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Electrical conductivity of nondegenerate, fully ionized plasmas

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Within a virial expansion of the electrical conductivity of a fully ionized plasma, which takes into account many-particle effects, different limiting cases are considered. An appropriate interpolation formula is compared with experimental values.

The conductivity σ of a (hydrogenic) plasma governed by the Coulomb interaction is given by a universal function

$$\sigma^*(n,T;m_l,m_h)=\frac{m_l^{1/2}e^2}{(k_B T)^{3/2}(4\pi\epsilon_0)^2}\sigma \ , \tag{1}$$

where T is the temperature, $n=n_l=n_h$ is the density of electrons (l) or protons (h) in the charge-neutral plasma, and m_l and m_h denote the masses of the light and the heavy particles, respectively.

The behavior of the electrical conductivity has been investigated by means of quantum-statistical approaches; see Refs. 1 and 2 and references given therein. Interpolation formulas for $\sigma(n,T)$ have been constructed which are valid in a wide range of the temperature-density plane.

The derivation of interpolation formulas for the conductivity implies the correct treatment of limiting cases. We consider the case $m_l/m_h \ll 1$ and perform a low-density expansion,[3]

$$\sigma^{-1}(n,T)=A(T)\ln n+B(T)+C(T)n^{1/2}\ln n+\cdots \ . \tag{2}$$

For the virial coefficients $A(T)$, $B(T)$, and $C(T)$, different results can be found in the literature.[1,3,4] Whereas $A(T)$ is given by the Spitzer result,[4] the evaluation of $B(T)$ includes the treatment of dynamic screening and ladder-T matrix approaches,[3] and $C(T)$ is determined by self-energy effects and nonequilibrium two-particle correlations. In particular, the quasiclassical limit of these coefficients has been considered in Ref. 3. Introducing the dimensionless parameters[1]

$$\Gamma=\frac{e^2}{4\pi\epsilon_0 k_B T}(4\pi n/3)^{1/3}, \quad \Theta=\frac{2m_l k_B T}{\hbar^2}(3\pi^2 n)^{-2/3} \ , \tag{3}$$

the expression for $\sigma^*(\Gamma,\Theta\to\infty)$ has been found,[3]

$$\sigma^*=0.591(\ln\Gamma^{-3/2}+1.102+0.239\Gamma^{3/2}\ln\Gamma^{-3/2}+\cdots)^{-1} \ , \tag{4}$$

which can be applied for $\Gamma<1$ and $\Gamma^{-2}\Theta^{-1}\ll 1$. We will relax the latter restriction so that we obtain results for the entire region $\Gamma<1$, $\Theta>1$. Obviously, the low-density expansion (4) breaks down for the strongly coupled, nondegenerate region $\Gamma\gtrsim 1$, $\Theta\gtrsim 1$, where new effects (bound-state formation,[5] structure-function effects) may become of importance. As discussed below, the discrepancies to Ref. 1 are clearly pointed out within our quantum-statistical approach.

A generalized linear Boltzmann equation[3] is derived from linear response theory which allows for a direct comparison with respective expressions of standard kinetic theory. Applying a finite moment expansion, the conductivity is given by[3,5]

$$\sigma=\frac{\beta}{\Omega}\begin{vmatrix}0 & N_m\\ N_n & D_{nm}\end{vmatrix}\Big/ \left| D_{nm} \right| \ , \tag{5}$$

with the correlation functions $N_m=(\dot{R},P_m)$, $D_{nm}=\langle \dot{P}_n(\eta);\dot{P}_m\rangle$, where

$$(A,B)=\frac{1}{\beta}\int_0^\beta d\tau\,\mathrm{Tr}[\rho_0 A(-i\hbar\tau)B] \ ,$$

$$\langle A(\eta);B\rangle=\int_{-\infty}^0 dt\, e^{\eta t}(A(t),B) \ ,$$

$$\dot{R}=-\frac{e}{m_l}P_0, \quad P_m=\sum_k \hbar k_z(\beta\hbar^2k^2/2m_l)^m a_k^\dagger a_k \ ;$$

the time dependence of operators and ρ_0 are determined by the system Hamiltonian. Expression (5) is fully equivalent to the Kubo expression for the conductivity and allows for the evaluation of the transport coefficients for arbitrary degrees of degeneration. Many-particle effects in a strongly coupled plasma can be taken into account in a systematic way. For this, the correlation functions in thermal equilibrium are related to thermodynamic Green functions which are evaluated by means of a diagram technique.

