Proton maser gain

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It is shown that in the low beam space-charge limit the small-signal gain for a proton maser, calculated from a dispersion relation, is proportional to $(\omega_p/\omega_0)^2$, and agrees to this order with that derived from orbital velocity perturbations, to order $(qE_x/m\omega_0 U_0)^2$. The rf current density in the interaction space is derived; "self-bunching" and the underlying physics of the gain ("self-catching") are explored and a comparison is made between the collective formulation and that of single-particle orbits.

Motivation for using protons to drive a simple transit-time oscillator lies in its potential ability for both producing very intense microwave beams as compared with those using electrons, and for fusion research in terms of obtaining the necessary very large energy densities that are needed.

A self-excited microwave oscillator, or free-particle maser, can be obtained simply by passing a beam of charged particles through a resonant cavity under appropriate boundary conditions.[1] This simple system represents a maser, equivalent to a purely quantum-mechanical treatment for stimulated emission from individual particles,[2] with the momentum balance taken up by the radiation mode itself acting against the cavity walls, exactly analogous to the role played by the heavy nucleus in bound-state systems.

Electron masers, sometimes referred to also as monotrons, studied heretofore,[1-7] have shown relatively low efficiencies at high power levels, principally because of relativistic effects, which even at relatively high energies have little effect on performance of a proton maser. This result remains unchanged even in those cases where a guiding magnetic field is applied to the electron beam in an effort to render the motion one dimensional, as it is in the case of the proton maser.

A dispersion relation will be derived by computing the linearized current density and using this as the source term for a solution of Maxwell's equations. We will compute in detail only the gain for the fundamental cavity mode. The methods developed, however, are general and could be used to compute operation in any higher modes as well. A rectangular cavity model is chosen for simplicity, but without loss of generality, with the fundamental TE_{011} mode corresponding to polarization of the electric field across the particle-transit gap, and the eigenfrequency independent of that gap and determined by lateral dimensions (see Fig. 1).

An energy exchange relation and associated gain will also be derived by directly perturbing the orbital velocity of the particles, with the resulting growth for the mode fields identical to that derived above using the dispersion relation.

The particle velocity V_x and density N are perturbed about the steady-state low-charge-density values U_0 and N_0, where U_0 is the beam streaming velocity and N_0 the initial uniform beam-particle density. Inside the cavity, these quantities are functions of (x,t) only, and the cavity gap extends from $x=0$ to $x=\Delta x$. The perturbed quantities, v and n, are defined by $V_x = U_0 + v$ and $N = N_0 + n$. The resulting linearized equations of motion and continuity are

$$\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} = \frac{qE_x}{m} e^{-i\omega_0 t}, \tag{1}$$

$$\frac{\partial n}{\partial t} + U_0 \frac{\partial n}{\partial x} + N_0 \frac{\partial v}{\partial x} = 0 . \tag{2}$$

From Eq. (1), a solution for v which satisfies the boundary condition $v(0,t)=0$ is

$$v = \frac{iqE_x}{m\omega}(1-e^{ikx})e^{-i\omega_0 t}, \tag{3}$$

where k is real and of magnitude $k=\omega_0/U_0$, ω_0 is an arbitrary real frequency, which we will identify later for convenience as the real part of the loaded cavity-mode frequency. E_x represents the fundamental-mode electric field amplitude in the absence of any beam loading. With this value of v, the perturbed beam density is found from Eq. (2) as

$$n = \frac{-N_0 qE_x kx}{m\omega_0 U_0} e^{ikx - i\omega_0 t}, \tag{4}$$

which also satisfies the boundary condition $n(0,t)=0$.

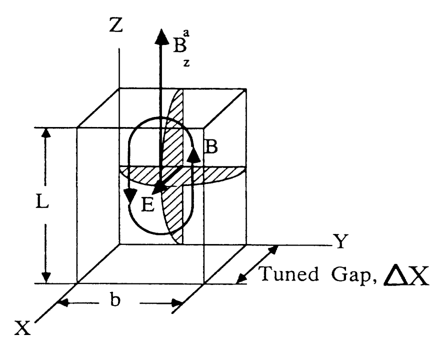

FIG. 1. TE_{011} mode geometry.

To first order in the perturbed quantities, the current density J_x is given by

$$J_x = qN_0U_0 + qN_0v + qU_0n$$
$$= J_0 + \frac{\omega_p^2}{4\pi\omega_0}[i(1-e^{ikx}) - kxe^{ikx}]E_x e^{-i\omega_0 t}, \quad (5)$$

where $\omega_p^2 = 4\pi N_0 q^2/m$ and J_0 is the initial unperturbed current density. The fundamental-mode electric field in this case is along x, is independent of x, and depends upon y and z. The field components for the unloaded mode are given by the following:

$$E_x = E_0 \sin(\pi y/b)\sin(\pi z/L),$$
$$B_y = \frac{\pi c}{\omega_0 L} E_0 \sin(\pi y/b)\cos(\pi z/L) i e^{-i\omega_0 t}, \quad (6)$$
$$B_z = -\frac{\pi c}{\omega_0 L} E_0 \cos(\pi y/b)\sin(\pi z/L) i e^{-i\omega_0 t},$$

where E_0 is the assumed initial excitation amplitude present in this mode, and the magnetic field interaction was neglected since the cyclotron radius for the protons is assumed much greater than the cavity gap Δx, and particle velocities are only along the x direction. Note that at energies of interest this simplification is not in general possible for an electron beam.[1]

The perturbed current density (5) is now used to derive the source term in order to solve Maxwell's equations for the cavity-beam interaction, which, for the case of a current density having only an x component and in terms of the electric field is given by

$$\frac{\partial}{\partial x}\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] - \nabla^2 E_x + \frac{\partial^2 E_x}{c^2 \partial t^2} = -\frac{4\pi}{c^2}\frac{\partial J_x}{\partial t}, \quad (7)$$

where, in general, the cavity fields are superpositions of the (l,m,n)th eigenmodes,

$$E_x = \sum_{l,m,n} E_{lmn}^x \cos(k_x x)\sin(k_y y)\sin(k_z z) e^{-i\omega_{lmn} t},$$
$$E_y = \sum_{l,m,n} E_{lmn}^y \sin(k_x x)\cos(k_y y) \times \sin(k_z z) e^{-i\omega_{lmn} t},$$
$$E_z = \sum_{l,m,n} E_{lmn}^z \sin(k_x x)\sin(k_y y) \times \cos(k_z z) e^{-i\omega_{lmn} t}$$

with

$$k_x = 1\pi/\Delta x, \quad k_y = m\pi/b, \quad k_z = n\pi/L,$$

and

$$B = -(i/\omega_{lmn})\nabla \times E,$$
$$E_{lmn}^x k_x + E_{lmn}^y k_y + E_{1mn}^z k_z = 0,$$
$$k_{lmn}^2 = k_x^2 + k_y^2 + k_z^2.$$

The usual procedure is to synthesize the current source term in Eq. (7) in a Fourier expansion in the orthogonal eigenmode series and determine the field-amplitude coefficients and eigenfrequencies produced by the source. The wave vectors are real, but ω_{lmn} is in general complex, and may show growth or damping in a particular mode. This is a particularly simple procedure in the present case, where the current density contains the y,z dependence of the TE_{011} mode. The only term from the Fourier expansion of the source which contributes to the TE_{011} cavity mode, which is independent of x, is therefore the constant or lowest-order term, essentially just the average of the source over the interval of orthogonality in x, and we arrive at the dispersion relation for this mode from Eq. (7),

$$\begin{aligned}(\omega_{011})^2 - (k_m c)^2 &= (\omega_p^2/\Delta x)\int_0^{\Delta x}(1 - e^{ikx} + ikxe^{ikx})dx \\ &= \omega_p^2\left[1 - \frac{2i}{k\Delta x} + e^{ik\Delta x} + \frac{2ie^{ik\Delta x}}{k\Delta x}\right] \\ &= \omega_p^2\left[\left[1 + \cos(k\Delta x) - \frac{2\sin(k\Delta x)}{k\Delta x}\right] + i\left[\frac{2\cos(k\Delta x)}{k\Delta x} + \sin(k\Delta x) - \frac{2}{k\Delta x}\right]\right] \\ &= \omega_p^2(\alpha + i\beta), \quad k_m^2 = \pi^2[(1/b)^2 + (1/L)^2]. \end{aligned} \quad (8)$$

It should be noted that the time factors occurring in Eq. (7) were cancelled on either side in arriving at Eq. (8). This comes about because we are comparing terms in the dispersion relation over many field oscillations, but still over less than one growth time, so the magnitude of the electric field is essentially the same on both sides. In addition, we identify the arbitrary ω_0 as the real part of ω_{011}, so the relative phase is stationary. The validity of this procedure, however, depends upon the smallness of the imaginary component in ω_{011} as compared to the real component, e.g., that there be only small percentage growth or damping per oscillation.

The roots of Eq. (8) can be extracted easily, and are given by

$$\omega_{011} = \pm(\gamma^2 + \delta^2)^{1/4}[\cos(\theta/2) + i\sin(\theta/2)], \quad (9)$$

where

$$\tan(\theta)=\frac{\omega_p^2\beta}{\omega_p^2\alpha+(k_m c)^2}=\frac{\beta}{\alpha+(k_m c/\omega_p)^2}\ ;$$

also, $\gamma=\omega_p^2\alpha+(k_m c)^2$ and $\delta=\omega_p^2\beta$.

For cases of interest, $(k_m c/\omega_p)^2 \gg 1$, so as to avoid critical density problems, so the phase angle θ is always small compared to unity. The imaginary root from Eq. (9) which determines field gain is then given by

$$W_E^I=(\gamma^2+\delta^2)^{1/4}\sin(\theta/2)\simeq(k_m c)(\omega_p/k_m c)^2\beta/2\ ,$$

or explicitly in terms of the cavity geometry by

$$W_E^I=\frac{\omega_p^2}{2k_m c}\left[\frac{2\cos(k\Delta x)}{k\Delta x}+\sin(k\Delta x)-\frac{2}{k\Delta x}\right]. \tag{10}$$

We can express this result in terms of the unperturbed transit angle through the cavity gap by noting that $k\Delta x=\omega_0\Delta x/U_0=\omega_0\tau$, where τ is the unperturbed transit time $\Delta x/U_0$. Note that growth or damping depends upon the sign of the oscillating function of transit angle above, growth if positive and damping if negative. Note also that this growth is for the cavity field strength, with the growth for cavity-stored energy twice as great and which can be written in terms of ϵ, with $\omega_0\simeq k_m c$ and $\epsilon=\omega_0\tau/2$, as

$$W_u^I=\frac{2\omega_p^2}{\omega_0}(\cot\epsilon-1/\epsilon)\sin^2\epsilon\ . \tag{11}$$

The cavity-stored energy growth for this same beam-cavity system can also be estimated from a perturbation in the particle velocity. Particles incident at $x=0$ with velocity U_0 and which exit the gap at $x=\Delta x$ with velocity $U_0+\delta v$ then have a change in energy given by

$$\Delta e=\tfrac{1}{2}m(U_0+\delta v)^2-\tfrac{1}{2}mU_0^2$$
$$=mU_0\delta v+\tfrac{1}{2}m(\delta v)^2\ .$$

The mode electric field is represented as $E_x\sin(\omega t+\phi)$, where ϕ is the arbitrary phase when the particle enters the cavity at $x=0$.

First calculate the perturbed transit time T in terms of the perturbed velocity and the perturbed velocity from the equation of motion as follows:

$$T=\int_0^{\Delta x}\frac{dx}{U_0+\delta v}$$
$$\simeq\int_0^{\Delta x}\frac{dx}{U_0}-\frac{1}{U_0^2}\int_0^{\Delta x}\delta v(x)dx\ , \tag{12}$$

$$\delta v(t)=\frac{qE_x}{m}\int_0^t\sin(\omega_0 t'+\phi)dt'$$
$$=-(qE_x/m\omega_0)[\cos(\omega_0 t+\phi)-\cos\phi]\ . \tag{13}$$

Then, using the zeroth-order orbit $t=x/U_0$ in (13) to define $\delta v(x)$ for use in (12) results in the following perturbed transit time to first order in the natural expansion parameter $\nu=qE_x/m\omega_0U_0$, the ratio of maximum particle velocity imparted by the mode field to the incident velocity. ν is thus arbitrarily small compared to unity.

$$T=\tau+(\nu/U_0)\int_0^{\Delta x}[\cos(\omega_0/U_0x+\phi)-\cos\phi]dx$$
$$=\tau+(\nu/\omega_0)[\sin(\omega_0\tau+\phi)-\sin\phi-\omega_0\tau\cos\phi]\ . \tag{14}$$

Next, we use the first-order transit time from (14) in (13) to compute the second-order perturbed velocity when the particle crosses $x=\Delta x$,

$$\delta v(T,\phi)=-(qE_x/m\omega_0)[\cos(\omega_0T+\phi)-\cos\phi]$$
$$=-\nu U_0\{\cos[\omega_0\tau+v\sin(\omega_0\tau+\phi)$$
$$-v\sin\phi-\nu\omega_0\tau\cos\phi+\phi]-\cos\phi\}\ . \tag{15}$$

We now compute the energy gain up to second order for a particle crossing the cavity gap, after averaging over the initial field phase parameter ϕ. The beam is composed of completely uncorrelated incident particle velocities, evenly distributed in phase and uniformly covering the cavity cross section, as in the previous dispersion calculation. Using the above value, Eq. (15), for δv we need to evaluate the following expression for the average energy gain

$$\overline{\Delta e}=\frac{mU_0}{2\pi}\int_0^{2\pi}\delta v\,d\phi+\frac{m}{4\pi}\int_0^{2\pi}(\delta v)^2d\phi\ . \tag{16}$$

The first term in (16), linear in δv, contributes the quantity

$$\frac{mU_0^2\nu^2}{2}[1-\omega_0\tau\sin(\omega_0\tau)-\cos(\omega_0\tau)]\ ,$$

while the second member produces the remaining second-order terms

$$\frac{mU_0^2\nu^2}{2}[1-\cos(\omega_0\tau)]\ .$$

By combining these terms, the phase average energy gain per incident particle through second order in ν can be written

$$\overline{\Delta e}=\frac{2q^2E_x^2}{m\omega_0^2}(\epsilon\cot\epsilon-1)\sin^2\epsilon\ . \tag{17}$$

Equation (17) is a result in agreement with an earlier quantum-mechanical calculation by Marcuse[2] for a transit-time oscillator. If we multiply this result by the incident uniform particle flux N_0U_0, we obtain the power gained by the cavity per unit of area. If we then integrate this over the cavity cross section, we obtain the total power delivered to the cavity. Denoting the cavity stored energy by W, we then have

$$\frac{dW}{dt}=\frac{\omega_p^2U_0}{2\pi\omega_0^2}(\epsilon\cot\epsilon-1)\sin^2\epsilon\int\int E_x^2\,dy\,dz\ .$$

We can also express the cavity stored energy at any instant in terms of the volume integral of the energy density,

$$W=\frac{1}{8\pi}\int\int\int E_x^2\,dx\,dy\,dz=\frac{\Delta x}{8\pi}\int\int E_x^2\,dy\,dz\ ,$$

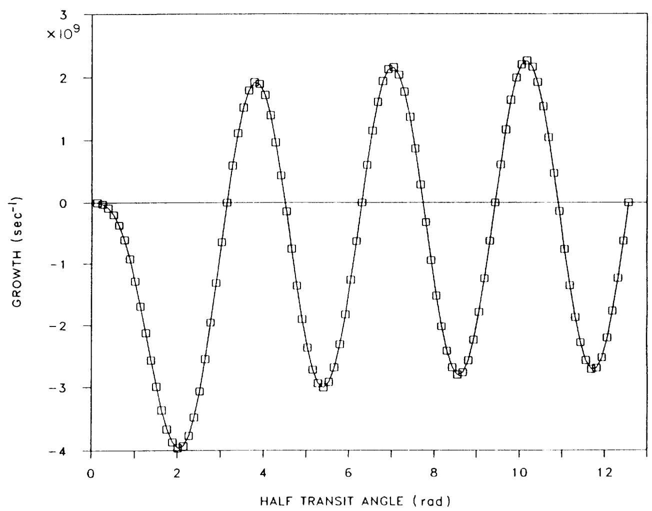

FIG. 2. Imaginary root [Eq. (11)] vs half-cavity transit angle $k\Delta x/2$ for $\omega_0=40$ GHz and $\omega_p/\omega_0=0.1$.

