Quantum nondemolition measurement of photon number in a lossy optical Kerr medium

N. Imoto and S. Saito
NTT Basic Research Laboratories, Midori-cho 3-9-11, Musashino-shi, 180 Tokyo, Japan
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A photon-number quantum nondemolition (QND) measurement theory that takes dissipative losses into account is presented. A QND measurement criterion is developed to determine the feasibility of QND measurements in a lossy optical medium. Using an analysis based on loss-error characteristics, we examine a lossy Kerr medium. We obtain restrictions on the losses and requirements for the nonlinearity and signal and probe powers necessary to observe the QND effect. We calculate the expected results for an experimental system meeting such requirements. We conclude that QND measurements are possible in existing media.

I. INTRODUCTION

After quantum nondemolition (QND) measurements were first discussed by Braginsky,[1] several QND measurement schemes in optics were proposed.[2-7] One such proposed suggested a method to make a QND measurement of the number of photons in an optical signal using an optical Kerr medium in a nonlinear Mach-Zehnder interferometer.[4] The assumption was made that the Kerr medium was totally transparent to optical signals. In practice, however, the optical Kerr medium has a finite transparency which changes the photon number in an optical signal. Under what conditions is a QND measurement practical in a medium with loss? To answer this question, we present a QND measurement theory that takes into account dissipative losses for a medium with finite transparency.

We first discuss a general criterion for lossy QND measurement. This criterion, which is independent of any particular scheme used, can be expressed as a requirement on a loss-error function. Next, we show that for an optical Kerr medium, we can take losses into account, and obtain a QND criterion using the loss-error function. This gives us a criterion for determining whether or not we can obtain QND measurements in a Kerr medium with known losses. To relate this criterion to experiment, we calculate the correlations between the photon number measured in a Kerr-QND measurement system and the photon number transmitted through the QND measurement system. We conclude that it is possible to make QND measurements using existing finite transparency media.

II. GENERAL CRITERION FOR LOSSY QND MEASUREMENT

The conventional method for measurement of the number of photons in a signal beam requires random deletion of photons from the beam. A large deletion rate is necessary for an accurate measurement, but this causes increased randomness in the number of photons. A low deletion rate is required to preserve photons, but this produces a low level of measurement accuracy. Thus, an accurate measurement of photon number would seem to require change in the photon number. This change in the photon number can be described by loss-error characteristics.

The loss-error characteristics for several different measurement schemes are shown in Fig. 1. Here, the figure shows the measurement error Δ plotted against the insertion loss η. Δ is defined as

$$\Delta \equiv \frac{\langle (\hat{n}_{\text{meas}} - \langle \hat{n}_{\text{in}} \rangle)^2 \rangle}{\langle \hat{n}_{\text{in}} \rangle} , \tag{1}$$

where $\hat{n}_{\text{in}}$ is the photon-number operator of the input signal beam, $\hat{n}_{\text{meas}}$ is the photon-number operator that is actually measured, and $\langle \hat{n}_{\text{in}} \rangle$ is the average signal photon number. We take Δ as our loss-error function, because usually Δ depends upon the losses in the system. The loss-error characteristics of any photon-number measurement scheme can be plotted in Fig. 1. For example, point A denotes ideal photon counting, where all the photons are absorbed. Point B denotes an ideal QND measurement, where there is neither loss nor error. However, there is no detector that is totally transparent, so point B cannot be achieved. Practical systems with finite transparency and finite accuracy give loss-error characteristics that fall elsewhere in Fig. 1. For these systems, we can establish a criterion for the possibility of a QND measurement.

The criterion depends on the existence of quantum correlations in the system. We base the criterion for QND measurement of photon number on *whether or not the measurement error and the transmitted photon-number fluctuations are smaller than those required by the quantum-mechanical fluctuation-dissipation theorem.* Because the fluctuation-dissipation theorem results from a linear-response theory, reduced photon-number fluctuations necessarily result from a nonlinear system. The role of the nonlinear effect is to produce quantum-mechanical correlations between the signal and the probe.

The beamsplitter is a linear-response device that is well understood and can be used to illustrate the basis for our criterion. A lossy QND measurement is one in which the measurement error is smaller than that for a photon-number measurement using a beamsplitter with a unity quantum efficiency photon counter. We illustrate the measurement scheme with a beamsplitter in Fig. 2. Here, $\hat{a}_{\text{in}}$ is the photon annihilation operator for the input field,

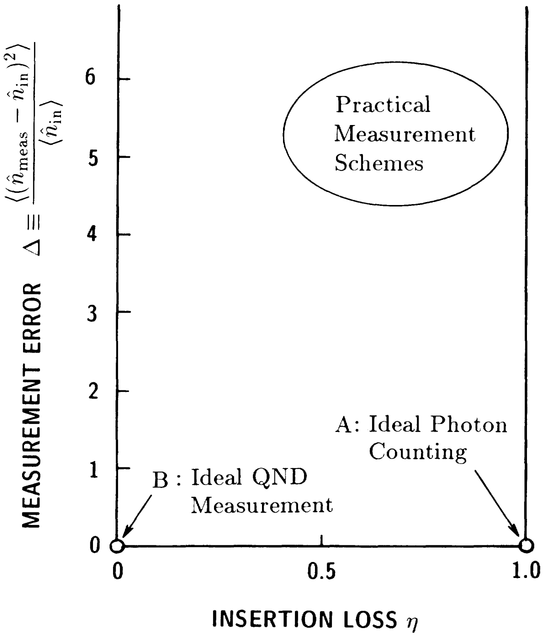

FIG. 1. Loss-error characteristics for photon-number measurement schemes. Any photon-number measurement scheme can be plotted within this figure. Point A denotes an ideal photon counting, and point B denotes an ideal QND measurement having no loss or error.

$\hat{a}_{out}$ is the annihilation operator for the transmitted field, $\hat{b}$ is the annihilation operator for the transmitter field for reflected field, and $\hat{c}$ is the annihilation operator for the transmitted field for the noise field coupled in through the unused portion of the beamsplitter. The input photon number $\hat{n}_{in} \equiv \hat{a}_{in}^{\dagger}\hat{a}_{in}$ can be estimated from the reflected photon number $\hat{n}_{refl} \equiv \hat{b}^{\dagger}\hat{b}$ for the reflected field. The reflected photon number is reduced (on average) by the reflectivity factor η of the beamsplitter, so we define measured photon number $\hat{n}_{meas}$ as

$$\hat{n}_{meas} \equiv \hat{b}^{\dagger}\hat{b}/\eta \ . \tag{2}$$

This allows us to directly compare the measured photon number with the input photon number. The expectation value for $\hat{n}_{meas}$ is thus equal to that for $\hat{n}_{in}$, $\langle(\hat{n}_{meas} - \hat{n}_{in})^2\rangle$ is the photon-number uncertainty added as a result of the measurement.

The input and output operators obey the operator equations

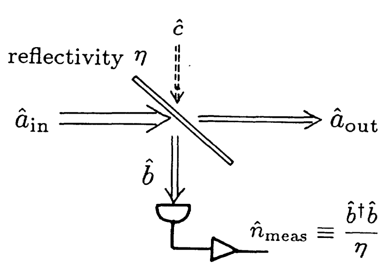

FIG. 2. A dissipative photon-number measurement with a beamsplitter and an ideal photon counting. Vacuum fluctuation $\hat{c}$ is linearly coupled to the incident field $\hat{a}_{in}$.

$$\hat{a}_{out} = \sqrt{1-\eta}\,\hat{a}_{in} + \sqrt{\eta}\,\hat{c} \tag{3}$$

and

$$\hat{b} = -\sqrt{\eta}\,\hat{a}_{in} + \sqrt{1-\eta}\,\hat{c} \ . \tag{4}$$

When the noise field $\hat{c}$ is in a thermal state (blackbody radiation), then Eqs. (3) and (4) give results in agreement with the usual fluctuation-dissipation theorem for a system with a heat bath. Quantum theory imposes a noise field $\hat{c}$ even when the temperature of the heat bath is at zero degrees, i.e., for vacuum fluctuations. Using Eqs. (3) and (4), the loss-error function Δ for the beamsplitter can be shown to be

$$\Delta = \frac{1-\eta}{\eta} \ , \tag{5}$$

where we assume that $\hat{c}$ is in the vacuum state. This gives us the desired QND measurement criterion, using Δ for a measurement scheme to be examined, as

$$\Delta < \frac{1-\eta}{\eta} \ . \tag{6}$$

This shows the tradeoff between the measurement error and the insertion loss. The loss-error function (= measurement error) goes to infinity when the insertion loss goes to zero, and the loss-error function goes to zero when the insertion loss goes to unity. This is shown in graph form in Fig. 3.

We can also obtain this result by two other methods. One is to consider photon-number variance. We define a normalized photon-number variance as $\langle(\Delta\hat{n})^2\rangle/\langle\hat{n}\rangle^2$, and define the increase in photon-number variance produced by a measurement as ΔV. For the reflected beam, the increase in variance ΔV_{refl} is given by

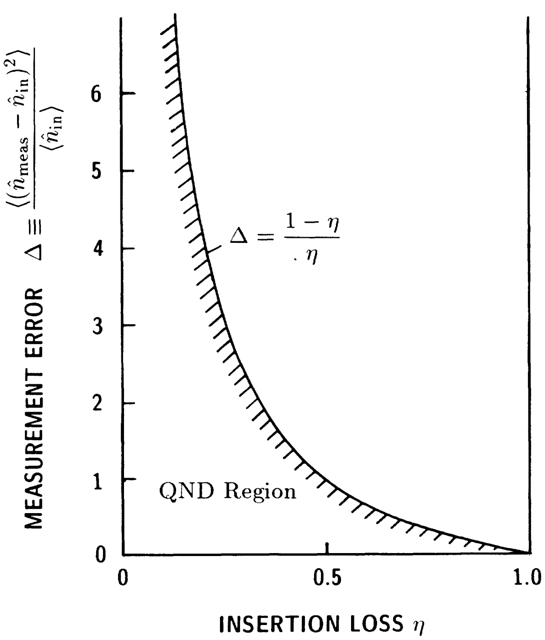

FIG. 3. General criterion for dissipative QND measurement of the photon number.

$$\Delta V_{\text{refl}}=\frac{\langle(\Delta\hat{n}_{\text{refl}})^2\rangle}{\langle\hat{n}_{\text{refl}}\rangle^2}-\frac{\langle(\Delta\hat{n}_{\text{in}})^2\rangle}{\langle\hat{n}_{\text{in}}\rangle^2}\ , \tag{7}$$

and for the transmitted beam, the increase in variance ΔV_{out} is given by

$$\Delta V_{\text{out}}=\frac{\langle(\Delta\hat{n}_{\text{out}})^2\rangle}{\langle\hat{n}_{\text{out}}\rangle^2}-\frac{\langle(\Delta\hat{n}_{\text{in}})^2\rangle}{\langle\hat{n}_{\text{in}}\rangle^2}\ . \tag{8}$$

Because of the normalization, the photon-number variance for the incoming beam, which we define as ΔV_{meas} is equal to ΔV_{refl}, i.e.,

$$\Delta V_{\text{meas}}=\Delta V_{\text{refl}}\ . \tag{9}$$

ΔV_{meas} and ΔV_{out} are uncertainties induced by the measurement because the photon number uncertainty $\langle(\Delta\hat{n}_{\text{in}})^2\rangle/\langle\hat{n}_{\text{in}}\rangle$, which depends on the input-photon state, is eliminated.

For a beamsplitter, the increased photon-number variances become

$$\Delta V_{\text{meas}}=\frac{1-\eta}{\eta}\frac{1}{\langle\hat{n}_{\text{in}}\rangle} \tag{10}$$

and

$$\Delta V_{\text{out}}=\frac{\eta}{1-\eta}\frac{1}{\langle\hat{n}_{\text{in}}\rangle}\ . \tag{11}$$

The product of the variances for the beamsplitter is, therefore,

$$\Delta V_{\text{meas}}\Delta V_{\text{out}}=\frac{1}{\langle\hat{n}_{\text{in}}\rangle^2}\ . \tag{12}$$

This product is in agreement with the results of the fluctuation-dissipation theorem.

For the general case of a measurement on an arbitrary system with linear loss η, the variance product for a photon-number measurements can be obtained as follows. The photon-number operator $\hat{n}_{\text{meas}}$, which we want to determine from an actual measurement, can be expressed as

$$\hat{n}_{\text{meas}}=\hat{n}_{\text{in}}+\Delta\hat{n}\ , \tag{13}$$

where $\Delta\hat{n}$ is a noise operator resulting from measurement and uncorrelated with $\hat{n}_{\text{in}}$. Here, $\hat{n}_{\text{meas}}$ has the same expectation value as $\hat{n}_{\text{in}}$, so that the expectation value of $\Delta\hat{n}$ is equal to zero. We define $\hat{n}_{\text{meas}}$ so that $\Delta\hat{n}$ is not correlated with $\hat{n}_{\text{in}}$. If a measurement somehow introduces a correlation between $\hat{n}_{\text{in}}$ and the noise of measurement, we include the correlated "noise" in $\hat{n}_{\text{meas}}$. The variance for $\hat{n}_{\text{meas}}$ then is the summation of the variances for $\hat{n}_{\text{in}}$ and $\Delta\hat{n}$. ΔV_{meas} is therefore expressed as

$$\Delta V_{\text{meas}}\equiv\frac{\langle(\Delta\hat{n}_{\text{meas}})^2\rangle-\langle(\Delta\hat{n}_{\text{in}})^2\rangle}{\langle\hat{n}_{\text{in}}\rangle^2}$$

$$=\frac{\langle(\Delta\hat{n})\rangle^2}{\langle\hat{n}_{\text{in}}\rangle^2}=\frac{\Delta}{\langle\hat{n}_{\text{in}}\rangle}\ , \tag{14}$$

where Δ is defined by Eq. (1). Because ΔV_{out} has the same form as in Eq. (11), the variance product is

$$\Delta V_{\text{meas}}\Delta V_{\text{out}}=\frac{1}{\langle\hat{n}_{\text{in}}\rangle^2}\frac{\eta}{1-\eta}\Delta\ , \tag{15}$$

and the result is general. If we require that this general result for a QND measurement should be smaller than the beamsplitter result given in Eq. (12), we obtain Eq. (6) as the QND measurement criterion.

By consideration of signal-to-noise (S/N) ratios, we can obtain Eq. (6) in yet another way. If we define S/N ratio as $\langle\hat{n}\rangle^2/\langle(\Delta\hat{n})^2\rangle$, the S/N ratios for the measured and transmitted photon numbers for a beamsplitter with input light in a coherent state can be shown to satisfy

$$(S/N)_{\text{meas}}+(S/N)_{\text{out}}=(S/N)_{\text{in}}\ . \tag{16}$$

Equation (6) can then be seen to be the case where the photon-number measurement scheme has a larger S/N ratio than that required by Eq. (16).

III. THE QND MEASUREMENT CRITERION FOR A LOSSY KERR MEDIUM

In this section we calculate the loss-error function Δ for a QND measurement with a lossy optical Kerr medium. We thereby obtain the QND measurement criterion in terms of the Kerr nonlinearity, the losses in the medium, and the input laser power. We use a measurement scheme previously proposed for loss-free conditions.[4] This scheme is diagrammed in Fig. 4 and consists of a Mach-Zehnder interferometer that senses the refractive index changes induced in a Kerr medium by a signal beam. In operation, signal photons (in the beam labeled by operators $\hat{a}_{\text{in}}$ and $\hat{a}_{\text{out}}$) pass through a loss-free dichroic mirror into the Kerr medium and then out through a second dichroic mirror. The probe beam is transmitted down the two legs of the interferrometer and recombined at the output beamsplitter. In one leg the probe acquires an additional optical phase proportional to the signal photon number. The signal number is determined by measuring the probe phase with a balanced mixer detector at the output beamsplitter. The result is QND measurement for the loss-free case, as signal photons pass unabsorbed through the interferometer.

For QND measurements using a nonlinear Mach-Zehnder interferometer with a lossy Kerr medium, we consider three different cases as shown in Fig. 5: (a) inser-

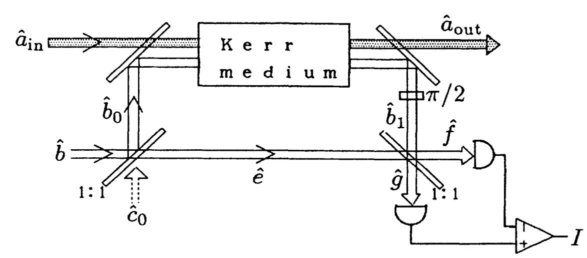

FIG. 4. Photon number QND measurement scheme with the Kerr effect under loss-free conditions (Ref. 4). The signal photon-number modulates the probe phase via the loss-free Kerr effect. The probe phase is measured by means of the Mach-Zehnder interferometer with a balanced mixer detector.

tion loss before the Kerr medium, (b) insertion loss after the medium, and (c) distributed insertion loss in the Kerr medium. Case (a) describes input coupling loss for a signal coupled into a single-mode optical fiber. This is the most severe case because a measurement is done after the loss of photons has occurred. Case (b) represents the least severe case because the measurement is done before the loss of photons occurs. Case (c) is the most important case, because its results are determined by the intrinsic characteristics of the medium. In case (c) the photons are measured while they propagated through the lossy medium. For simplicity we assume that losses for the probe beam are the same in the two legs of the interferometer.

The loss error functions for cases (a) and (b) are obtained as follows. Using field operators as defined in Fig. 5(a), we have

$$\hat{a}'=\sqrt{1-\eta}\,\hat{a}+\sqrt{\eta}\,\hat{c}_1\ , \tag{17}$$

$$\hat{e}_1=(\tfrac{1}{2})^{1/2}\hat{b}+\hat{c}_0\ , \tag{18}$$

$$\hat{b}_0=(\tfrac{1}{2})^{1/2}\hat{b}-\hat{c}_0\ , \tag{19}$$

$$\hat{b}_1=\sqrt{1-\eta_p}\,\hat{b}_0+\sqrt{\eta_p}\,\hat{c}_2\ , \tag{20}$$

$$\hat{b}_2=\hat{b}_1 e^{i\hat{\theta}}\ , \tag{21}$$

$$\hat{\theta}=\sqrt{F}\,\hat{a}'^{\dagger}a'\quad\text{(the Kerr effect)}\ , \tag{22}$$

$$\hat{b}_3=i\hat{b}_2\quad(\pi/2\ \text{phase shift})\ , \tag{23}$$

$$\hat{f}=(\tfrac{1}{2})^{1/2}\hat{b}_3+\hat{e}_2\ , \tag{24}$$

and

$$\hat{g}=(\tfrac{1}{2})^{1/2}\hat{b}_3-\hat{e}_2\ . \tag{25}$$

Here η is the insertion loss for the signal beam, η_p is the insertion loss for the probe beam, and $\sqrt{F}$ is proportional to the magnitude of the Kerr effect, and is defined as[4]

$$\sqrt{F}\equiv\frac{2\pi^2\hbar\chi^{(3)}L}{\epsilon_0 n^2\lambda_s\lambda_p A_{\text{eff}}\tau}\ , \tag{26}$$

where $\chi^{(3)}$ is the third-order nonlinear susceptibility, L the medium length, A_{eff} the beam cross-sectional area, τ the time interval for photon counting, and $\langle\hat{n}_p\rangle$ the average probe photon number. We take the photon number of the input signal beam to be $\hat{n}_{\text{meas}}$, which we define as

$$\hat{n}_{\text{meas}}\equiv\frac{1}{(1-\eta)(1-\eta_p)}\frac{\hat{g}^{\dagger}\hat{g}-\hat{f}^{\dagger}\hat{f}}{\sqrt{F}\,\langle\hat{n}_p\rangle}\ . \tag{27}$$

This reduces to

$$\hat{n}_{\text{meas}}\cong\frac{1}{\sqrt{F}\,\langle\hat{n}_p\rangle(1-\eta)}\times\Bigg[(\hat{b}^{\dagger}\hat{b}-\hat{c}_0^{\dagger}\hat{c}_0)\sin\hat{\theta}-i(\hat{b}^{\dagger}\hat{c}_0-\hat{c}_0^{\dagger}\hat{b})\cos\hat{\theta}-i\left[\frac{\eta_p}{2(1-\eta_p)}\right]^{1/2}(\hat{b}^{\dagger}\hat{c}_2-\hat{c}_2^{\dagger}\hat{b}-\hat{b}^{\dagger}\hat{c}_3+\hat{c}_3^{\dagger}\hat{b})\Bigg]\ , \tag{28}$$