Whereas the values $N_m=-eN\Gamma(m+\frac{5}{2})/\beta\Gamma(\frac{5}{2})$ are immediately obtained in the nondegenerate case ($N=n\Omega$, particle number), a perturbative treatment is needed to evaluate the correlation functions D_{nm}. In lowest order with respect to the density, the D_{nm} are given by a ladder sum of diagrams. This T-matrix expression relates the correlation functions to the two-particle scattering process. We introduce the transport cross sections $Q_{ei}^T(k)$ and $Q_{ee}^T(k)$ for electron-ion and electron-electron scattering, respectively,

$$D_{nm}=D_{nm}^{ei}+D_{nm}^{ee}\ ,$$
$$D_{nm}^{ei}=\frac{8}{3\sqrt{\pi}}\frac{\hbar}{\beta}n^2\Omega\left[\frac{\beta\hbar^2}{2m_l}\right]^{n+m+5/2}\times\int_0^\infty dk\,k^{2n+2m+5}e^{-\beta\hbar^2k^2/2m_l}Q_{ei}^T(k)\ ,$$
$$D_{nm}^{ee}=\frac{8\sqrt{2}}{3\sqrt{\pi}}\frac{\hbar}{\beta}n^2\Omega\left[\frac{\beta\hbar^2}{m_l}\right]^{7/2}\times\int_0^\infty dk\,k^7R_{nm}\left[\beta\frac{\hbar^2k^2}{m_l}\right]e^{-\beta\hbar^2k^2/m_l}Q_{ee}^T(k)\ ;$$
$$R_{\mathrm{om}}(x)=0,\quad R_{11}(x)=1\ ,$$
$$R_{12}(x)=\tfrac{7}{2}+x^2,\quad R_{22}(x)=\tfrac{77}{4}+7x^2+x^4\ . \tag{6}$$

The transport cross sections $Q^T(k)$ can be represented by integrals over the scattering angle χ [or the transfer momentum $q=2k\sin(\chi/2)$, respectively] in the following way:[6]

$$Q_{ei}^T(k)=2\pi\int_0^\pi\frac{d\sigma}{d\Omega}(1-\cos\chi)\sin\chi\,d\chi\ , \tag{7}$$

where $d\sigma/d\Omega$ denotes the differential cross section. For the Coulomb interaction, the behavior of the integrand in the region of small χ values leads to the well-known Coulombic divergences, and we have to introduce the concept of dynamic screening. However, at present a rigorous treatment of a dynamically screened T-matrix approach, where the ladder sum with respect to the dynamically screened Coulomb interaction is considered, is not in reach.

The regularization of the Coulomb divergences in the region of small scattering angles $[2k\sin(\chi/2)<(2ne^2\beta/\epsilon_0)^{1/2}]$ is already obtained from the Born approximation which is justified for the evaluation of the differential cross section in this region. The treatment of the linearized Lenard-Balescu equation yields,[3] e.g.,

$$D_{00}^{ei,\mathrm{LB}}=\pi\hbar\sum_{h,l,q}\int d\omega\frac{q^2}{3}\left|\frac{V(q)}{\epsilon(q,\omega)}\right|^2\delta(E_h+E_l-\omega)\times\delta(E_{h+q}+E_{l-q}-\omega)f_hf_l\ , \tag{8}$$

if the dielectric function is taken in random-phase approximation; f_h and f_l denote the distribution functions of ions and electrons, respectively. Numerical evaluation of the collision terms within the Lenard-Balescu approximation yields the low-density expansions

$$D_{nm}^{ei,\mathrm{LB}}=(n+m)!d(\ln\zeta+c_{nm}^{ei,\mathrm{LB}}+\cdots)\ ,$$
$$D_{11}^{ee,\mathrm{LB}}=\sqrt{2}d(\ln\zeta+c_{11}^{ee,\mathrm{LB}}+\cdots)\ , \tag{9}$$

with

$$\zeta=\frac{1}{4}\left[\frac{3}{2\pi}\right]^{1/6}(\Theta/\Gamma)^{1/2}\ ,$$
$$d=\tfrac{4}{3}(2\pi)^{1/2}n^2\Omega m_l^{1/2}\beta^{1/2}\frac{e^4}{(4\pi\epsilon_0)^2}\ ,$$

and $c_{00}^{ei,\mathrm{LB}}=1.323$, $c_{01}^{ei,\mathrm{LB}}=1.823$, $c_{11}^{ei,\mathrm{LB}}=2.073$, and $c_{11}^{ee,\mathrm{LB}}=1.590$.

For $2k\sin(\chi/2)\gg(2ne^2\beta/\epsilon_0)^{1/2}$, the differential Lenard-Balescu cross section converges to the differential cross section of the Coulomb interaction in Born approximation. The ladder-T matrices are immediately evaluated for effective static Debye potentials

$$V_{ei}(r)=-\frac{e^2}{4\pi\epsilon_0 r}\exp(-r/R_{nm}^{ei})\ ,$$
$$V_{ee}(r)=\frac{e^2}{4\pi\epsilon_0 r}\exp(-r/R_{nm}^{ee})\ . \tag{10}$$

Usually, the Debye radii R_{nm}^{ei} and R_{nm}^{ee} are taken by putting $\omega=0$ in the dielectric function of Eq. (8). We will determine the open parameters R_{nm}^{ei} and R_{nm}^{ee} by the condition that the Born approximations in evaluating the collision terms D_{nm} coincide with the Lenard-Balescu results. In evaluating the D_{nm}, the T-matrix results with respect to the effective potentials (10) are expected to be a reasonable approximation for the dynamically screened T-matrix results because at small scattering angles the correct Lenard-Balescu behavior is reproduced, whereas at large scattering angles the differential cross section of the Coulomb potential comes out.

The evaluation of the collision terms D_{nm} for the effective potentials (10) in Born approximation gives the results

$$D_{nm}^{ei}=(n+m)!d[\ln\zeta+\tfrac{1}{2}\ln(2ne^2\beta/\epsilon_0)+\ln R_{nm}^{ei}+c_{nm}^{ei,D}+\cdots]\ ,$$
$$D_{11}^{ee}=\sqrt{2}d[\ln\zeta+\tfrac{1}{2}\ln(2ne^2\beta/\epsilon_0)+\ln R_{11}^{ee}+c_{11}^{ee,D}+\cdots] \tag{11}$$

with $c_{00}^{ei,D}=1.170$, $c_{10}^{ei,D}=1.670$, $c_{11}^{ei,D}=1.920$, and $c_{11}^{ee,D}=0.8235$. Comparing with the Lenard-Balescu results (9), we find for the effective Debye radii

$$(R_{nm}^{ei})^{-2}=1.4727ne^2\beta/\epsilon_0\ ,$$
$$(R_{11}^{ee})^{-2}=0.4313ne^2\beta/\epsilon_0\ . \tag{12}$$

This result indicates that the potential strength of dynamic screening corresponds to a static screening by about half of the density of electrons, if e-e collisions are considered, whereas in e-i collisions the total electron density and about one half of the ion density contribute to the effective static screening.

Taking the Debye potentials (10) with the effective Debye radii (12) as an optimized static potential, the transport cross sections $Q^T(k)$ can be evaluated from a phase-shift calculation. Introducing the classical parameter $\epsilon=k^2a_BR/2$ and the quantum parameter $\kappa=\pi^{-2}k^2a_B^2$, we propose the following interpolation formula for the transport cross section:

$$\frac{k^4a_B^2}{4\pi}Q^T(k)=\alpha_0\ln\left\{\epsilon^2\frac{1+\alpha_4\kappa}{\alpha_1+\alpha_2\kappa+\alpha_3\kappa^2}\times\left[1+\alpha_5\ln\left(\frac{\alpha_6}{\epsilon}+1\right)\right]^2+1\right\}\ , \tag{13}$$

with $\alpha_0=\frac{1}{2}$ for e-i and $\alpha_0=\frac{1}{4}$ for e-e collisions. The correct classical asymptote[7] gives $\alpha_1=\gamma^2 e/16=0.5389$ for e-i and $\alpha_1=\gamma^2 e^2/64=0.3662$ for e-e collisions, γ denoting Euler's constant. The coefficients α_2, α_3, and α_4 of the Padé approximation (13) can be chosen to reproduce the Born approximation for $\kappa \gg 1$, and the first and second WKB approximation for small quantum corrections $\kappa \ll 1$. The parameters α_5 and α_6 are introduced to fit the behavior of $Q^T(k)$ in the quasiclassical limit for small values of ϵ. In contrast to the interpolation formulas given in Refs. 7 and 8, the general form (13) yields a strictly positive behavior of $Q^T(k)$. In detail, we have for e-i collisions $\alpha_2=1.9804$, $\alpha_3=3.4035$, $\alpha_4=2.0298$, $\alpha_5=0.6276$, and $\alpha_6=4.5$. For e-e collisions, we find $\alpha_2=1.4644$, $\alpha_3=102.69$, $\alpha_4=3.4166$, $\alpha_5=0.25$, and $\alpha_6=2.25$. We mention that the estimations for α_5 and α_6 are not of relevance for the evaluation of the virial coefficients $A(T)$, $B(T)$, and $C(T)$, but are introduced to give a good fit (relative error $<1\%$ for $\epsilon>1$, $<15\%$ in the entire ϵ region given in Ref. 7 for the quasiclassical limit of e-i collisions) to the numerical results.

In analogy to (13), we can also give interpolation formulas for the collision terms D_{nm} which reproduce the correct limiting cases in the quasiclassical and in the Born approximation. However, we discuss here immediately the results for the inverse conductivity.

The values of the virial coefficients of σ^{-1} (2) depend on the number L of moments taken in Eq. (5), and a quick convergence is expected with increasing numbers L. The Spitzer value

$$A(T)=-32.676T^{-3/2} \tag{14}$$

is obtained with an accuracy of 0.2% in a five-moment approximation. (We use the units m^{-3} for the density, K for temperature, and $(\Omega\,m)^{-1}$ for the conductivity.) For the evaluation of $B(T)$, we consider the three-moment approximation ($L=3$) which yields the Padé approximation

$$B(T)=98.030T^{-3/2}\left[\ln T+10.612+\frac{1}{3}\ln\frac{1-(2.139\times10^{-5})T}{1-(1.833\times10^{-5})T-(1.038\times10^{-10})T^2}\right]. \tag{15}$$

For $C(T)$, we refer to the value given in Ref. 3 for the single-moment approximation ($L=1$, Debye-Onsager relaxation effect) which is also given in Eq. (4).

The virial coefficient $B(T)$ interpolates in the region $\Gamma<1$, $\Theta>1$ between the quasiclassical limit $\Gamma^2\Theta\gg1$ and the Born limit $\Gamma^2\Theta\ll1$, where in both cases the correct behavior $\beta_1\ln T+\beta_2$ for the bracket in (15) is reproduced. Comparing with the results for the quasiclassical limit given in Ref. 3, our more rigorous recalculation of the constant β_2 gives a lower value (10.612 instead of 11.256).

An interpolation formula for the conductivity has already been given in Ref. 1 which correctly reproduces the Born limit of the Coulomb logarithm. Our fit furthermore reproduces the quasiclassical limit $[(\sigma^*)^{-1}\sim\ln(T^3/n)]$. As mentioned in Ref. 1, a single-moment approximation is not able to give the correct prefactor $A(T)$ in the low-density limit of the conductivity. Adopting the prefactor from the Spitzer theory[4] [cf. Eq. (14)] as proposed in Ref. 1, already the next-order term β_2 of the virial expansion is not correctly obtained. Our fit, however, is based also on the correct value for β_2.

