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Analytically Solvable Problems in Radiative Transfer. III

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The density of excited atoms and the diffusely reemitted radiation is calculated for a homogeneous radiation field incident on a slab. The solutions of the Biberman-Holstein integral equation are used in the calculations. The optical thickness is assumed to be large. The line shapes of both the absorption line and the incident radiation are arbitrary.

I. INTRODUCTION

A homogeneous radiation field is incident on a homogeneous layer of gas. The atoms of the gas are excited by absorption of radiation from the incident field. The excited atoms decay again and the reemitted radiation can be absorbed elsewhere in the slab. The expression for the reemitted radiation must contain the density of excited atoms everywhere in the slab. Therefore, the equation describing the density of excited atoms is an integral equation. In Sec. II the pertinent equation will be studied and its solution derived for large optical thickness and for a wide class of shapes of the absorption line $k(\nu)$. This class includes all line shapes of interest (Doppler, Lorentz, Voigt, and statistical line shape). The solutions depend on the profile of the incident radiation compared to the shape of $k(\nu)$. Three cases are considered and dealt with in Sec. II: (i) The intensity of the radiation tion is constant over the absorption line; (ii) the profile of the incident radiation is a δ function in frequency; (iii) the profile is arbitrary. Cases (i) and (ii) admit of simple closed-form solutions. Case (iii) is deduced from (ii) by integration.

It will be shown that it is possible to classify an arbitrary profile of the incident radiation as a "broad" or "narrow" line. This depends on its frequency behavior compared to that of $k(\nu)$, both far from the line center. The density of excited atoms for a narrow line is indiscernable from the one for a δ function in frequency [case (ii)]. The slab reemits diffusely the radiation absorbed from the incident field. The reemitted radiation has various interesting features, in particular, a characteristic line shape. Expressions for the reemitted radiation as a function of the frequency and direction of emission will be derived in Secs. III A and III B for cases (i) and (ii), respectively. These situations have closed-form solutions. Situation (iii) is not dealt with. The solution for (iii) can be deduced directly from the one for (ii) according to the methods given in Sec. II C. For deriving the solutions of Sec. II (as for the treatment of self-absorption problems in general) the eigenvalues and eigenfunctions of the Biberman-Holstein integral equation[1,2] are needed. They will be given in Appendix A for the class of line shapes treated here, in the limit of large optical depth.

This paper has been organized such that the method of solution is explained in detail for case (i), which is of the most practical interest (Secs. II A and III A). It may be advantageous to a reader to pass directly from Sec. II A to Sec. III A. Reading of the other subsections can be postponed to a later stage. Appendix A can be read independently of the other results. The relation of the paper to the previous ones[3,4] (hereafter denoted by I and II) is as follows. Section II of this paper generalizes the solution for a Doppler profile obtained in Sec. III of Paper II. Appendix A simplifies and generalizes the results in I. The proofs in this appendix are sketched only. Details will be given elsewhere.[5] The results admit of the straightforward generalization of the solutions of the problems dealt with in Secs. II and IV of Paper II. This will be indicated briefly in Appendix B. In Appendix B will be calculated as well the mean number of scatterings $\bar{N}$ a photon experiences before leaving the slab.

II. DENSITY OF EXCITED ATOMS

Let us consider a slab of thickness L, containing atoms with a resonance frequency ν_0. Only transitions between the ground state and the resonance state will be considered. Therefore, a two-level model is adopted for the atoms. The density in the ground state, $n(1)$, is assumed to be independent of position and to be so high that the optical thickness $k_0 L \gg 1$. The spectral line shape, normalized to unity, is assumed to exhibit for certain values of the parameters D and α the following behavior at frequencies far away from the line center ν_0:

$$\mathfrak{L}(\nu)d\nu \equiv \mathfrak{L}'(u)du \sim D\,|u|^{1/(\alpha-1)}du,$$
$$|u| = 2\,|(\nu-\nu_0)/\Delta\nu| \gg 1. \tag{1}$$

The prime in $\mathfrak{L}'(u)$ is henceforth omitted for convenience. To enable $\mathfrak{L}(u)$ to be normalized, we have $0<\alpha<1$. In particular, Lorentz or Voigt profiles give $D=\pi^{-1}$ and $\alpha=\frac{1}{2}$. Asymmetric line shapes will be considered in Appendix A. It follows that the (asymmetric) statistical profile[6] is described by $D=1$ and $\alpha=\frac{1}{3}$.

A Doppler profile is not described by Eq. (1) for any α. It has to be treated separately. This has been done in Sec. III of Paper II. However, it appears to exhibit many features of the limiting case $\alpha \to 1$, $\alpha < 1$. It should be noted that in Eq. (1) neither D nor α are affected by possible hyperfine structure of the line. Therefore, this case is included at once. The absorption line is defined by

$$k(\nu) \equiv k(u) = k_0\,\mathfrak{L}(u), \qquad k_0 = \frac{2\pi e^2}{mc}\,\frac{n(1)f}{\Delta\nu} \equiv \frac{h\nu_0 B(1,2)n(1)}{2\pi\Delta\nu}; \tag{2}$$

f is the oscillator strength, and $B(1, 2)$ the Einstein coefficient for absorption. The homogeneous radiation field is incident at an angle η with the normal at the left side $(x=-\frac{1}{2}L)$ of the slab. The incident energy per cm^2 and per sec in a frequency interval $d\nu$ within an element of the solid angle $d\Omega$ is denoted by $\tilde{I}[2(\nu-\nu_0)/\Delta\nu]d\nu$. Stimulated emission is neglected. By equating the loss due to spontaneous emission and the gain due to absorption of the incident and the reemitted radiation, the following integral equation for the density in the excited state $n(2)$ is obtained $(-\frac{1}{2}L \leq x \leq \frac{1}{2}L)$:

$$A(2,1)n(2;x) = \frac{1}{h\nu_0}\int_0^\infty \tilde{I}(\nu)\,k(\nu)\exp\left[-\frac{k(\nu)}{\cos\eta}\left(\frac{L}{2}+x\right)\right]d\nu + A(2,1)\int K(|\vec{r}-\vec{r}\,'|)\,n(2;x')\,d\vec{r}\,'. \tag{3}$$

The integration in the second term of the right-hand side of Eq. (3) extends over the entire volume of the slab. $K(r)$ is the integral kernel derived by Biberman[1] and Holstein,[2]

$$K(r) = \int_0^\infty \mathfrak{L}(\nu)\,k(\nu)(e^{-k(\nu)r}/4\pi r^2)d\nu.$$

It is useful to outline briefly the principle of solving Eq. (3). The main difficulty in Eq. (3) obviously arises from the second term in the right-hand side. This difficulty can be overcome if we expand as follows:

$$\frac{n(2;\xi)}{n(1)} = \sum_{j=0}^\infty a_j\psi_j(\xi), \quad \xi = \frac{2x}{L}. \tag{4}$$

The coefficients a_j have to be determined from Eq. (3). The functions $\psi_j(\xi) \equiv n_j(2;\xi)/n(1)$ are the eigenfunctions of the following eigenvalue problem for the Biberman-Holstein integral equation:

$$A(2,1)n_j(2;x) - A(2,1)\int K(|\vec{r}-\vec{r}\,'|)n_j(2;x')\,d\vec{r}\,' = \tilde{A}_j(2,1)\,n_j(2;x); \tag{5}$$

$\tilde{A}_j(2,1)$ are the eigenvalues. The eigenvalue problem is considered in Appendix A for the class of line shapes defined in Eq. (1) and for $k_0L \gg 1$. The requirement $k_0L \gg 1$ means physically that the photons are assumed to experience a large number of scatterings before leaving the volume (see Appendix B). It should be noted that procedure (4) can also be applied if Eq. (3) is generalized by adding a term $n_e \times n(2)K(2,1)$ to the left-hand side and $n_e n(1)K(1,2)$ to the right-hand side of Eq. (3). These terms describe the loss and gain due to collisional deexcitation and excitation by particles whose density is n_e. $K(2,1)$ and $K(1,2)$ are the rate constants. For simplicity, these processes will not be dealt with in the following. However, the calculations are analogous, although, in general, closed-form solutions are not possible. See Sec. II of Paper II for details.

As has been said, the expansion coefficients a_j in Eq. (4) have to be determined.

To this end, Eq. (4) is substituted into Eq. (3). By using the definition of the $\psi_j(\xi)$ [Eq. (5)], the orthogonality relation [Eq. (A11)], and the definition of k_0 [Eq. (2)], the following expression is obtained for the a_j:

$$\tilde{A}_j(2,1)a_j = \frac{B(1,2)}{4\pi k_0}\int_{-\infty}^{+\infty} du\,k(u)\,\tilde{I}(u) \times \int_{-1}^{+1}\psi_j(\xi)\exp\left[-\frac{k(u)L}{2\cos\eta}(1+\xi)\right]d\xi. \tag{6}$$

The integral with respect to ξ is readily evaluated by Eq. (A14). In this formula, applied to Eq. (6), we get a Bessel function of imaginary argument and order $n+\frac{1}{2}+\frac{1}{2}\alpha$, denoted by $I_{n+1/2+\alpha/2}$.[7] Equation (6) becomes $[w = k_0\,\mathfrak{L}(u)L/2\cos\eta]$

$$\tilde{A}_j(2,1)a_j = \frac{B(1,2)\cos\eta}{\pi k_0 L}\sum_{n=0}^\infty(-1)^n c_{j,n}(\alpha) \times \int_{-\infty}^{+\infty} w^{1/2-\alpha/2}\,e^{-w}\,I_{n+1/2+\alpha/2}(w)\,\tilde{I}(u)du. \tag{7}$$

This expression cannot be reduced further analytically if $\tilde{I}(u)$ is not specified. We shall first consider two cases: (i) $\tilde{I}(u) = I_0$, independent of u; (ii) $\tilde{I}(u) = I_0\,\delta(u-u')$. The case of arbitrary $\tilde{I}(u)$ will be considered in Sec. II C.

A. Very Broad Exciting Line

We put $\tilde{I}(u) = I_0$ in Eq. (7). The change of variables $w = k(u)L/2\cos\eta$ is easily performed by using Eq. (1). The resulting expression is expanded asymptotically for $k_0L \gg 1$. If $\mathfrak{L}(u)$ is symmetric it takes the form

$$\tilde{A}_j(2,1)a_j = \frac{B(1,2)I_0(1-\alpha)}{\pi D^{\alpha-1}}\left(\frac{2\cos\eta}{k_0L}\right)^\alpha \sum_{n=0}^\infty(-1)^n c_{j,n}(\alpha) \times \int_0^\infty w^{\alpha/2-3/2}\,e^{-w}\,I_{n+1/2+\alpha/2}(w)dw$$

$$= \frac{B(1,2) I_0 (1-\alpha)}{\pi^{3/2} D^{\alpha-1}} \left(\frac{2\cos\eta}{k_0 L}\right)^{\alpha} \frac{\Gamma(1-\frac{1}{2}\alpha)}{2^{\alpha/2-1/2}} \times \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha)}{\Gamma(n+2)} c_{j,n}(\alpha). \quad (8)$$

Substitution of the expression for $\tilde{A}_j(2,1)$, Eq. (A5), furnishes

$$a_j = \frac{B(1,2)}{A(2,1)} I_0 (\cos\eta)^{\alpha} \frac{(1+\alpha)\, 2^{1/2-\alpha/2}}{\pi^{3/2}\Gamma(\frac{1}{2}\alpha)} \lambda_j(\alpha) \times \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha)}{\Gamma(n+2)} c_{j,n}(\alpha). \quad (9)$$

Note that, in fact, Eq. (9) gives the solution of Eq. (3) since the eigenvalues $\lambda_j(\alpha)$ and the expansion coefficients can be found numerically (see Appendix A). A closed-form solution of Eq. (3) can also be obtained. Equation (9) is substituted back into Eq. (4) in which we have first replaced the eigenfunctions $\psi_j(\xi)$ by their expansion in Gegenbauer polynomials[8] $C_m^{1/2+\alpha/2}(\xi)$ [Eq. (A8)]. We then have

$$\frac{n(2;\xi)}{n(1)} \equiv \sum_{j=0}^{\infty} a_j \psi_j(\xi) = \frac{B(1,2)}{A(2,1)} I_0 (\cos\eta)^{\alpha} \frac{4}{\pi^{5/2}} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha)} \times (1-\xi^2)^{\alpha/2} \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda_j(\alpha) c_{j,m}(\alpha)\, c_{j,n}(\alpha) \times \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+2)} (-1)^n C_m^{1/2+\alpha/2}(\xi).$$

The summation over j can be carried out by the orthogonality relation Eq. (A13). The summation over n then becomes trivial and we are left with the expression

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} I_0 (\cos\eta)^{\alpha} \frac{2}{\pi^{3/2}} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha)} (1-\xi^2)^{\alpha/2} \times \sum_{m=0}^{\infty} (-1)^m \frac{(m+\frac{1}{2}+\frac{1}{2}\alpha)}{(m+\alpha)(m+1)} C_m^{1/2+\alpha/2}(\xi).$$

It can be verified[9] that this expression is precisely the expansion in Gegenbauer polynomials of the function

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} I_0 (\cos\eta)^{\alpha} \times \frac{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}{\pi^{3/2}\Gamma(\frac{1}{2}\alpha)} \int_{\xi}^{1} (1-\xi^2)^{\alpha/2-1} d\xi. \quad (10)$$

Equation (10) is the solution of Eq. (3) for $k_0 L \gg 1$. The behavior in the wings, Eq. (1), is not affected by possible hyperfine structure. Since all the derivations use this behavior only, Eq. (10) is also valid for lines with hfs. If $\mathfrak{L}(u)$ is asymmetric, the right-hand side of Eq. (8) has to be divided by 2. But $\tilde{A}_j(2,1)$ now takes half the value of a symmetric line (see Appendix A). Both factors cancel in Eq. (9). Therefore, Eq. (10) is also valid in this case. For the choice of α, see Appendix A.

For a simple Doppler profile an analogous treatment given in Sec. III of Paper II yields

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} I_0 \cos\eta \left(\frac{\ln(k_0 L/2\pi^{1/2})}{\ln(k_0 L/2\pi^{1/2}\cos\eta)}\right)^{1/2} \frac{\cos^{-1}\xi}{\pi^2} \sim \frac{B(1,2)}{A(2,1)} I_0 \cos\eta \frac{\cos^{-1}\xi}{\pi^2}. \quad (11)$$

The second term in the right-hand side of Eq. (11) is the particular case $\alpha \to 1$ of Eq. (10). It is valid for a line with hfs too (the first term is not). In Fig. 1, $n(2;\xi)/n(1)$ has been displayed for $\alpha = 1$ (Doppler profile), $\alpha = \frac{1}{2}$ (Lorentz, Voigt profile), and the limiting case $\alpha \to 0$, $\alpha > 0$. Its behavior as a function of α can readily be understood. If $\alpha (0<\alpha<1)$ decreases, the contribution of the wing to the total line increases [see Eq. (1)]. Since the radiation for $k_0 L \gg 1$ is predominantly transmitted through the wings [see Eq. (A3)], the slab must become increasingly transparent. Gradients in the density of excited atoms must, therefore, become less steep. Indeed, in the limiting case $\alpha \to 0$, the gradient vanishes for $-1<\xi<+1$. The singularity at $\xi = \pm 1$ for all α should be noted.

Thermodynamic equilibrium. Suppose that radiation from a blackbody at a temperature T_r is incident at $x = -\frac{1}{2}L$ from all directions $(0 \leq \eta < \frac{1}{2}\pi)$. For $k_0 L \gg 1$, local thermodynamic equilibrium must exist between the radiation field and the two-level atoms. Therefore, at $\xi = -1$ the relative density $n(2)/n(1)$ must be given by the Boltzmann factor corresponding to the radiation temperature T_r. We want to check whether this is indeed the case in Eq. (10). Carrying out the integration over a half-sphere in the right-hand side of Eq. (10) and making

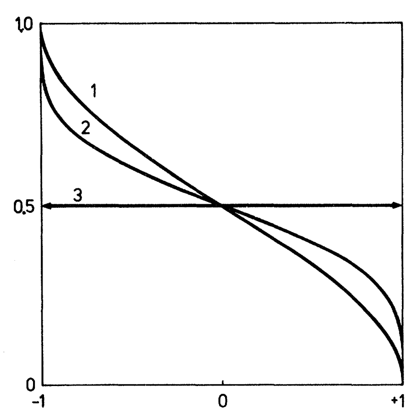

FIG. 1. Density of excited atoms $n(2;\xi)/n(1)$ for $\tilde{I}(u) = I_0$ as a function of $\xi = 2x/L$. Curve 1: $\alpha = 1$ (Doppler profile); curve 2: $\alpha = \frac{1}{2}$ (Lorentz, Voigt profile); curve 3: limiting case $\alpha \to 0$. In the end points curve 3 is equal to 1 $(\xi = -1)$ and to 0 $(\xi = +1)$. See Eq. (10).

use of the fact that blackbody radiation is isotropic, we obtain

$$\frac{n(2;\xi)}{n(1)}=\frac{B(1,2)}{A(2,1)}\,I_0\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{\pi^{1/2}\,\Gamma(\frac{1}{2}\alpha)}\int_\xi^1(1-\zeta^2)^{\alpha/2-1}d\zeta$$

$$=\frac{g_2}{g_1}\,e^{-h\nu_0/kT_r}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{\pi^{1/2}\,\Gamma(\frac{1}{2}\alpha)}\int_\xi^1\,(1-\zeta^2)^{\alpha/2-1}d\zeta. \tag{12}$$

Since the value of the integral for $\xi=-1$ is $\pi^{1/2}\times\Gamma(\frac{1}{2}\alpha)/\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)$, $n(2;\,-1)/n(1)$ is, indeed, equal to the Boltzmann factor. In deriving Eq. (12) the term caused by stimulated emission in Planck's law for I_0 was neglected. In fact, Wien's law, valid if $h\nu_0 \gg kT_r$, was used. This is consistent with the requirement—always imposed here—that $n(2;\xi)/n(1) \ll 1$.

Now let blackbody radiation also be incident from all directions at $x=+\frac{1}{2}L$. From Eq. (12) it follows directly that the relative density is

$$\frac{n(2)}{n(1)}=\frac{n(2;\xi)+n(2;-\xi)}{n(1)}=\frac{g_2}{g_1}\,e^{-h\nu_0/kT_r}\,. \tag{13}$$

In other words, $n(2;\xi)/n(1)$ is constant everywhere and equal to the Boltzmann factor. Equation (13) can be inferred also from thermodynamic principles (it is even valid for all values of k_0L [10]). It has been verified, therefore, that Eq. (10) yields the correct thermodynamic limits.

B. Very Narrow Exciting Line

As a model for a very narrow exciting line $\tilde{I}(u)=\tilde{I}(u')\,\delta(u-u')$ is taken. The substitution of this expression into Eq. (7) yields ($w'=k_0L\;\mathfrak{L}(u')/2\cos\eta$)

$$\tilde{A}_j(2,1)a_j=\frac{B(1,2)\tilde{I}(u')}{\pi k_0L}\;\cos\eta\;w'^{\,1/2-\alpha/2}\,e^{-w'}\times\sum_{n=0}^{\infty}(-1)^n\,c_{j,n}(\alpha)\,I_{n+1/2+\alpha/2}(w'). \tag{14}$$

The formula for $\tilde{A}_j(2,1)$ [Eq. (A5)] is substituted into Eq. (14). We have

$$a_j=\frac{B(1,2)}{A(2,1)}\,\tilde{I}(u')\,\frac{(1+\alpha)}{(1-\alpha)}\,\frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha\pi^2}\;\cos\eta\;(k_0LD)^{\alpha-1}\times w'^{\,1/2-\alpha/2}\,e^{-w'}\,\lambda_j(\alpha)\sum_{n=0}^{\infty}(-1)^n\,c_{j,n}(\alpha)\,I_{n+1/2+\alpha/2}(w'). \tag{15}$$

Equation (15) is substituted back into Eq. (4), in which the eigenfunctions have been replaced by their expansion [Eq. (A8)]. In the same manner as in Sec. II A, the summation over j can be carried out by the orthogonality relation [Eq. (A13)], and the summation over n as a consequence. Because of the complete analogy with Sec. II A the details are omitted here. The result is

$$\frac{n(2;\xi)}{n(1)}=\frac{B(1,2)}{A(2,1)}\,\tilde{I}(u')\,\frac{1+\alpha}{1-\alpha}\,\frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha\pi}\times\cos\eta\;(k_0LD)^{\alpha-1}\,g(w',\xi). \tag{16}$$

The function $g(w',\xi)$ is given by

$$g(w',\xi)=\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{\pi 2^{1/2-\alpha/2}}\;(1-\xi^2)^{\alpha/2}\,w'^{\,1/2-\alpha/2}\,e^{-w'}\times\sum_{m=0}^{\infty}(-1)^m\,(m+\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,\frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}\times I_{m+1/2+\alpha/2}(w')\,C_m^{1/2+\alpha/2}(\xi).$$

By using the addition theorem for Bessel functions[11] it is not difficult to sum this series. The expression for $g(w',\xi)$ now becomes

$$g(w',\xi)=\frac{2^{-1/2-\alpha/2}}{\pi^{1/2}\Gamma(\frac{1}{2}\alpha)}\;(1-\xi^2)^{\alpha/2}\,w'\;e^{-w'}\times\int_0^\infty e^{-t}\,\omega^{-1/2-\alpha/2}\,I_{1/2+\alpha/2}(\omega)\,dt. \tag{17}$$

In this formula $\omega=(t^2-2\xi\,w't+w'^2)^{1/2}$. It can be shown that Eq. (16) is also valid for a Doppler profile (without hfs) if $(1+\alpha)/(1-\alpha)$ is replaced by $4[\ln(k_0L/2\pi^{1/2})]^{1/2}$ and the limit $\alpha\to 1$ is taken. It is useful to discuss the differences between Eqs. (16) and (10). In contradistinction to Eq. (10), $n(2;\xi)/n(1)$ in Eq. (17) is a function of w' and $w'\xi=k_0x\times\mathfrak{L}(u')/\cos\eta$. Therefore, the relative density in the slab varies already over distances of the microscopic order of $[k_0\mathfrak{L}(u')/\cos\eta]^{-1}$ and not only over macroscopic distances of the order of L as in Eq. (10).

As shown by Fig. 2 and also by the analysis below, $n(2;\xi)/n(1)$ for the situation of Eq. (16) remains concentrated mainly within a layer of the order of a few optical depths k_0^{-1}. However, in Eq. (10), $n(2;\xi)/n(1)$ "fills" the whole volume. The reason for the different behavior is the following. The assumption, made in deriving Eq. (10), that $\tilde{I}(u')$ does not vary over the absorption line, requires an energy input increasing without limit for k_0L increasing. In contradistinction to this, the absorbed energy for Eq. (16) is at most $\tilde{I}(u')$. Equation (16) is, of course, only approximately valid within a sheet of the order of a few optical depths k_0^{-1} near by $x=-\frac{1}{2}L$. The photons have experienced an insufficient number of scattering events within this thin layer. Therefore, the asymptotic solutions of the Bibermann-Holstein integral equation do not apply within this sheet, and hence, neither does Eq. (16). In Eq. (17), $w'=k_0L\;\mathfrak{L}(u')/2\cos\eta$ is not necessarily large when $k_0L\gg 1$ since $\mathfrak{L}(u')$ may be small. However, let w' be large too. The new variables $w's=t$ and $\omega'=(s^2-2\xi\,s+1)^{1/2}$ are introduced in Eq. (17). $I_{1/2+\alpha/2}(w'\omega')$ is replaced by the first term of its asymptotic expansion.[7] The Laplace method[12] is applied to the resulting integral for $\xi\neq-1$ in a straightforward manner. Equation (17) then takes

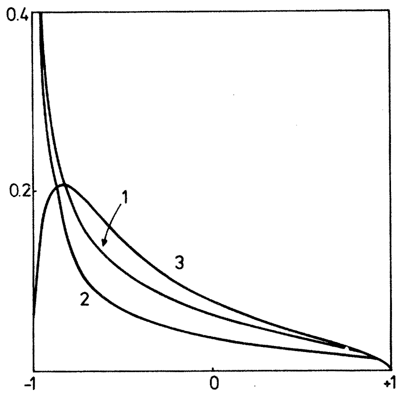

FIG. 2. Density of excited atoms for $\tilde{I}(u)=\tilde{I}(u')\times\delta(u-u')$ as a function of $\xi=2x/L$. Curve 1: $w'^{\alpha/2}g(w',\xi)$ for $w'\gg 1$ and $\alpha=1$ (Doppler profile); curve 2: $w'^{\alpha/2}\times g(w',\xi)$ for $w'\gg 1$ and $\alpha=\frac{1}{2}$ (Lorentz, Voigt profile); curve 3: $w'^{\alpha/2}g(w',\xi)$ for $w'=5$ and $\alpha=1$. See Eqs. (16) and (18).

the form

$$g(w',\xi)\sim\frac{2^{-1-\alpha/2}}{\pi\Gamma(\frac{1}{2}\alpha)}(1-\xi^2)^{\alpha/2}w'^{1-\alpha/2}\times\int_0^\infty w'^{-1-\alpha/2}e^{-w'(s+1-\omega')}ds$$

$$\sim\frac{2^{-1-\alpha/2}}{\pi\Gamma(\frac{1}{2}\alpha)}(1-\xi^2)^{\alpha/2}w'^{1-\alpha/2}\int_0^\infty e^{-w'(1+\xi)s}ds$$

$$=\frac{2^{-1-\alpha/2}}{\pi\Gamma(\frac{1}{2}\alpha)}w'^{-\alpha/2}\frac{(1-\xi^2)^{\alpha/2}}{1+\xi}. \qquad (18)$$

It should be noted that the next order term in Eq. (18) contains terms which are smaller than the first by a factor $[w'(1+\xi)]^{-1}$. The asymptotic development is therefore not uniform in ξ. In particular, Eq. (18) cannot be used for the calculation of the reemitted radiation. Again, it is seen that the behavior of Eq. (16) is quite different from that of Eq. (10), if we substitute Eq. (18) into Eq. (16) and let $k_0L\to\infty$. The ratio $n(2;\xi)/n(1)$ now approaches zero everywhere outside a thin layer of the order of $2\cos\eta/k_0L$ near $\xi=-1$. As has been said already, the detailed description of the physical reality within this sheet by Eq. (16) is only approximate. Hence, not much can be inferred about $n(2;\xi)/n(1)$ in that region. The fact that Eq. (16) describes correctly only the density outside such a layer, explains also the following result, surprising at first sight: From the factor $w'^{-\alpha/2}$ in Eq. (18), it follows that $n(2;\xi)/n(1)$ increases when u' increases, although, in fact, less energy is absorbed. This effect, however, is negligibly small compared to the following one. When u' increases, less radiation is absorbed within the thin sheet near $\xi=-1$. Hence, more radiation is absorbed in the rest of the volume, causing an increase of $n(2;\xi)/n(1)$ there.

C. Arbitrary Exciting Line

From the solution for $\tilde{I}(u)=\tilde{I}(u')\,\delta(u-u')$ [Eq. (16)], the solution for arbitrary $\tilde{I}(u)$ can be deduced by integration. This point will be considered now. The main result is that it is possible to classify the line shape of the incident radiation as a "broad" or "narrow" line. The densities of excited atoms for both classes exhibit a relationship with the special cases considered in Secs. II A and II B, respectively.

For simplicity it will be assumed that $\tilde{I}(u)$ is symmetric. As will be shown, the behavior of $\tilde{I}(u)$ for $|u|\gg 1$ is particularly important. Therefore, the additional assumption is introduced that $\tilde{I}(u)$ exhibits a negative power law dependence in the line wings:

$$\tilde{I}(u)\sim I_0D_0|u|^{1(\beta-1)}, \qquad |u|\gg 1.$$

Here I_0 is the peak intensity and D_0 and β are constants. Since $I(u)$ does not need to be integrable, only $\beta<1$ is required. $\mathfrak{L}(u)$ shows a similar behavior [see Eq. (1)]. Therefore, both can be compared by writing

$$\tilde{I}(u)\sim I_0D_1\mathfrak{L}^{\gamma}(u-u_0), \quad \gamma\geqslant 0, \quad |u|\gg 1. \qquad (19)$$

Here u_0 describes a possible shift of the center of $\tilde{I}(u)$ from the center $u=0$ of $\mathfrak{L}(u)$. It will be assumed that Eq. (19) holds good for a Doppler profile too. In that case the parameter γ has a direct physical interpretation. It describes that the incident radiation is due to a source at a temperature different from that of the slab if $\gamma\neq 1$. In Table I, $\tilde{I}(u)$ is given for some values of γ and D_1 which are of special interest.

It should be noted that a different half-width of $\tilde{I}(u)$ caused by more or less pressure broadening in the source emitting $\tilde{I}(u)$, is described by D_0 and therefore by D_1 and not by γ.[13]

Equation (19) is substituted into Eq. (16) and the integration over u' is carried out. The new variable $w'=k_0L\mathfrak{L}(u')/2\cos\eta$ is introduced. We wish to expand the resulting expression asymptotically. It appears that this has to be done in two different ways according to whether $\frac{1}{2}\alpha+\gamma<1$ or $\frac{1}{2}\alpha+\gamma>1$. The former case will be dealt with first. If $\tilde{I}(u)$ obeys Eq. (19) with $\frac{1}{2}\alpha+\gamma<1$, it will be called a broad line. The asymptotic calculation in this case

TABLE I. $\tilde{I}(u)$ for special values of D and γ.

γ	0	1	$\to\infty$
$D_1=1$	I_0	$I_0\,\mathfrak{L}(u-u_0)$	0
$\pi^{(\gamma-1)/2}\gamma^{1/2}$	0	$I_0\,\mathfrak{L}(u-u_0)$	$I_0\delta_{(u-u_0)}$ (Doppler)
$\dfrac{\pi^{\gamma}\Gamma(\gamma)}{\pi^{1/2}\Gamma(\gamma-\frac{1}{2})}$	...	$I_0\,\mathfrak{L}(u-u_0)$	$I_0\delta_{(u-u_0)}$ (Lorentz)

is as described in Sec. II A. We have for $k_0L \gg 1$

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} I_0 D_1 \frac{\sin(\frac{1}{2}\alpha\pi)}{\pi^2} (\cos\eta)^\alpha \left(\frac{2\cos\eta}{k_0 LD}\right)^\gamma h_1(\xi), \tag{20}$$

with

$$h_1(\xi) = 2^{1/2+\alpha/2}\,\Gamma(\tfrac{3}{2}+\tfrac{1}{2}\alpha)\,(1-\xi^2)^{\alpha/2} \sum_{m=0}^{\infty} (-1)^m (m+\tfrac{1}{2}+\tfrac{1}{2}\alpha) \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} C_m^{1/2+\alpha/2}(\xi) \int_0^\infty w'^{-3/2+\alpha/2+\gamma} e^{-w'} I_{m+1/2+\alpha/2}(w')\,dw'$$

$$= \frac{2^{1-\gamma}}{\pi^{1/2}}\,\Gamma(\tfrac{3}{2}+\tfrac{1}{2}\alpha)\,\Gamma(1-\tfrac{1}{2}\alpha-\gamma)(1-\xi^2)^{\alpha/2} \sum_{m=0}^{\infty} (-1)^m (m+\tfrac{1}{2}+\tfrac{1}{2}\alpha) \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} \frac{\Gamma(m+\alpha+\gamma)}{\Gamma(m+2-\gamma)} C_m^{1/2+\alpha/2}(\xi).$$

Equation (20) is valid for a Doppler profile too if we put $u_0 = 0$, multiply by $[\ln(k_0L/2\pi^{1/2})/\ln(k_0L/2\pi^{1/2} \times \cos\eta)]^{1/2}$, and take the limit $\alpha \to 1$. Though the factor between the square brackets is asymptotically equal to 1, it makes the formulas somewhat more precise. For $D_1 = 1$ and $\gamma = 0$ in Eq. (20), Eq. (10) is immediately recovered. Only in the case $\alpha = 1$ has it appeared possible to obtain a relatively simple expression for $h_1(\xi)$ with the aid of the generating function of the Gegenbauer polynomials.[14] In all other cases $h_1(\xi)$ does not seem to be reducible to a known function.

We turn now to the case that $\frac{1}{2}\alpha + \gamma > 1$. If $\tilde{I}(u)$ obeys Eq. (19) for these values of γ, it is called a narrow line. We now have, for $k_0L \gg 1$,

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} I_0 \frac{1+\alpha}{1-\alpha} \frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha \pi} \times\cos\eta\ (k_0LD)^{\alpha-1} h_2(\xi), \tag{21}$$

with

$$h_2(\xi) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{\pi\, 2^{1/2-\alpha/2}} (1-\xi^2)^{\alpha/2} \times \sum_{m=0}^{\infty} (-1)^m (m+\tfrac{1}{2}+\tfrac{1}{2}\alpha) \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} C_m^{1/2+\alpha/2}(\xi) \times D_1 \int_{-\infty}^{+\infty} w'^{1/2-\alpha/2} e^{-w'} I_{m+1/2+\alpha/2}(w')\, \mathfrak{L}^\gamma(u'-u_0)\,du'.$$

For $\frac{1}{2}\alpha + \gamma > 1$, the asymptotic expansion of $h_2(\xi)$ is now derived by substituting for $I_{m+1/2+\alpha/2}(w')$ the first term of its asymptotic expansion for $w' = k_0L\,\mathfrak{L}(u')/2\cos\eta \gg 1$. The expression for $h_2(\xi)$ then takes the form

$$h_2(\xi) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{\pi^{3/2}} 2^{-1+\alpha/2} \left(D_1 \int_{-\infty}^{+\infty} \mathfrak{L}^\gamma(u'-u_0)\, w'^{-\alpha/2} du'\right) \times (1-\xi^2)^{\alpha/2} \sum_{m=0}^{\infty} (-1)^m (m+\tfrac{1}{2}+\tfrac{1}{2}\alpha) \times \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} C_m^{1/2+\alpha/2}(\xi).$$

It can be verified[15] that this is precisely the expansion of the function

$$h_2(\xi) = \frac{2^{-1-\alpha/2}}{\pi\,\Gamma(\frac{1}{2}\alpha)} \left(D_1 \int_{-\infty}^{+\infty} \mathfrak{L}^\gamma(u'-u_0)\, w'^{-\alpha/2} du'\right) \frac{(1-\xi^2)^{\alpha/2}}{1+\xi}.$$

By substituting this expression into Eq. (21), the final result for a narrow line becomes

$$\frac{n(2;\xi)}{n(1)} = \frac{B(1,2)}{A(2,1)} \left(\int_{-\infty}^{+\infty} \tilde{I}(u')\, w'^{-\alpha/2} du'\right) \frac{1+\alpha}{1-\alpha} \times \frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha \pi^2} \cos\eta\,(k_0LD)^{\alpha-1} \frac{2^{-1-\alpha/2}}{\Gamma(\frac{1}{2}\alpha)} \frac{(1-\xi^2)^{\alpha/2}}{1+\xi}. \tag{22}$$

Hence, for $\frac{1}{2}\alpha + \gamma > 1$, the solution is indiscernible from the solution for $\tilde{I}(u) = \tilde{I}(u')\,\delta(u-u')$, $w' = k(u')L/2\cos\eta \gg 1$. This is true for $\gamma = 1$ in particular $[\tilde{I}(u) \propto \mathfrak{L}(u-u_0)$, u_0 arbitrary$]$. If we let $\gamma \to \infty$ in Eq. (22) and adapt D_1 such that $\tilde{I}(u) \to I_0\,\delta(u-u_0)$ (see Table I), then Eq. (16) for $w' \gg 1$ [i.e., Eq. (18)] is immediately recovered.

III. REEMITTED RADIATION

Once the density of excited atoms is obtained, it is a relatively simple matter to calculate the reemitted radiation. This will be done in this section. We shall confine ourselves to the two cases dealt with in Secs. II A and II B. The reason for this choice is that a very broad exciting line is of particular physical interest. The case of $\tilde{I}(u) = \tilde{I}(u') \times \delta(u-u')$ shows a few interesting features of the problem. Moreover, the general solution can be immediately deduced from it, according to the methods of Sec. II C.

The intensity of the radiation emitted at $x = \pm\frac{1}{2}L$ into an element of the solid angle $d\Omega$ at the angle ϑ with the normal on the right-hand plane of the slab at the frequency ν is designated by $I_\nu(\pm\frac{1}{2}L, \vartheta)d\nu$. It should be noted that, according to this definition, for $I_\nu(+\frac{1}{2}L, \vartheta)$ only $0 \le \vartheta < \frac{1}{2}\pi$, and for $I_\nu(-\frac{1}{2}L, \vartheta)$ only $\frac{1}{2}\pi < \theta \le \pi$ are allowed. A well-known expression relating the emitted radiation to the relative density $n(2;\xi)/n(1)$ reads[16]

$$I_\nu(\pm\tfrac{1}{2}L, \theta) = \frac{A(2,1)}{B(1,2)} w e^{-w} \int_{-1}^{+1} e^{w\xi} \frac{n(2;\pm\xi)}{n(1)} d\xi,$$

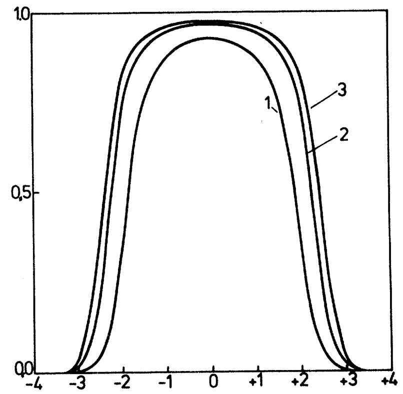

FIG. 3. Line shape of the radiation reemitted in backward direction, i.e., $S_1(w)$, $w=k(u)L/2|\cos\theta|$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)(\ln 2)^{1/2}/\Delta\nu_D$ for $\alpha=1$ and a Doppler profile ($\Delta\nu_D$ Doppler half-width); curve 1: $\pi^{1/2}w(0)=50$; curve 2: $\pi^{1/2}w(0)=250$; curve 3: $\pi^{1/2}w(0)=500$. See Eq. (23).

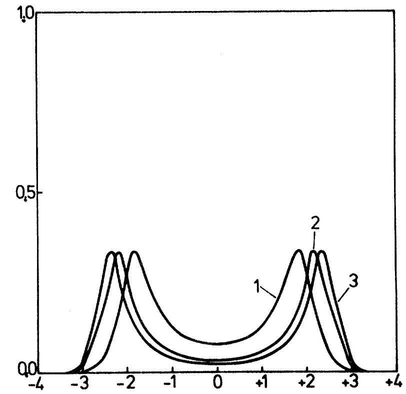

FIG. 4. Line shape of the radiation reemitted in forward direction, i.e., $S_2(w)-S_1(w)$, $w=k(u)L/2|\cos\theta|$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)\times(\ln 2)^{1/2}/\Delta\nu_D$ for $\alpha=1$ and a Doppler profile; curve 1: $\pi^{1/2}w(0)=50$; curve 2: $\pi^{1/2}w(0)=250$; curve 3: $\pi^{1/2}w(0)=500$. See Eq. (24).

$$w=k(u)L/2|\cos\theta|,\quad u=[2(\nu-\nu_0)]/\Delta\nu. \tag{23}$$

We have now to substitute particular expressions for $n(2;\xi)/n(1)$ in Eq. (23) and therefore turn to the special cases treated in Sec. II.

A. Very Broad Exciting Line

Substitution of Eq. (10) into Eq. (23) yields immediately for the backward scattered radiation $(\frac{1}{2}\pi<\theta\leq\pi)$

$$I_\nu(-\tfrac{1}{2}L,\theta)=I_0(1+\alpha)\cos^\alpha\eta\, S_1(\omega)/2\pi, \tag{24}$$

and for the radiation reemitted in the forward direction $(0\leq\theta<\frac{1}{2}\pi)$

$$I_\nu(+\tfrac{1}{2}L,\theta)=I_0(1+\alpha)\cos^\alpha\eta\,[S_2(w)-S_1(w)]/2\pi, \tag{25}$$

where

$$S_1(w)=1-2^{\alpha/2-1/2}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,e^{-w}\,w^{1/2-\alpha/2}\,I_{\alpha/2-1/2}(w),$$

$$S_2(w)=1-e^{-2w}.$$

The spectral line shape described by Eqs. (24) and (25) is of particular interest (see Figs. 3–6). For $k(u)L/2|\cos\theta|\gg1$ (i.e., $k_0L/2|\cos\theta|\gg1$ and in the center of the line), the asymptotic behavior of the modified Bessel function[7] yields

$$I_\nu(-\tfrac{1}{2}L,\vartheta)=I_0\,\frac{1+\alpha}{2\pi}\,\cos^\alpha\eta\left[1-\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}{2^{1-\alpha}\pi^{1/2}}\left(\frac{|\cos\vartheta|}{k(u)L}\right)^{\alpha/2}\right], \tag{26}$$

$$I_\nu(+\tfrac{1}{2}L,\vartheta)=I_0\,\frac{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}{2^{1-\alpha}\pi^{3/2}}\,\cos^\alpha\eta\left(\frac{|\cos\vartheta|}{k(u)L}\right)^{\alpha/2}. \tag{27}$$

Equations (26) and (27) show that $I_\nu(+\frac{1}{2}L,\vartheta)$ has a dip in the center and therefore exhibits self-reversal, while $I_\nu(-\frac{1}{2}L,\vartheta)$ does not, as is also shown by the figures. The behavior in the far wings of the line (defined as those values of u for which $k(u)\,L/2\times|\cos\vartheta|\ll1$) is found from the series representation of the exponential and Bessel functions. We have

$$I_\nu(\pm\tfrac{1}{2}L,\vartheta)=I_0\,\frac{1+\alpha}{4\pi}\,\cos^\alpha\eta\,\frac{k_0L\,\mathfrak{L}(u)}{|\cos\vartheta|}. \tag{28}$$

Equations (23)–(28) are equally valid for a Doppler profile with or without hfs, if we substitute the appropriate line shape and put $\alpha=1$. It should be noted that in the far wings the slab is optically thin so that $I_\nu(\pm\frac{1}{2}L,\vartheta)$ is directly determined by the line shape of the excited atoms, and is not perturbed by self-absorption. The line shape is $\mathfrak{L}(\nu)$ *by assumption*. Where this assumption is only approximately valid (for example, a Doppler profile[17]) deviations and sometimes even strong deviations[18] from Eq.

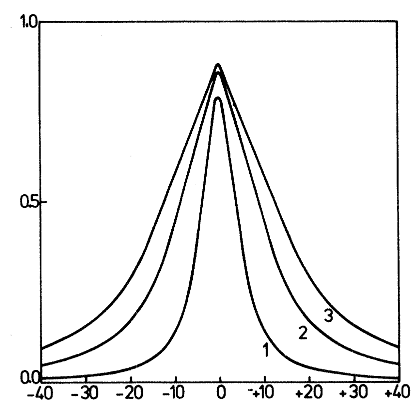

FIG. 5. Line shape of the radiation reemitted in backward direction, i.e., $S_1(w)$, $w=k(u)L/2|\cos\theta|$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)/\Delta\nu_L$ for $\alpha=\frac{1}{2}$ and a Lorentz profile ($\Delta\nu_L$ Lorentz half-width); curve 1: $\pi w(0)=50$; curve 2: $\pi w(0)=250$; curve 3: $\pi w(0)=500$. See Eq. (23).

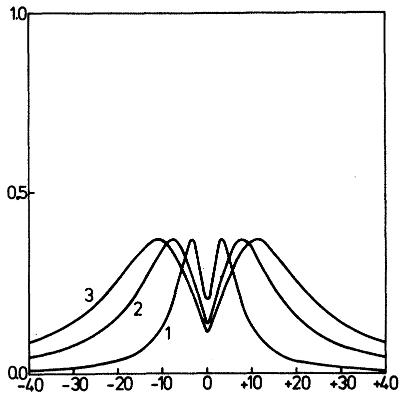

FIG. 6. Line shape of the radiation reemitted in forward direction, i.e., $S_2(w)-S_1(w)$, $w=k(u)L/2|\cos\theta|$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)/\Delta\nu_L$ for $\alpha=\frac{1}{2}$ and a Lorentz profile; curve 1: $\pi w(0)=50$; curve 2: $\pi w(0)=250$; curve 3: $\pi w(0)=500$. See Eq. (24).

(28) can occur. It is convenient to compare for $k_0L \gg 1$ the total backward and forward scattered intensities in the direction ϑ with the total absorbed energy. The latter quantity denoted by E_{abs} is given by[19] $[w'=k_0L\,\mathfrak{L}(u')/2\cos\eta]$

$$E_{abs}\equiv I_0\cos\eta\int_0^\infty (1-e^{-2w'})\,d\nu'=I_0\Delta\nu\cos\eta\int_0^\infty S_2(w')\,du'$$

$$\sim -I_0\Delta\nu\cos\eta\left(\frac{k_0LD}{2\cos\eta}\right)^{1-\alpha}\int_0^\infty S_2(w')\,dw'^{\,\alpha-1}.$$

Hence, by partial integration,

$$E_{abs}\sim I_0\Delta\nu\Gamma(\alpha)\cos^\alpha\eta\,(k_0LD)^{1-\alpha}. \qquad (29)$$

For a simple Doppler profile it is shown in similar manner that[20]

$$E_{abs}\sim I_0\Delta\nu\cos\eta[\ln(k_0L/\pi^{1/2}\cos\eta)]^{1/2}.$$

Equation (29) derived for a symmetric line is equally valid for lines with hfs and if divided by 2 for asymmetric lines. For the choice of α in the latter case, see Appendix A. For a Doppler profile with hfs, see Ref. 20. The expressions for the total reemitted intensities in backward and forward directions contain integrals of the functions $S_1(w)$ and $S_2(w)$. The asymptotic treatment of the integral of $S_2(w)$ has been given above and the other expression is dealt with analogously. The results, for $k_0L\gg 1$, are

$$\int_0^\infty I_\nu(-\tfrac{1}{2}L,\vartheta)\,d\nu\sim\frac{1+\alpha}{2\pi^{3/2}}\;2^{\alpha-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)$$

$$\times\Gamma(1-\tfrac{1}{2}\alpha)\,|\cos\vartheta|^{\alpha-1}E_{abs}, \qquad (30a)$$

$$\sim I_0\frac{\Delta\nu}{\pi}\;\cos\eta[\ln(k_0L/2\pi^{1/2}|\cos\vartheta|)]^{1/2},$$

$$\text{(Doppler)} \qquad (30b)$$

$$\int_0^\infty I_\nu(+\tfrac{1}{2}L,\vartheta)\,d\nu\sim\frac{1+\alpha}{2\pi}\left(1-\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma(1-\frac{1}{2}\alpha)}{2^{1-\alpha}\,\pi^{1/2}}\right)$$

$$\times|\cos\vartheta|^{\alpha-1}E_{abs}, \qquad (31a)$$

$$\sim 0.5306\,I_0\Delta\nu\,\frac{\cos\eta}{\pi}\,[\ln(k_0L/2\pi^{1/2}|\cos\vartheta|)]^{-1/2}.$$

$$\text{(Doppler)} \qquad (31b)$$

Equations (30a) and (31a) are also valid for asymmetric line shapes (see Appendix A) and for lines with hfs.

For a Doppler profile, the first asymptotic terms in the integrals of S_1 and S_2 in Eq. (31b) appear to cancel. The next order gives Eq. (31b) in which the numerical factor[21] $\Gamma'(1)-\Gamma'(\frac{1}{2})/\pi^{1/2}=0.5306$ has been inserted. For the method of calculation see Ref. 20. It should be noted that, in contradistinction to Eq. (31a), for a Doppler profile the forward scattered intensity decreases as a function of k_0L (though weakly). By multiplying Eqs. (30) and (31) with $\cos\vartheta$ and integrating over a half-sphere, the total reemitted intensities in backward and forward directions, denoted by $E_{em}(\pm\frac{1}{2}L)$, are obtained. It is easily verified that $E_{em}(+\frac{1}{2}L)+E_{em}(-\frac{1}{2}L)=E_{abs}$, so that all absorbed radiation is indeed reemitted. Furthermore, $E_{em}(-\frac{1}{2}L)=R(\alpha)E_{abs}$; $R(\alpha)=2^{\alpha-1}\times\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\,\Gamma(1-\frac{1}{2}\alpha)/\pi^{1/2}$. $R(\alpha)$ is an increasing function of α. Hence, the slab increasingly diffusely "reflects" the incident radiation. This is in accordance with the discussion at the end of Sec. II A, in which it was shown that the slab becomes less transparent with increasing α. In particular $R(0)=\frac{1}{2}$. Therefore, for $\alpha\to 0$, equal amounts of energy are reradiated in forward and backward directions: $E_{em}(+\frac{1}{2}L)=E_{em}(-\frac{1}{2}L)=\frac{1}{2}E_{abs}$. This result could also have been inferred directly from Eq. (10) or Fig. 1 since $n(2;\xi)/n(1)$ becomes constant as a function of position for $\alpha\to 0$. For a Doppler profile we have $R(1)=1$. Hence, in first approximation for $k_0L\gg 1$, the slab reemits in backward direction all the absorbed energy. No radiation leaves the volume through the surface at $x=\frac{1}{2}L$ as also shown by Eq. (31b) for $k_0L\to\infty$.

The reemission of radiation should be clearly distinguished from what is known as specular reflection,[22] which is caused by the change of the index of refraction at the glass- (or quartz-) vapor interface. Diffuse scattering has been experimentally observed.[23] The experimental conditions described in Ref. 23 seem not to allow of an easy theoretical interpretation of the results. The author is not aware of other similar experimental papers.

B. Very Narrow Exciting Line

Equation (16) is substituted into Eq. (23). The integration is easily performed[24] and we get, with $w'=k_0L\,\mathfrak{L}(u')/2\cos\eta$ and $w=k_0L\,\mathfrak{L}(u)/2\,|\cos\vartheta|$

$$I_\nu(\pm\tfrac{1}{2}L,\vartheta)=\tilde{I}(u')\,\frac{1+\alpha}{1-\alpha}\,\frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha\pi}\times(k_0LD)^{\alpha-1}\cos\eta\,S_\pm(w,w'),$$
$$S_\pm(w,w')=(ww')^{1/2-\alpha/2}\,e^{-w-w'}\times\sum_{m=0}^{\infty}(\mp)^m(m+\tfrac{1}{2}+\tfrac{1}{2}\alpha)I_{m+1/2+\alpha/2}(w)I_{m+1/2+\alpha/2}(w'). \quad (32)$$

It should be noted that $S_\pm(w,w')$ is symmetric in the variables w and w': $S_\pm(w',w)=S_\pm(w,w')$. See Figs. 7 and 8 for a display of $S_\pm(w,w')$ for $\alpha=1$ (Doppler profile).

It can be proved[5] that for w or w' (or both) large $S_\pm(w,w')$ is expressible in a confluent hypergeometric function[21] $M(a,b;x)$:

$$S_-(w,w')=\frac{2^{-1-\alpha/2}}{\pi^{1/2}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}\,\frac{ww'}{(w+w')^{1+\alpha/2}}\times M\left(\frac{\alpha}{2},1+\alpha;\frac{-2ww'}{w+w'}\right),$$
$$S_+(w,w')=\frac{2^{-1-\alpha/2}}{\pi^{1/2}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}\,\frac{ww'}{(w+w')^{1+\alpha/2}}\times\exp\left(\frac{-2ww'}{w+w'}\right)M\left(\frac{\alpha}{2},1+\alpha;\frac{2ww'}{w+w'}\right). \quad (33)$$

In Eqs. (33), the symmetry in the variables w and w' as present in Eq. (32) has been preserved. Equations (33) comprise the special result w fixed, $w'\to\infty$ [corresponding to the density of excited atoms Eq. (18)] which can also directly be deduced from Eq. (32). This is done by substituting for the modified Bessel functions the first term of their asymptotic expansion.[7] The remaining series is identified as the expansion of the confluent hypergeometric function[21] of argument $2w$ [i.e., Eq. (33) for w fixed, $w'\to\infty$]. We discuss the line shape of the emitted radiation, i.e., the properties of $S_\pm(w,w')$ in Eq. (32) for w' fixed and w variable. For $w=k_0L\times\mathfrak{L}(u)/2|\cos\vartheta|\gg1$ (i.e., $k_0L/2|\cos\vartheta|\gg1$ and in the center of the line), Eq. (33) is applicable. Three regions of values of w' have to be distinguished, namely, $w'\ll w$, $w'\approx w$ and $w'\gg w$. It is readily deduced from the asymptotic properties[21] of $M(a,b;x)$ whether self-reversal occurs or not. The results are given in Table II.

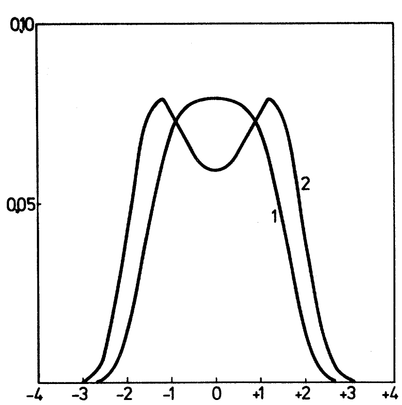

FIG. 7. Line shape of the radiation reemitted in backward direction, i.e., $S_-(w,w')$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)(\ln2)^{1/2}/\Delta\nu_D$ for $\alpha=1$ and a Doppler profile; curve 1: $\pi^{1/2}w(0)=\pi^{1/2}w^1=50$; curve 2: $\pi^{1/2}w(0)=250$, $\pi^{1/2}w'=50$. The curve for $\pi^{1/2}w(0)=\pi^{1/2}w'=250$ is indiscernable from curve 1. See discussion in the text and Eq. (32) and Table II.

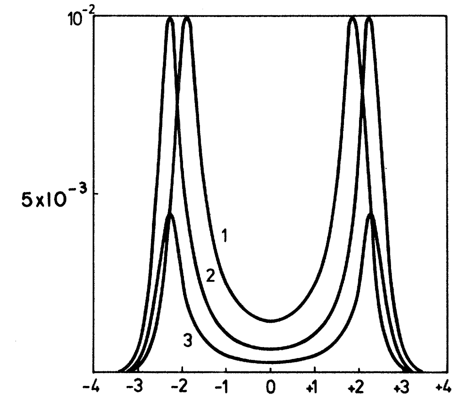

FIG. 8. Line shape of the radiation reemitted in forward direction, i.e., $S_+(w,w')$ as a function of the dimensionless frequency $u=2(\nu-\nu_0)(\ln2)^{1/2}/\Delta\nu_D$ for $\alpha=1$ and a Doppler profile; curve 1: $\pi^{1/2}w(0)=\pi^{1/2}w'=50$; curve 2: $\pi^{1/2}w(0)=250$, $\pi^{1/2}w'=50$; curve 3: $\pi^{1/2}w(0)=\pi^{1/2}w'=250$. See Eq. (32) and Table II.

However, it should be noted that the intensity for frequencies $\nu\approx\nu_0$ of the backward reemitted radiation $I_\nu(-\frac{1}{2}L,\vartheta)$ is directly determined by the density of excited atoms $n(2;\xi)/n(1)$ for $\xi\approx-1$. As discussed in Sec. II B, the description of $n(2;\xi)/n(1)$ is only approximate in a thin layer of the order of a few optical depths k_0^{-1} near $\xi=-1$. This is brought about because the photons undergo an insufficient number of scatterings in this sheet, and, therefore, the asymptotic solutions of the Bibermann-Holstein equation do not apply there. Hence, the description of $I_\nu(-\frac{1}{2}L,\vartheta)$ for $\nu\approx\nu_0$ by Eq. (32) is

TABLE II. Behavior of $S_\pm(w,w')$ in the line center, $w\gg1$.

	$S_+(w,w')$	$S_-(w,w')$
$w'\ll w$	self-reversal	self-reversal
$w'\approx w$	self-reversal	no self-reversal, flat for $\alpha=1$
$w'\gg w$	self-reversal	no self-reversal

only approximate.

The behavior of $I_\nu(\pm\frac{1}{2}L, \vartheta)$ for frequencies in the far wings, $w \ll 1$, is found by substituting the first term of the power-series expansion of the pertinent functions. It is found that the intensity decreases proportional to $\mathfrak{L}(u)$. We repeat the remark made in Sec. III A that this behavior is due to the *assumption* that the line shape of the excited atoms is $\mathfrak{L}(\nu)$ (independent of the frequency of the absorbed photon).

Finally, we want to obtain the total emitted intensity for the situation described by Eq. (32). It can be found directly by using the symmetry of $S_\pm(w, w')$. We put $\tilde{I}(u') = I_0$ in Eq. (32) and integrate over all u'. Clearly, we obtain the reemission due to excitation by a very broad line. Comparison of Eqs. (32) and (24) shows that (for $k_0 L \gg 1$) we must have, in the case of $S_-(w, w')$,

$$\int_{-\infty}^{+\infty} S_-(w, w')\, du' = \frac{1-\alpha}{\sin(\frac{1}{2}\alpha\pi)}\left(\frac{k_0 LD}{2\cos\eta}\right)^{1-\alpha} S_1(w').$$

Because $S_-(w, w')$ is symmetric in the variables w and w' we have

$$\int_{-\infty}^{+\infty} S_-(w, w')\, du = \frac{1-\alpha}{\sin(\frac{1}{2}\alpha\pi)}\left(\frac{k_0 LD}{2|\cos\vartheta|}\right)^{1-\alpha} S_1(w') .$$

With the aid of the latter integral and Eq. (32) the total reemitted energy in the direction ϑ is $(\frac{1}{2}\pi < \vartheta \leq \pi)$

$$\int_0^\infty I_\nu(-\tfrac{1}{2}L, \vartheta)\, d\nu = \tilde{I}(u')\, \Delta\nu[(1+\alpha)/4\pi] \times \cos\eta\, |\cos\vartheta|^{\alpha-1} S_1(w'). \quad (34)$$

In the same fashion, we have $(0 \leq \vartheta < \frac{1}{2}\pi)$

$$\int_0^\infty I_\nu(+\tfrac{1}{2}L, \vartheta)\, d\nu = \tilde{I}(u')\, \Delta\nu[(1+\alpha)/4\pi] \times \cos\eta\, |\cos\vartheta|^{\alpha-1}[S_2(w') - S_1(w')]. \quad (35)$$

By multiplying Eqs. (34) and (35) by $\cos\vartheta$, integrating over a half-sphere and adding the results, the total reemitted energy $E_{\rm em}(-\frac{1}{2}L) + E_{\rm em}(\frac{1}{2}L)$ is obtained:

$$E_{\rm em}(-\tfrac{1}{2}L) + E_{\rm em}(+\tfrac{1}{2}L) = \tilde{I}(u')\, \tfrac{1}{2}\Delta\nu \cos\eta\left(1 - e^{-k_0 L\, \mathfrak{L}(u')/\cos\eta}\right). \quad (36)$$

This expression equals the total absorbed energy.[19] The reemitted radiation corresponding to the situations treated in Sec. II C can be deduced directly by integrating $\tilde{I}(u')\, S_\pm(w, w')$ [Eq. (32)] with respect to u' (see Sec. II C for details). In particular, for a narrow line we have

$$I_\nu(+\tfrac{1}{2}L, \vartheta) = \frac{1+\alpha}{1-\alpha}\, \frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha \pi^{3/2}}\, \frac{2^{-1-\alpha/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}\, (k_0 LD)^{\alpha-1} \cos\eta \times\left(\int_{-\infty}^{+\infty} \tilde{I}(u')\, w'^{-\alpha/2} du'\right) w e^{-2w}\, M(\tfrac{1}{2}\alpha, 1+\alpha; 2w), \quad (37a)$$

$$I_\nu(-\tfrac{1}{2}L, \vartheta) = \frac{1+\alpha}{1-\alpha}\, \frac{\sin(\frac{1}{2}\alpha\pi)}{2^\alpha \pi^{3/2}}\, \frac{2^{-1-\alpha/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)}\, (k_0 LD)^{\alpha-1} \cos\eta \times\left(\int_{-\infty}^{+\infty} \tilde{I}(u')\, w'^{-\alpha/2} du'\right) w\, M(\tfrac{1}{2}\alpha, 1+\alpha; -2w). \quad (37b)$$

Equations (37a) and (37b) are indistinguishable from the results for $\tilde{I}(u) = \tilde{I}(u')\ \delta(u-u')$ [Eq. (32)] for $w' \to \infty$ [i.e., Eq. (33) for $ww'/(w+w') \to w$], in accordance with the conclusions of Sec. II C. It should be noted that for $\tilde{I}(u')$ arbitrary the total reemitted energy is equal to Eq. (36) integrated over all u'. The resulting expression precisely equals the one for the total absorbed energy,[19] as it should.

IV. SUMMARY AND CONCLUSIONS

In Secs. II and III the density of excited atoms and reemitted radiation has been calculated for different profiles of the incident radiation. It is not difficult to extend the treatment to the situation where collision processes are involved. Again, $n(2)/n(1)$ can be written as a superposition of the eigenfunctions of the Biberman-Holstein integral equation (see Secs. II and III of Paper II). Expressions for the reemitted radiation are readily obtained with the aid of the results derived in Appendix B. An interesting application of the theory is possibly the following: In an experimental arrangement simulating a slab, gas or vapor is introduced consisting of atoms capable of deexciting the atoms $n(2)$. Of course, $n(2; \xi)/n(1)$ changes compared to the situation in which the gas or vapor is absent, in particular, as a function of position. Consequently, the reemitted radiation changes also, in particular, its spectral line shape. The total cross section for deexcitation or excitation transfer can therefore be derived from an analysis of the changing spectral line shape of the reemitted radiation.

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APPENDIX A

In this appendix the method of solving the Biberman-Holstein integral equation is given. The results are a generalization of the calculations of Paper I.[3] Detailed proofs are omitted; they are given elsewhere.[5]

For the solution of self-absorption problems in general we need the eigenvalues $\tilde{A}_j(2,1)$ and eigenfunctions $\psi_j(\vec{r}) \equiv n_j(2; \vec{r})/n(1)$ of the Biberman-Holstein integral equation[1,2]

$$\left(1 - \frac{\tilde{A}_j(2,1)}{A(2,1)}\right) n_j(2; \vec{r}) = \int K(|\vec{r} - \vec{r}\,'|)\, n_j(2; \vec{r}\,')\, d\vec{r}\,', \quad (A1)$$

$$K(r)=\int_0^\infty \mathcal{L}(\nu)\,k(\nu)\,e^{-k(\nu)r}/4\pi r^2\,d\nu.$$

The densities in the state 2 and in the ground state are denoted by $n(2;\vec{r})$ and $n(1)$ [$n(1)$ independent of position].

$A(2,1)$ is the Einstein constant for spontaneous emission $2\to 1$. The integration is over the volume V, which will be assumed to be a slab of thickness $L(-\frac{1}{2}L\leq x\leq\frac{1}{2}L$ and $-\infty<y,\,z<+\infty)$. $\mathcal{L}(\nu)$ is the line shape of the emitted radiation and $k(\nu)$ the absorption coefficient. The solutions of Eq. (A1) are treated in the case that the symmetric line shape $\mathcal{L}(\nu)$, properly normalized to unity, exhibits the following behavior in the wings:

$$\mathcal{L}(\nu)\,d\nu\equiv\mathfrak{L}(u)\,du\sim D\,|u|^{1/(\alpha-1)}\,du,$$

$$|u|\equiv 2\,|(\nu-\nu_0)|/\Delta\nu\gg 1, \tag{A2}$$

$$k(\nu)\equiv k(u)=k_0\mathcal{L}(u),\quad k_0=(2\pi e^2/mc)n(1)f/\Delta\nu.$$

D is a constant determined by the normalization; $\Delta\nu$ is a characteristic breadth. $\mathcal{L}(u)$ must be integrable and therefore we have that $0<\alpha<1$. An important case as the Doppler profile cannot be included in the class of line shapes defined by Eq. (A2) and has to be treated separately. However, it will appear to exhibit many features of the limiting case $\alpha\to 1$ $(\alpha<1)$. It should be noted that the behavior in the wings (i.e., D and α) [Eq. (A2)] is not affected by possible hfs of the line. We shall need the behavior of the Fourier transform of $K(\vec{r})$ for small values of $\sigma=|\vec{\sigma}|$. A formula for this asymptotic behavior has been derived in Paper I.[3] It reads

$$\int e^{i\vec{\sigma}\cdot\vec{r}}\,K(r)\,d\vec{r}\sim 1-(Ck_0/\sigma)\int_{\Delta(\sigma)}^\infty \mathfrak{L}^2(u)du,\quad \sigma/k_0\ll 1\,. \tag{A3}$$

C is a constant (dependent on α) and $\Delta(\sigma)>0$ is the solution for $\sigma/k_0\ll 1$ of the equation

$$\mathfrak{L}(\Delta(\sigma))=\sigma/k_0\,.$$

For further details and for a physical interpretation, see Paper I. For the class of line shapes defined by Eq. (A2) a simple calculation yields immediately

$$\int e^{i\vec{\sigma}\cdot\vec{r}}\,K(r)\,d\vec{r}\sim 1-\frac{1-\alpha}{1+\alpha}\,\frac{\pi D^{1-\alpha}}{\sin(\frac{1}{2}\alpha\pi)}\left(\frac{\sigma}{k_0}\right)^\alpha,\quad \frac{\sigma}{k_0}\ll 1. \tag{A4}$$

It has been shown by Widom[25] that the eigenvalues and eigenfunctions of an integral equation of the type of Eq. (A1) can be obtained from the Fourier transform of the integral kernel. The result is an asymptotic one. Applied to Eq. (A1) the theorem shows that a relation exists between the eigenvalues and the eigenfunctions of Eq. (A1) for $k_0L\gg 1$ and the Fourier transform of $K(\vec{r})$ for $\sigma/k_0\ll 1$ [Eq. (A4)]. More precisely, the eigenvalues $\tilde{A}_j(2,1)$ of Eq. (A1) are asymptotically

$$\frac{\tilde{A}_j(2,1)}{A(2,1)}\sim\frac{1-\alpha}{1+\alpha}\,\frac{\pi D^{1-\alpha}}{\sin(\frac{1}{2}\alpha\pi)}\,\frac{2^\alpha\lambda_j^{-1}(\alpha)}{(k_0L)^\alpha},\quad k_0L\gg 1, \tag{A5}$$

and the eigenfunctions $\psi_j(\xi)$, $\xi=2x/L$, $-1\leq\xi\leq+1$, are

$$\frac{n_j(2;\xi)}{n(1)}\equiv\psi_j(\xi)\sim f_j^{(\alpha)}(\xi),\quad k_0L\gg 1.$$

The numbers $\lambda_j(\alpha)$ and the functions $f_j^{(\alpha)}(\xi)$ are solutions of the following integral equation:

$$\lambda_j(\alpha)f_j^{(\alpha)}(\xi)=\int_{-1}^{+1}K^{(\alpha)}(\xi,\xi')f_j^{(\alpha)}(\xi')\,d\xi'. \tag{A6}$$

An expression for $K^{(\alpha)}(\xi,\xi')$ has been given by Widom.[25] For our purposes a different expression, derived elsewhere,[5] is more appropriate. It reads, for $|\xi|,|\xi'|\leq 1$, $\mathrm{Re}\alpha>0$,

$$K^{(\alpha)}(\xi,\xi')=(2^\alpha/\pi)\,[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)]^2\,(1-\xi^2)^{\alpha/2}\,(1-\xi'^2)^{\alpha/2}\times\sum_{k=0}^\infty(k+\tfrac{1}{2}+\tfrac{1}{2}\alpha)\left[\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\right]^2\times C_k^{1/2+\alpha/2}(\xi)\,C_k^{1/2+\alpha/2}(\xi')$$

$$=0,\quad |\xi|,\,|\xi'|>1. \tag{A7}$$

$C_k^{1/2+\alpha/2}(\xi)$ is a Gegenbauer polynomial.[8] The particular case $\alpha=1$ of Eq. (A7) has been derived previously by Kac and Pollard.[26] We mention in passing that though the expression for $K^{(\alpha)}(\xi,\xi')$ looks complicated, its Fourier transform[24] is simpler:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i(\xi t+\xi't')}\,K^{(\alpha)}(\xi,\xi')\,d\xi\,d\xi'$$

$$=(tt')^{-1/2-\alpha/2}\sum_{k=0}^\infty(k+\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,(-1)^k\times J_{k+1/2+\alpha/2}(t)\;J_{k+1/2+\alpha/2}(t').$$

Here $J_{k+1/2+\alpha/2}(t)$ is a Bessel function of order $k+\frac{1}{2}+\frac{1}{2}\alpha$. Before proceeding to the method of solving Eq. (A6) we pay attention to an extension of the results given above. For a simple Doppler profile it can be shown[3] from Eq. (A3) that Eqs. (A4)–(A7) remain valid if $(1+\alpha)/(1-\alpha)$ is replaced by $4\times[\ln(k_0L/2\pi^{1/2})]^{1/2}$ and the limit $\alpha\to 1$, $\alpha<1$ is taken. For a Doppler profile with hfs, see Paper I. For asymmetric line shapes it can be proved that the parameters α and D are defined by the most slowly decreasing wing and that the second term in Eqs. (A3)–(A5) has to be divided by 2.[3] No change occurs in Eqs. (A6) and (A7), or in the results derived from them. We try to solve Eq. (A6) numerically by expanding the eigenfunctions $f_j^{(\alpha)}(\xi)$ in the Gegenbauer polynomials $C_m^{1/2+\alpha/2}(\xi)$ with expansion coefficients $c_{j,m}(\alpha)$ to be determined

$$f_j^{(\alpha)}(\xi)=\begin{cases}\dfrac{2^{1/2+\alpha/2}}{\pi}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,(1-\xi^2)^{\alpha/2}\sum_{m=0}^{\infty}\dfrac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}\\ \qquad\times c_{j,m}(\alpha)\,C_m^{1/2+\alpha/2}(\xi),\quad |\xi|\le 1\\ 0,\quad |\xi|\ge 1.\end{cases}\tag{A8}$$

