasma column in such a way as to again annihilate that part of the magnetic field which gives rise to $F''/F < 0$.

We see that both modes involve global displacement of the plasma column; they both derive their energy from the same source, only the detailed motions are different. This difference is of course made possible by the presence of resistivity in the case of the tearing mode.

Another way of looking at it is to consider neighboring equilibria.[13,14] Here we have available, in addition to the initial equilibrium, a neighboring equilibrium of lower energy than the original one. Then the plasma will tend to move to the lower energy state, if the motion is possible. In the plane case this type of motion is possible only in the form of a tearing mode. In the cylindrical case both forms are possible, but the tearing modes can go unstable over a greater area in the (k/α, $|\alpha| r_0$) plane than kinks can (see Fig. 3). In the toroidal case the picture must still be approximately the same as in the cylindrical case, especially for large aspect ratio.

Notes added in proof. The fact that the plane analysis for the aurora breaks down was recognized by Hasegawa in a later Erratum [Phys. Rev. Letters 24, 1468 (1970)], where it is remarked that the dispersion relation was wrong and also that the most unstable mode has a long wavelength, violating the local assumption. Therefore, the problem is essentially non-one-dimensional.

Dr. D. C. Robinson has brought to our attention that the tearing-mode instability of the Lundquist field was studied before by Gibson and Whiteman [Plasma Physics 10, 1101 (1968)].

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Multiple Scattering of Neutrons in Gaseous and Liquid Methane*

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A "factor method" of evaluating the double-scattering correction for inelastic-neutron-scattering experiments with an infinite-plane-slab specimen has been developed. Using appropriate dynamical models, calculations have been made both for highly compressed room-temperature gaseous methane and for liquid methane at 98 °K. As a result, the agreement between analytical results with measurements is remarkably improved. In addition, the model sensitivity of the correction and the advantages of reflection-type experimental geometry over transmission-type are investigated.

I. INTRODUCTION

The use of slow neutrons as a probe to investigate molecular dynamics is considered to be a very powerful technique. The experimentally observed scattered-neutron intensities are usually interpreted with a phenomenological molecular-dynamics model. The success of such an interpretation depends on measuring single-scattering intensity for a monoenergetic incident neutron beam and an infinite-resolution detector. Such idealized experimental requirements are never achieved; hence, analytical results should include the effects of second- and higher-order scatterings and of the finite resolution of the incident beam and the detector. Of these three factors, multiple scattering appears to be the most complicated one to account for. The problems mentioned here are not peculiar to inelastic-neutron-scattering experiments, but rather are common to a wide variety of different experiments. We will confine ourselves only to inelastic-neutron-scattering experiments.

A workable scheme for evaluating the multiple-scattering correction was first given by Vineyard,[1] who solved an energy-independent neutron-transport equation for an infinite-plane-slab specimen. Subsequently, Blech and Averbach[2] have extended the method to infinite cylindrical samples. Both of these investigations were restricted to elastic scattering of neutrons under the assumption of a quasi-isotropic scattering cross section. Cocking and Heard[3] have calculated second-order scattering without making use of the quasi-isotropic approximation.

Recently, the effects of second-order scattering have been studied by Slaggie,[4] as well as by others,[5] who solved the energy-dependent neutron-transport equation for transmission-type experiments. A common conclusion of these workers is that the second-order scattering correction is very important and quite sensitive to the details of the scattering cross section.

In this paper we describe in Sec. II a "factor method" of evaluating second-order corrections for both transmission- and reflection-type experimental geometries. Resulting expressions are then programmed[6] to evaluate the correction for an arbitrary scattering law. In Sec. III numerical results are obtained for three different geometrical sample thicknesses for room-temperature gaseous methane with a liquid density. These results are directly applicable to Larsson's experiment.[7] Section III also deals with a comparison of our results with cold-neutron liquid-methane data of Dasannacharya and Venkataraman.[8] The finite resolution of the incident neutron beam is accounted for in a suggested approximate manner. Scattering-law model sensitivity and advantages of reflection geometry over transmission geometry are discussed along with some concluding remarks in Sec. IV.

II. THEORY

We consider a scattering sample in which the scatterers may or may not be uniformly distributed. We assume that a monoenergetic neutron beam of energy E_0 is incident at a direction defined by the vector $\vec{s}_0$. The nth-order scattered-neutron density is then the solution of the following transport equation[1]: