four Eqs. (4.15) are *closed* equations with respect to $g^<$ and $g^<$ in contrast to the hierarchy equations such as (2.6) from which they are derived. The equations are highly nonlinear and non-Markovian. However, fortunately for the purpose of formulating transport coefficients, these equations can be greatly simplified without the loss of rigor, which will be discussed in the forthcoming papers.

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Forward-Scattering Amplitudes and Fermi-Liquid Factors in Dilute Solutions of He^4 in Liquid He^3

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A derivation of exact expressions, given in terms of thermodynamic quantities, for the *s*-wave parts of the He^3-He^4 and He^4-He^4 quasiparticle forward-scattering amplitudes in very dilute solutions of He^4 in liquid He^3 is given. The result for the *p*-wave part of the He^3-He^4 amplitude is related to the He^4 quasiparticle effective mass. The associated Fermi-liquid parameters are also derived.

I. INTRODUCTION

In this paper we use a Green's-function formalism to derive exact equations for the He^3-He^4 and He^4-He^4 quasiparticle forward-scattering amplitudes in dilute solutions of He^4 in liquid He^3. The *s*-wave parts of the He^3-He^4 and He^4-He^4 amplitudes are explicitly evaluated in terms of the thermodynamic quantities, and the *p*-wave part of the He^3-He amplitude is related to the He^4 quasiparticle effective mass. Only the cases of one and two He^4 atoms in liquid He^3 at $T=0$ are considered. However, the results should be useful at all temperatures and He^4 concentrations where a quasiparticle picture is valid (i.e., for $T \leq 0.1$ K in the one phase region on the He^3 rich side of the phase-separation curve). Qualitative arguments leading to many of the results herein have been presented elsewhere.[1] The present work thus provides a quantitative justification of these results within the limits of the quasiparticle description.

The two major results of this work are those for the *s*-wave parts of the He^4-He^3 and He^4-He^4 quasiparticle forward-scattering amplitudes (a^0_{43} and a^0_{44}, respectively) for He^3 quasiparticles on the Fermi surface and He^4 quasiparticles of very small momenta. We find (see also Ref. 1)

$$a^0_{43} = (1+\alpha)/\nu(0) , \tag{1}$$

$$a^0_{44} = \left(\frac{\partial \mu_4}{\partial n_4}\right)_{\mu_3} . \tag{2}$$

Here α is the fractional excess volume occupied by a He^4 atom in liquid He^3, and $\nu(0)$ is the density of He^3 quasiparticle states at the Fermi surface. μ_3 and μ_4 are the He^3 and He^4 chemical potentials, and n_4 is the He^4 number density. Using the result $\alpha \approx 0.32$,[2] we obtain $a^0_{43} \approx 0.68/\nu(0)$. The analogous quantities for parallel and antiparallel spin-quasiparticle scattering in pure He^3 are,[3] respectively, $a_0^{\uparrow\uparrow} = 2.9/\nu(0)$ and $a_0^{\uparrow\downarrow} = -1.1/\nu(0)$. It follows then that the temperature range over which a quasiparticle picture for He^4's in He^3 may be expected to be valid is the same as that for He^3's, namely, $T \leq 0.1$ K. a^0_{44} has been evaluated[1] using a rather plausible assumption concerning the nature of the phase-separation curve as the He^4 number concentration x and T approach zero. The result is

$$a^0_{44} = 0 . \tag{3}$$

It follows from (3) that the phase separation at small x is closely related to that in a noninteracting Bose gas.

Section II is devoted to a derivation of the exact integral equations relating the forward-scattering amplitudes and their associated Fermi-liquid parameters. In Sec. III we evaluate the Fermi-liquid parameters, and in Sec. IV we use these results to obtain (1) and (2) as well as the p-wave part of a_{43}. In Sec. V we reproduce, for the sake of completeness, the arguments given in Ref. 1 leading to (3) and give some further discussion of our theory.

II. INTEGRAL EQUATIONS

To derive the necessary integral equations we consider the variations of the He^3 and He^4 self-energies, denoted collectively by Σ_i, in the presence of a set of external nonlocal potentials V_i coupled to the various particle fields. The perturbing Hamiltonian is, then,

$$H_{\text{ext}}(t_1)=\Sigma_i \int_{-\infty}^{\infty} dt_2 \int d^3r_1 \int d^3r_2 \psi_i^\dagger(1)\psi_i(2)V_i(12) , \tag{4}$$

where $1\equiv(\vec{r}_1, t_1)$, and the index i takes on three values corresponding to spin-up and spin-down He^3's and He^4's. The i-j scattering amplitude is then defined in terms of Σ_i, V_j, and Green's function G_j as[4]

$$\Gamma_{ij}(12, 34)=G_j^{-1}(3\bar{3})\frac{\delta\Sigma_i(12)}{\delta V_j(\bar{4}\bar{3})}G_j^{-1}(\bar{4}4) . \tag{5}$$

Here we have introduced the convention that repeated barred variables are summed (or integrated) over. The variational derivative is to be taken holding V_l constant for $l\neq j$ and at $H_{\text{ext}}=0$.

Considering Σ_i to be a functional of the G_l's, we next write

$$\frac{\delta\Sigma_i(12)}{\delta V_j(43)}=\frac{\delta\Sigma_i(12)}{\delta G_{\bar{l}}(\bar{5}\bar{6})}\frac{\delta G_{\bar{l}}(\bar{5}\bar{6})}{\delta V_j(43)} . \tag{6}$$

Using the easily proven symmetry property

$$\frac{\delta G_{\bar{l}}(\bar{5}\bar{6})}{\delta V_j(43)}=s_l s_j\frac{\delta G_j(34)}{\delta V_l(\bar{6}\bar{5})} , \tag{7}$$

where $s_i=+1(-1)$ for fermions (bosons) together with the relation

$$G_i(1\bar{1})\delta G_i^{-1}(\bar{1}\bar{2})G_i(\bar{2}2)=-\delta G_i(12) \tag{8}$$

and Dyson's equation

$$G_i^{-1}(12)=G_{0i}^{-1}(12)-V_i(12)-\Sigma_i(12) , \tag{9}$$

where G_{0i} is a free-particle Green's function, we easily put (6) in the form

$$\frac{\delta\Sigma_i(12)}{\delta V_j(43)}=\frac{\delta\Sigma_i(12)}{\delta G_j(\bar{5}\bar{6})}G_j(3\bar{6})G_j(\bar{5}4) + \frac{\delta\Sigma_i(12)}{\delta G_{\bar{l}}(\bar{5}\bar{6})}\, G_j(3\bar{3})s_{\bar{l}}s_j\frac{\delta\Sigma_j(\bar{3}\bar{4})}{\delta V_{\bar{l}}(\bar{6}\bar{5})}G_j(\bar{4}, 4) . \tag{10}$$

Next, using (5) in (10), we arrive at

$$\Gamma_{ij}(12, 34)=\frac{\delta\Sigma_i(12)}{\delta G_j(43)}+\frac{\delta\Sigma_i(12)}{\delta G_{\bar{l}}(\bar{5}\bar{6})}G_{\bar{l}}(\bar{5}\bar{3})G_{\bar{l}}(\bar{4}\bar{6}) \times s_{\bar{l}}s_j\,\Gamma_{j\bar{l}}(34, \bar{3}\bar{4}) . \tag{11}$$

It follows directly from (5) and (7) that

$$\Gamma_{ij}(12, 34)=s_i s_j\,\Gamma_{ji}(34, 12) . \tag{12}$$

Putting (9) into (8), we obtain

$$\Gamma_{ij}(12, 34)=\frac{\delta\Sigma_i(12)}{\delta G_j(43)}+\frac{\delta\Sigma_i(12)}{\delta G_{\bar{l}}(\bar{5}\bar{6})}G_{\bar{l}}(\bar{3}\bar{6})G_{\bar{l}}(\bar{5}\bar{4})\Gamma_{\bar{l}j}(\bar{3}\bar{4}, 34). \tag{13}$$

After Fourier transforming in the variable differences $1-2$, $3-4$, and $4-2$ corresponding to the four-momenta p, p', and k, Eq. (13) becomes

$$\Gamma_{ij}(p,p';k)=I_{ij}(p,p';k)+I_{i\bar{l}}(p,\bar{q}+k;k)G_{\bar{l}}(\bar{q}) \times G_{\bar{l}}(\bar{q}+k)\,\Gamma_{\bar{l}j}(\bar{q},p';k) , \tag{14}$$

where to simplify notation we have defined

$$I_{ij}(12, 34)\equiv\delta\Sigma_i(12)/\delta G_j(43) . \tag{15}$$

It is convenient to write (14) in matrix form as

$$\Gamma(p,p';k)=I(p,p';k)+I(p,\bar{q}+k;k)G(\bar{q}) \times G(\bar{q}+k)\,\Gamma(\bar{q},p';k), \tag{16}$$

where the product $G(\bar{q})G(\bar{q}+k)$ is to be interpreted as the diagonal matrix

$$[G(\bar{q})G(\bar{q}+k)]_{ij}=\delta_{ij}G_i(\bar{q})G_i(\bar{q}+k) . \tag{17}$$

Equation (16) has the easily understandable diagrammatic form shown in Fig. 1.

We will be interested in the $k=(\vec{k}, k_0)\to 0$ limit of (16). As is well known,[5] a study of this limit requires some care since the product $G(\bar{q})G(\bar{q}+k)$ is singular in this limit, the results depending upon the limit which $r\equiv|\vec{k}|/k_0$ is allowed to take. We have, in fact, for the He^3's, for small k (σ is a spin index)[5]

$$G_\sigma(q)G_\sigma(q+k)=G_\sigma^2(q)+\frac{2\pi i z_3^2}{V_F}\,\frac{\vec{k}\cdot\vec{V}_q}{k_0-\vec{k}\cdot\vec{V}_q}$$

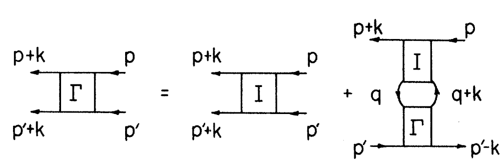

FIG. 1. Graphical form for the integral equation for the matrix scattering amplitude. The external propagators are included for visual clarity.

$$\times\ \delta(q_0-\mu_3)\delta(|\vec{q}|-p_F)\ . \qquad (18)$$

Here V_F is the Fermi velocity, p_F is the Fermi momentum, $\vec{V}_q$ is the quasiparticle velocity, and μ_3 is the He^3 chemical potential. Since we consider only the dilute limit for the He^4, we have

$$G_4(q)G_4(q+k)=G_4^2(q) \qquad (19)$$

for small k; singular terms would come from the presence of He^4's in the ground state in the definition of the Green's function. The function $I=\delta\Sigma/\delta G$ is not singular as $k\to 0$, as discussed in Ref. 5.

We now take the $r\to\infty$, $k\to 0$ limit of (16) to find

$$\Gamma^\infty(p,p')=I(p,p')+I(p,\bar{q})[G^2(\bar{q})+R(\bar{q})]\Gamma^\infty(\bar{q},p')\ . \qquad (20)$$

Here,

$$R_{\sigma\sigma'}(\bar{q})=-\frac{2\pi i z_3}{V_F}\delta(\bar{q}_0-\mu_3)\delta(|\vec{q}|-p_F)\delta_{\sigma\sigma'}\ , \qquad (21)$$

$$R_{44}(\bar{q})=0,\quad I(p,p')=\lim_{k\to 0}\frac{\delta\Sigma}{\delta G}(p,p';k)\ .$$

Taking the $r\to 0$, $k\to 0$ limit gives

$$\Gamma^0(p,p')=I(p,p')+I(p,\bar{q})G^2(\bar{q})\Gamma^0(\bar{q},p')\ . \qquad (22)$$

It is convenient to eliminate G^2 and I from the two equations (20) and (22). Solving (22) symbolically, we have

$$\Gamma^0=[1-IG^2]^{-1}I\ . \qquad (23)$$

Doing the same with (20), and using (23),

$$\Gamma^\infty=[1-IG^2]^{-1}I[1+R\Gamma^\infty]=\Gamma^0+\Gamma^0R\Gamma^\infty\ . \qquad (24)$$

Written in more complete form, (24) is

$$\Gamma^\infty(p,p')=\Gamma^0(p,p')+\Gamma^0(p,\bar{q})R(\bar{q})\Gamma^\infty(\bar{q},p')\ , \qquad (25)$$

the result at which we were aiming. Γ^0 and Γ^∞ are (within constant factors) the Fermi-liquid factor and the quasiparticle forward-scattering amplitude.[5] We now turn to an evaluation of Γ^0.

III. EVALUATION OF FERMI-LIQUID FACTORS

Let us first evaluate $\Gamma^0_{4\sigma}$. This is most easily done by directly deriving a matrix equation for conventionally defined Fermi-liquid parameters. Recall that in the quasiparticle approximation the quasiparticle energies are given by

$$\epsilon_{p_i}=\frac{p^2}{2m_i}+\Sigma_i(\vec{p},\epsilon_{p_i})\ . \qquad (26)$$

Denoting by n_{p_i} the quasiparticle occupation number, we define the Fermi-liquid parameter as

$$f_{p_ip_j'}\equiv\left(\frac{\delta\epsilon_{p_i}}{\delta n_{p_j}}\right)_{n_{pl},\,l\neq j}\ . \qquad (27)$$

From Eq. (26) we may then derive an equation for $f_{p_ip_j'}$:

$$f_{p_ip_j'}=\frac{\delta\Sigma_i(p,p_0)}{\delta n_{p_j}}\bigg|_{p_0=\epsilon_{p_i}}+\frac{\partial\Sigma_i(\vec{p},p_0)}{\partial p_0}\bigg|_{p_0=\epsilon_{p_i}}f_{p_ip_j'}\ . \qquad (28)$$

Denoting as usual[5] the wave-function renormalization constant by

$$z_{ip}=\left(1-\frac{\partial\Sigma_i(\vec{p},p_0)}{\partial p_0}\bigg|_{p_0=\epsilon_{p_i}}\right)^{-1}, \qquad (29)$$

and assuming that z_{ip} may be considered to be independent of $\vec{p}$, we find

$$f_{p_i,p_j'}=z_i\frac{\delta\Sigma_i(p,p_0)}{\delta n_{p_j'}}\bigg|_{p_0=\epsilon_{p_i}}\ . \qquad (30)$$

Considering Σ_i as a functional of the Green's functions, we write

$$\frac{\delta\Sigma_i(p)}{\delta n_{p_j'}}=\frac{\delta\Sigma_i(\vec{p},p_0)}{\delta G_{\bar{l}}(\bar{q})}\frac{\delta G_{\bar{l}}(\bar{q})}{\delta n_{p_j}}. \qquad (31)$$

Noting that, when the quasiparticle picture is valid,[5]

$$\frac{\delta G_i(p)}{\delta n_{p_j'}}=2\pi i z_i s_i\delta_{ij}\delta(p_0-\epsilon_{p_j'})\delta_{\vec{p},\vec{p}'}+G_i^2\frac{\delta\Sigma_i(p)}{\delta n_{p_j'}}\ , \qquad (32)$$

where $s_i=+1(-1)$ for fermions (bosons), Eq. (31) becomes

$$\frac{\delta\Sigma_i(p)}{\delta n_{p_j'}}=iz_js_j\frac{\delta\Sigma_i(p)}{\delta G_j(p')}\bigg|_{p_0'=\epsilon_{p_j'}}+\frac{\delta\Sigma_i(p)}{\delta G_{\bar{l}}(\bar{q})}G_{\bar{l}}^2(\bar{q})\frac{\delta\Sigma_{\bar{l}}(\bar{q})}{\delta n_{p_j'}}\ . \qquad (33)$$

Comparing (33) with the i-j element of (22), we see that we have

$$\delta\Sigma_i(p)/\delta n_{p_j'}=iz_js_j\Gamma^0_{ij}(p,p')\big|_{p_0'=\epsilon_{p_j}}\ , \qquad (34)$$

since both sides of (34) satisfy the same integral equation. Further, comparing (34) with (30), we find

$$f_{p_i,p_j'}=iz_iz_js_j\Gamma^0_{ij}(p,p')\big|_{p_0=\epsilon_{p_i},p_0'=\epsilon_{p_j'}}\ . \qquad (35)$$

For our purposes, (35) may be evaluated at zero He^4 concentration. In this case, $f_{p\sigma,p'\sigma'}$ is the same as in pure He^3. Also, because

$$f_{p4,p'\sigma'}=\delta\epsilon_{p4}/\delta n_{p'\sigma'}=f_{p'\sigma',p4}\ , \qquad (36)$$

it is clear that at constant n_{p4},

$$\delta\epsilon_{p4}=\sum_{p'\sigma'}f_{p4,p'\sigma'}\delta n_{p'\sigma'}=\sum_{l\sigma}\frac{1}{2(2l+1)}f^l_{p4,p_F\sigma'}\delta n^l_{p_F\sigma'}\ ; \qquad (37)$$

here we have noted that $\delta n_{p'\sigma'}$ is restricted to the Fermi surface and have performed Legendre polynomial expansions for both $\delta n_{p'\sigma'}$ and $f_{p\sigma,p'\sigma'}$. Since $f_{p4,p'\sigma'}$ must be independent of σ', (37) reduces to

$$\delta\epsilon_{p4}=\sum_l \frac{1}{(2l+1)} f^l_{p4,p_F3}\,\delta n^l_{p_F3} \,. \tag{38}$$

For a uniform change in n_3 we have $\delta n^l_{p_F3}=\delta_{l,0}\,\delta n_3$ so that

$$\left(\frac{\partial\epsilon_{p4}}{\partial n_3}\right)_{n_4=0}=f^0_{p4,p_F3} \,. \tag{39}$$

For small $\vec{p}$, $\epsilon_{p4}\approx\mu_4$; hence

$$f^0_{p4,p_F3}=\left(\frac{\partial\mu_4}{\partial n_3}\right)_{n_4=0}=iz_3z_4\,{}^0\Gamma^0_{43}(p,p'_F)\big|_{p0=\epsilon_{p4},\,p'_0=\mu_3} \,, \tag{40}$$

where ${}^l\Gamma^0_{43}$ is used to denote the lth expansion coefficient of Γ^0_{43} in a Legendre polynomial expansion. Similarly, for small $\vec{p}$ and $\vec{p}\,'$,

$$f^0_{p4,p'4}=-iz_4^2\,{}^0\Gamma^0_{44}(p,p')\big|_{p0=\epsilon_{p4},\,p'_0=\epsilon_{p'4}}\approx\left(\frac{\partial\mu_4}{\partial n_4}\right)_{n_3} \,. \tag{41}$$

As one might expect, it is not difficult to relate $f_{p4,p'\sigma'}$ to the effective mass of a He[4] quasiparticle. One can employ either a formalism based on the use of the quasiparticle occupation numbers n_{p_i} or a vertex-operator technique similar to that of Ref. 5. We merely quote the result of the calculation without going into detail:

$$f^1_{p4,p_F3}=\frac{3m_3^*}{m_3}\left(1-\frac{m_4}{m_4^*}\right)\frac{1}{\nu(0)} \,, \tag{42}$$

for small $\vec{p}$.

This completes our discussion of Γ^0. We now turn our attention to Γ^∞, the forward-scattering amplitude.

IV. FORWARD-SCATTERING AMPLITUDES

In the dilute limit the He[3] scattering amplitudes may be taken equal to those in pure He[3]. The He[3]-He[4] scattering amplitude is, from (25), the solution of the equation (all four-momenta are put on the quasiparticle energy shell in this section)

$$\Gamma^\infty_{4\sigma'}(p,p')=\Gamma^0_{4\sigma'}(p,p')+\sum_{\bar{q}\bar{\sigma}}\Gamma^0_{4\bar{\sigma}}(p,\bar{q})R_{\bar{\sigma}\bar{\sigma}}(\bar{q})\Gamma^\infty_{\bar{\sigma}\sigma'}(\bar{q},p') \,. \tag{43}$$

Using (21), we may do the integral in (43), finding

$$\Gamma^\infty_{4\sigma'}(p,p')=\Gamma^0_{4\sigma'}(p,p')-\frac{i}{2\pi^2}\frac{p_F^2}{V_F}z_3^2 \times\sum_{\bar{\sigma}}\int\frac{d\Omega_{\bar{q}_F}}{4\pi}\,\Gamma^0_{4\bar{\sigma}}(p,\bar{q}_F)\Gamma^\infty_{\bar{\sigma}\sigma'}(\bar{q}_F,p'). \tag{44}$$

Since $\Gamma_{4\sigma'}\equiv\Gamma_{43}$ is independent of the spin of the He[3] atom, we may write (44) as

$$\Gamma^\infty_{43}(p,p')=\Gamma^0_{43}(p,p')-\frac{i}{\pi^2}\frac{p_F^2}{V_F}z_3^2 \times\int\frac{d\Omega_{\bar{q}_F}}{4\pi}\,\Gamma^0_{43}(p,\bar{q}_F)\Gamma^\infty_{3s}(\bar{q}_F,p'). \tag{45}$$

Here

$$\Gamma^\infty_{3s}\equiv\tfrac{1}{2}\textstyle\sum_{\sigma'}\Gamma^\infty_{\sigma\sigma'} \tag{46}$$

is the spin-symmetric part of the He[3]-He[3] scattering amplitude. Next, we use the Legendre polynomial expansions

$$\Gamma_{43}(p,\bar{q})=\sum_{l=0}^{\infty}{}^l\Gamma_{43}(p)P_l(\cos\theta_{p\bar{q}}) \,,$$
$$\Gamma_{3s}(\bar{q},p')=\sum_{l=0}^{\infty}{}^l\Gamma_{3s}P_l(\cos\theta_{\bar{q}p'}) \,, \tag{47}$$

and the addition theorem to put (45) in the form

$${}^l\Gamma^\infty_{43}(p)={}^l\Gamma^0_{43}\left(1-\nu(0)\,\frac{iz_3^2\,{}^l\Gamma^\infty_{3s}}{2l+1}\right) . \tag{48}$$

Defining the quasiparticle scattering amplitude by[5]

$$a^l_{43}(p)=iz_4z_3\,{}^l\Gamma^\infty_{43}(p),$$
$$a^l_s=iz_3^2\,{}^l\Gamma^\infty_{3s}\equiv A^s_l/\nu(0) \,, \tag{49}$$
$$a^l_{44}(p,p')=-iz_4^2\,{}^l\Gamma^\infty_{44}(p,p') \,,$$

and using the results of Sec. III, we obtain

$$a^l_{43}(p)=f^l_{43}(p)\left(1-\frac{A^s_l}{2l+1}\right) . \tag{50}$$

Using the relations[3]

$$1-A^s_0=\frac{1}{\nu(0)}\left(\frac{\partial n_3}{\partial\mu_3}\right)_{n_4} \,, \tag{51}$$

$$\left(\frac{\partial\mu_4}{\partial n_3}\right)_{n_4}=(1+\alpha)\left(\frac{\partial\mu_3}{\partial n_3}\right)_{n_4} \,, \tag{52}$$

together with (40) we arrive at the result (1):

$$a^0_{43}(p=0)=(1+\alpha)/\nu(0) \,. \tag{1}$$

Combining (50) and (42) with the known relation[3]

$$m_3/m_3^*=1-\tfrac{1}{3}A^s_1 \tag{53}$$

gives

$$a^1_{43}(p=0)=3\left(1-\frac{m_4}{m_4^*}\right)\frac{1}{\nu(0)} \,. \tag{54}$$

A completely parallel derivation yields

$$a^l_{44}(p,p')=f^l_{44}(p,p')-\frac{f^l_{43}(p)a^l_{43}(p')}{2l+1} \,. \tag{55}$$

Finally, for small p and p' we may use (40) and (1) to find (2):

$$a_{44}^0(0,0)=\left(\frac{\partial\mu_4}{\partial n_4}\right)_{\mu_3} . \tag{2}$$

We now turn to an explicit, albeit nonrigorous, evaluation of a_{44}^0.

V. EVALUATION OF a_{44}^0 AND DISCUSSION

With the aid of a very plausible assumption[6] concerning the nature of the phase-separation curve, we may evaluate a_{44}^0. The assumption is that the end point of the phase-separation line (at $x=0$, $T=0$) is a consolute point in the sense that $(\partial\mu_4/\partial n_4)_{P,T}=0$ there.[7] P is the pressure. In order to understand the import of this assumption, let us examine the possible alternatives: (i) The end point is neither a λ point, nor a consolute point. In this case there is a metastable region at $T=0$ and $x\neq 0$ in which we have a nonsuperfluid gas of Bose quasiparticles. Also, in this region $(\partial\mu_4/\partial n_4)_{P,T=0}$, which for small x is equal to $(\partial\mu_4/\partial n_4)_{\mu_3,T=0}$, must be greater than zero. Thus, from (2), a_{44}^0 is positive (the He^4-He^4 interaction is then repulsive). Since existing theory[8] indicates that a dilute Bose gas with repulsive interactions should be superfluid at $T=0$, we reject alternative (i). (ii) The endpoint is a λ point and not a consolute point. In this case there will be some region for small x and T where there exists a dilute solution of superfluid He^4 in liquid He^3. Such a phenomenon has never been observed, and we will assume that it does not occur. Finally, we call attention to the calculation of van Leeuwen and Cohen[9] on a dilute gas model of He^3-He^4 solutions which verifies our assumption in a special case.

Proceeding on the assumption that the end point of the phase-separation curve is indeed a consolute point, and noting again that $(\partial\mu_4/\partial n_4)_{P,T}$ is equal to $(\partial\mu_4/\partial n_4)_{\mu_3,T}$ in the limit as $x\to 0$, we immediately arrive at

$$a_{44}^0=\left(\frac{\partial\mu_4}{\partial n_4}\right)_{\mu_3,T=0}=0 . \tag{56}$$

This result is important since it implies that at low temperatures there are, to order n_4, no corrections to the free Bose gas result for μ_4.[10] It then follows that the phase separation, for small x and T, is closely related to the λ transition in a free Bose gas. In fact, if (56) is correct, since the fluctuations in number density in a free Bose gas become infinite at its λ line, the corresponding line for very dilute solutions of He^4 in liquid He^3 is spinodal line for the phase-separation transition (insofar as spinodal lines may be assigned physical meaning). The line is located within the two-phase region and ends at the point $x=0$, $T=0$.

Finally, we must note the difference between (56) and the corresponding nonzero result for dilute solutions of He^3 in superfluid He^4.[1,11] This difference is a consequence of the differing roles of Bose and Fermi statistics in the two cases. In our derivation of (56) the fact that He^4's obey Bose rather than Fermi statistics has been of central importance. However, the He^3-He^3 quasiparticle s-wave forward-scattering amplitude in dilute solutions of He^3 in superfluid He^4 is apparently (for opposite-spin He^3's) determined in large part by just the difference between the masses of He^3 and He^4 atoms.

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[10]A more detailed discussion of the points in this paragraph, which is included for completeness, is given in Ref. 1. Note that we assume m_4^* to be very weakly dependent on n_4 (at constant μ_3) for small n_4.

[11]J. Bardeen, G. Baym, and D. Pines, Phys. Rev. 156, 207 (1967). See also W. F. Saam, Ann. Phys. (N. Y.) 53, 239 (1969).