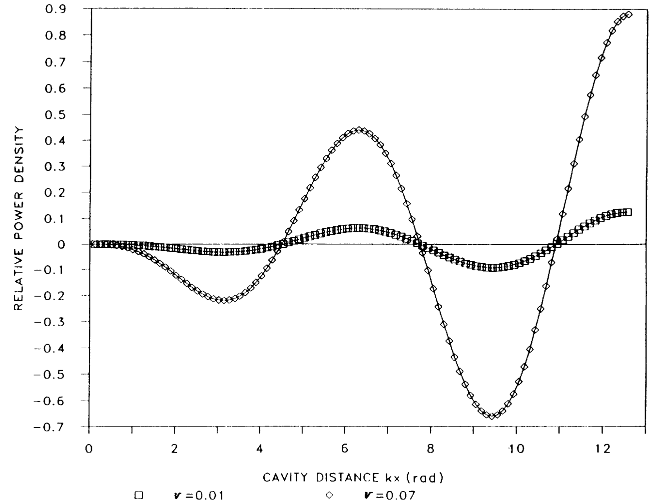

FIG. 3. Relative power transfer to cavity mode per unit volume $\frac{1}{2}\mathrm{Re}\mathbf{J}\cdot\mathbf{E}/J_0E_x$ vs dimensionless cavity-gap coordinate kx for $\omega_0=40$ GHz and $\omega_p/\omega_0=0.1$.

where, as indicated, the TE_{011}-mode field E_x depends upon y and z but not on x. From the above two relations, the equation describing the time dependence of cavity energy can be solved for the exponential growth, with the result

$$\begin{aligned}\frac{1}{w}\frac{dW}{dt} &= \frac{4\omega_p^2}{\omega_0^2\Delta x/U_0}(\epsilon\cot\epsilon-1)\sin^2\epsilon \\ &= \frac{2\omega_p^2}{\omega_0}(\cot\epsilon-1/\epsilon)\sin^2\epsilon \\ &= W_U^I\ .\end{aligned}$$

The above result is identical to the growth obtained earlier from the dispersion relation.

The exact agreement for cavity-energy growth between the individual particle model and the dispersion relation is somewhat remarkable. In the collective model, both the induced particle density and velocity must be taken into account explicitly. In the orbit model, a separate calculation of the density variations would have to be made in order to arrive at a comparable current density showing the same spatial growth. Nevertheless, the orbit model arrives at a comparable net-gain expression for energy transfer across the gap without requiring the calculation or use of an induced current density within the interaction volume.

The cavity growth Eq. (11) is shown graphically in Fig. 2 and demonstrates the influence of the transit-angle resonance function on gain, e.g., self-catching. Note that operation over a wide range of frequencies and geometries is indicated by the fact that gain or damping is determined only by this function, for low space-charge densities.

In order to understand in detail the above gain result, we can form the local rate of conversion into electromagnetic energy by the beam $-\mathbf{J}\cdot\mathbf{E}$ using Eq. (5) for the rf current density. This quantity, divided by the product of the magnitudes E_xJ_0 is, after time averaging over an rf period, given by

$$\frac{-1}{2E_xJ_0}\mathrm{Re}(\mathbf{J}\cdot\mathbf{E}^*)=\nu[kx\cos(kx)-\sin(kx)]/2\ , \quad (18)$$

where the coefficient in equation (5) was written in terms of ν.

Equation (18) is shown in Fig. 3. The self-bunching from the rf current is apparent. The local value of the rate of conversion of energy from beam to mode field from the cavity entrance at $x=0$ forward is an increasing function of x, and even though it is an oscillating function, each region has an increasing contribution which the preceding regions cannot completely cancel out. The integral of this function over x, with the appropriate constant, yields the gain function of Eq. (11).

It is also interesting to observe the dependence of growth, as opposed to the expansion parameter $\nu=qE_x/m\omega_0U_0$, upon the momentum of the incident beam particles. Note that ν is inversely proportional to the momentum, for a given mode field strength and frequency, while the growth depends only upon the parameter $(\omega_p/\omega_0)^2$, a quantity proportional to the *ratio* of beam-particle density to mass. This indicates that, as pointed out in an earlier paper,[1] a proton beam having the same gain as an electron beam would require a mode field much larger than that for the electrons before the expansion parameter would approach unity, and before the rf current within the cavity would approach 100% bunching. Also, from Eq. (4) the effective density perturbation parameter is the quantity νkx which, even for small ν, can reach values near unity within the cavity-transit angle $k\Delta x$, depending upon the value of $k=\omega_0/U_0$.

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