Note the singularity at $\xi=\pm 1$. The eigenfunctions of Eq. (A6) are even ($j=0, 2, \ldots$) or odd ($j=1, 3, \ldots$). Consequently, the expansion coefficients $c_{j,2m+1}(\alpha)$ of the even eigenfunctions, and the $c_{j,2m}$ of the odd ones are zero. By substituting Eq. (A8) into Eq. (A6), performing the integration and using the orthogonality relation of the Gegenbauer polynomials,[8] Eq. (A6) is reduced to a matrix problem[5]

$$\lambda(\alpha)\,\vec{c}(\alpha)=K(\alpha)\,\vec{c}(\alpha).\tag{A9}$$

The expansion coefficients have been put together into the vector $\vec{c}(\alpha)\equiv[c_0(\alpha), c_1(\alpha), \ldots]$. The matrix $K(\alpha)$ has the elements for $m,\ n=0, 1, \ldots$:

$$K_{m,n}(\alpha)=\begin{cases}\dfrac{(i)^{m+n}\,\Gamma(1+\alpha)\,(m+\frac{1}{2}+\frac{1}{2}\alpha)\,\Gamma[\frac{1}{2}(m+n+1)]\,(-1)^m}{2^\alpha\,\Gamma[1+\frac{1}{2}(\alpha+m-n)]\,\Gamma[1+\frac{1}{2}(\alpha+n-m)]\,\Gamma[\alpha+\frac{1}{2}(3+m+n)]}, & m+n=\text{even}\\ 0, & m+n=\text{odd}.\end{cases}\tag{A10}$$

Equation (A10) appears to be very suitable for numerical purposes.[5]

For many applications the particular values of $\lambda_j(\alpha)$ and the expansion coefficients are not needed, but a few orthogonality relations only. First, it is noticed that since $K^{(\alpha)}(\xi, \xi')$ is symmetric, the $f_j^{(\alpha)}(\xi)$ constitute an orthogonal set. By normalizing the $f_j^{(\alpha)}(\xi)$ to unity this set is made orthonormal:

$$\int_{-1}^{+1} f_i^{(\alpha)}(\xi)\,f_j^{(\alpha)}(\xi)\,d\xi=\delta_{ij}\,.\tag{A11}$$

As a consequence of Eq. (A11) the following orthogonality relation holds true for the expansion coefficients[5]:

$$\frac{2}{\pi}\,\lambda_i(\alpha)\sum_{m=0}^{\infty}\frac{c_{j,m}(\alpha)\,c_{i,m}(\alpha)}{m+\frac{1}{2}+\frac{1}{2}\alpha}=\delta_{ij}\,;\tag{A12}$$

another relation is[5]

$$\frac{2}{\pi}\sum_{j=0}^{\infty}\lambda_j(\alpha)\,c_{j,m}(\alpha)\,c_{j,n}(\alpha)=(m+\tfrac{1}{2}+\tfrac{1}{2}\alpha)\,\delta_{m,n}\,.\tag{A13}$$

Finally, it should be noted that the Fourier transform of $f_j^{(\alpha)}(\xi)$ is particularly simple[24]:

$$\int_{-\infty}^{+\infty} e^{i\xi t} f_j^{(\alpha)}(\xi)\,d\xi=2\sum_{m=0}^{\infty} i^m\,c_{j,m}(\alpha)\times t^{-1/2-\alpha/2}\,J_{m+1/2+\alpha/2}(t).\tag{A14}$$

The relation with previously derived results is the following. The representation of the asymptotic eigenfunctions of Eq. (A1) for a Doppler profile ($\alpha=1$) derived in Paper I[3] is the same as in Eq. (A8) apart from a factor $2\,[\pi(m+1)]^{-1}$. The representation of the eigenfunctions for a Lorentz or Voigt profile ($\alpha=\frac{1}{2}$) used here is somewhat different from the one derived in I but similar to the one in Sec. IV of Paper II. The present approach makes it possible to solve various problems generally, and leads to analogous formulas for different cases and to substantial simplifications.

APPENDIX B

This appendix is devoted to two special applications of the theory of Appendix A: (i) calculation of the mean number of scatterings $\overline{N}$ a photon experiences before leaving the slab [a distribution $n(2;\xi)$ being given at $t=0$], and (ii) the general form of the line shape emitted by a slab.

Any distribution of excited atoms $n(2;\xi)$ present at $t=0$ can be expanded in the eigenfunctions for $k_0L\gg 1$ of the Biberman-Holstein integral equation as follows:

$$n(2;\xi)=\sum_{j=0}^{\infty}\psi_j(\xi)\int_{-1}^{+1} n(2;\xi')\,\psi_j(\xi')\,d\xi'.\tag{B1}$$

According to the definition of $\tilde{A}_j(2,1)$, an eigenfunction $\psi_j(\xi)$ decays at a rate $\exp[-\tilde{A}_j(2,1)t]$.[2] Since a photon has an average lifetime $A^{-1}(2,1)$, the photons caused by an initial distribution $n(2;\xi)\propto\psi_j(\xi)$ experience a mean number of scatterings $A(2,1)/\tilde{A}_j(2,1)$. The mean number of scatterings for the distribution Eq. (B1) normalized to one photon is therefore

$$\overline{N}=\frac{1}{[n(2)]_{\text{av}}}\sum_{j=0}^{\infty}\frac{A(2,1)}{\tilde{A}_j(2,1)}\int_{-1}^{+1}\psi_j(\xi)\,d\xi\times\int_{-1}^{+1} n(2;\xi')\,\psi_j(\xi')\,d\xi',\tag{B2}$$

$$[n(2)]_{\text{av}}=\int_{-1}^{+1} n(2;\xi')\,d\xi'.$$

From the representation of the eigenfunctions [Eq. (A8)] it follows that[8]

$$\int_{-1}^{+1}\psi_j(\xi)\,d\xi=\frac{2^{1/2-\alpha/2}}{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}\,c_{j,0}(\alpha).\tag{B3}$$

By inserting into Eq. (B2) the expression for $\tilde{A}_j(2,1)$, Eq. (A5), and Eq. (B3) we obtain

$$\overline{N}=\frac{1+\alpha}{1-\alpha}\,\frac{\sin(\frac{1}{2}\alpha\pi)}{\pi D^{1-\alpha}}\left(\frac{k_0L}{2}\right)^{\alpha}\frac{2^{1/2-\alpha/2}}{\Gamma(\frac{3}{2}+\frac{1}{2}\alpha)}\times\sum_{j=0}^{\infty}\lambda_j(\alpha)\,c_{j,0}(\alpha)\int_{-1}^{+1}\frac{n(2;\xi')}{[n(2)]_{\mathrm{av}}}\,\psi_j(\xi')\,d\xi'. \tag{B4}$$

The representation of the eigenfunctions [Eq. (A8)] is substituted into Eq. (B4). The summation over j is carried out by the orthogonality relation [Eq. (A13)]. The summation over m [resulting from Eq. (A8)] becomes then trivial and we have as the final result for $k_0L \gg 1$

$$\overline{N}=\frac{1+\alpha}{1-\alpha}\,\frac{\sin(\frac{1}{2}\alpha\pi)}{\pi D^{1-\alpha}}\,\frac{2^{-\alpha}}{\Gamma(1+\alpha)}\,(k_0L)^{\alpha}\times\int_{-1}^{+1}\frac{n(2;\xi')}{[n(2)]_{\mathrm{av}}}\,(1-\xi'^2)^{\alpha/2}\,d\xi'. \tag{B5}$$

In the same manner $\overline{N}$ is calculated for a Doppler profile. It appears to be the special case $\alpha \to 1$ of Eq. (B5) if $(1+\alpha)/(1-\alpha)$ is replaced by $4[\ln(k_0L/2\pi^{1/2})]^{1/2}$. The special case of Eq. (B5), $n(2;\xi)$ independent of position, has been derived previously by Ivanov.[27]

For the calculation of the radiation due to a certain density of excited atoms $n(2;\xi)$, we start again with the representation of $n(2;\xi)$ as a sum of eigenfunctions of the Biberman-Holstein integral equation for $k_0L \gg 1$:

$$\frac{n(2;\xi)}{n(1)}=\sum_{j=0}^{\infty}a_j\psi_j(\xi)\ . \tag{B6}$$

The expansion coefficients a_j are characteristic for the problem under investigation. Obviously, it is sufficient to calculate for every eigenfunction $\psi_j(\xi)$ the corresponding radiation field. By substituting the representation of the eigenfunctions [Eq. (A8)] into Eq. (22) and by using Eq. (A14), we obtain the radiation emitted in the direction ϑ corresponding to the eigenfunction $n(2;\xi)/n(1)=\psi_j(\xi)$ for $k_0L \gg 1$:

$$I_\nu(\pm\tfrac{1}{2}L,\vartheta)=2\,\mathrm{sgn}[(\pm 1)^j]\,\frac{A(2,1)}{B(1,2)}\,e^{-w}\,w^{1/2-\alpha/2}\times\sum_{m=0}^{\infty}c_{j,m}(\alpha)\,I_{m+1/2+\alpha/2}(w), \tag{B7}$$

$$w=k_0L\,\mathfrak{L}(u)/2\,|\cos\vartheta|\ ,\quad u=[2(\nu-\nu_0)]/\Delta\nu\ .$$

$I_{m+1/2+\alpha/2}$ is a modified Bessel function of order $m+\frac{1}{2}+\frac{1}{2}\alpha$. For a discussion of the factor $\mathrm{sgn}[(\pm 1)^j]$, see Sec. IV B of Paper II.

The analysis of the spectral line shape described by Eq. (B7) proceeds along the lines explained already in Sec. III. It appears that for $\alpha > 0$, $I_\nu(\pm\frac{1}{2}L,\vartheta)$ shows a dip in the center and, therefore, self-reversal.

In the far wings $w \ll 1$, $I_\nu(\pm\frac{1}{2}L,\vartheta)$ decays proportionally to $\mathfrak{L}(u)$ (see the remark about this latter behavior made in Sec. III).

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[10]As a consequence, the higher-order asymptotic corrections to Eq. (10) can contain odd functions only.

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[14]Reference 7, p. 177, Eq. (29).

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