To give an estimation for the Coulombic part of the

TABLE I. Experimental values $\sigma_{\rm expt}$ of the electrical conductivity for different plasmas by Ivanov *et al.* (Ref. 9) and the theoretical predictions $\sigma_{\rm theor}$ based on Eq. (16). For comparison, the values $\sigma_{\rm IT}$ according to Ref. 1 are also given.

Gas	T (10^3 K)	n_e (10^{25} m^{-3})	Γ	Θ	$\sigma_{\rm expt}$ ($10^2\ \Omega^{-1}m^{-1}$)	$\sigma_{\rm theor}$ ($10^2\ \Omega^{-1}m^{-1}$)	$\sigma_{\rm IT}$ ($10^2\ \Omega^{-1}m^{-1}$)
Ar	22.2	2.8	0.368	56.9	190	238	200
	20.3	5.5	0.505	33.2	155	248	203
	19.3	8.1	0.604	24.4	170	257	209
	19.0	14	0.736	16.7	255	288	234
	17.8	17	0.838	13.7	245	288	232
Xe	30.1	25	0.564	17.9	450	474	442
	27.5	59	0.822	9.24	680	542	506
	27.0	79	0.922	7.47	740	580	546
	26.1	140	1.150	4.93	690	679	657
	25.1	160	1.260	4.34	780	699	660
	24.6	200	1.380	3.66	1040	753	728
	22.7	200	1.500	3.38	930	735	694
Ne	19.8	1.1	0.303	94.6	130	182	148
	19.6	1.9	0.367	65.0	165	197	160
Air	11.0	0.13	0.267	218	60	71	53.1

conductivity also for $\Gamma \sim 1$, we propose the following interpolation formula which is suggested by the expression (13) for the transport cross section:

$$\sigma = \gamma_0 \frac{T^{3/2}}{m\Omega K^{3/2}} \left[\ln \left[\Gamma^{-3} \frac{1-\gamma_4/\Gamma^2\Theta}{1-\gamma_2/\Gamma^2\Theta - \gamma_3/\Gamma^4\Theta^2} \times [\gamma_1 + \gamma_5 \ln(\gamma_6 \Gamma^{3/2}+1)]^2 + 1 \right] \right]^{-1} . \tag{16}$$

As already discussed for $B(T)$, the constants $\gamma_0, \ldots, \gamma_4$ are fixed by the Born and the second WKB limit. The parameters γ_5 and γ_6 are fitted to the numerical evaluation of the conductivity using the expression (13) for the transport cross section. We find $\gamma_0 = 3.060 \times 10^{-2}$, $\gamma_1 = 1.1586$, $\gamma_2 = 3.1355$, $\gamma_3 = 3.0369$, $\gamma_4 = 3.6588$, $\gamma_5 = 0.9$, and $\gamma_6 = 5.1$.

Comparison between expression (16) and experimental values for the conductivity can be performed with data for extremely nonideal plasmas such as given in Ref. 9. The results are shown in Table I. Taking into account an experimental error of about 30%, theory and experiment are not in contradiction. However, the experimental values are systematically smaller for the lower-density plasmas ($\Gamma \sim 0.2, \ldots, 0.7$), but for the higher-density plasmas, they are higher than the theoretical ones. This situation is slightly improved (about 5% correction) if the Debye-Onsager relaxation effect according to Eq. (4) is taken into account. For comparison, we present also the results of Ichimaru and Tanaka[1] in Table I which include, especially, the effects of degeneration and structure factor. They found also an overall agreement with the experimental results within the error bars of about 30%. More precise experiments on the electrical conductivity are needed in order to verify the region of validity of the different fit formulas.

The present approach can be immediately generalized by including further effects as bound-state formation and degeneration.[5] In this way, the theory can be extended to describe, e.g., partially ionized plasmas and is able to give an improved approach to the strongly coupled case.

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