when we make the approximations of large photon number

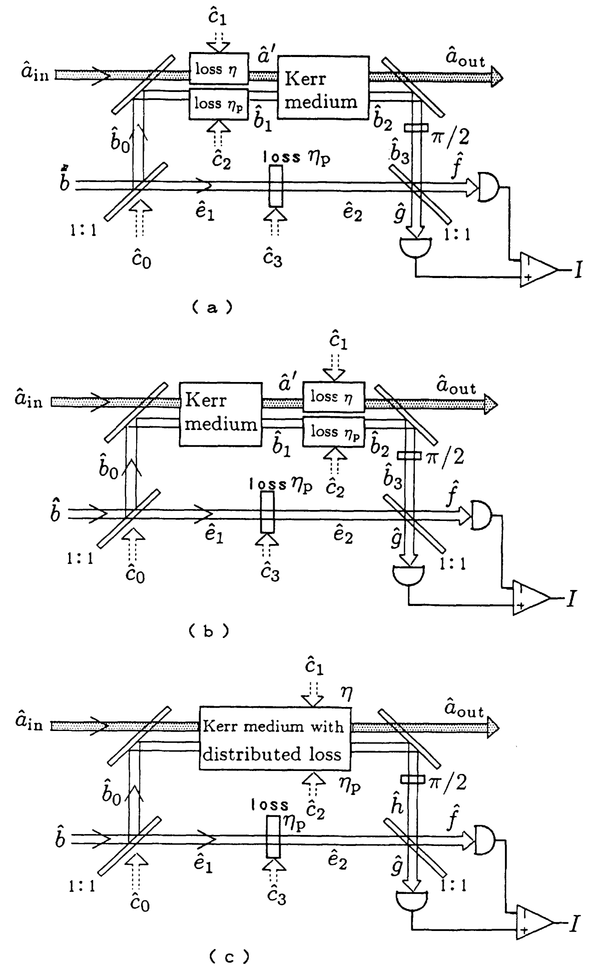

FIG. 5. Three cases used in the analysis of lossy Kerr-QND measurement scheme. (a) Loss before the Kerr medium, (b) loss after the medium, and (c) distributed loss.

$$\langle\hat{n}_p\rangle,\langle\hat{n}_{\text{in}}\rangle\gg1\ , \tag{29}$$

moderate Kerr effect,

$$\langle\hat{n}_p\rangle^2F,\ \langle\hat{n}_{\text{in}}\rangle^2F\ \text{and}\ \langle\hat{n}_p\rangle\langle\hat{n}_{\text{in}}\rangle F\ll1\ , \tag{30}$$

and we assume we are in the linear-response regime of the interferometer, so that

$$\sin\hat{\theta}\cong\hat{\theta},\quad\cos\hat{\theta}\cong1-\tfrac{1}{2}\hat{\theta}^2\ . \tag{31}$$

Using Eq. (28), the expectation value of $\hat{\eta}_{\text{meas}}$ can be shown to be equal to that of the input photon number. The loss-error function Δ for case (a) is then

$$\Delta=\frac{1}{F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle}\frac{1}{(1-\eta)^2(1-\eta_p)}+\frac{\eta}{(1-\eta)}\ . \tag{32}$$

The loss-error function for case (b) can similarly be shown to be

$$\Delta=\frac{1}{F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle}\frac{1}{(1-\eta_p)}\ . \tag{33}$$

One can see that the values of the loss-error functions

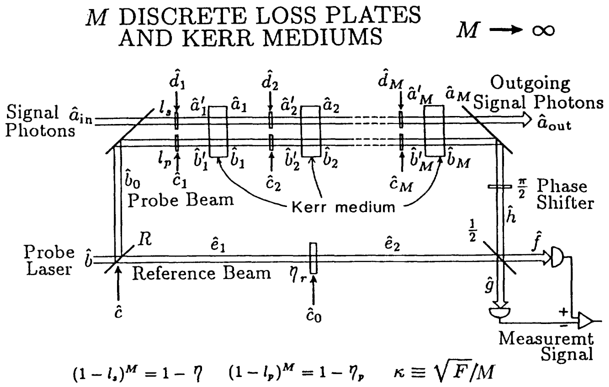

FIG. 6. Model of the QND measurement scheme with a lossy Kerr medium. Alternating loss plates and Kerr media represent a lossy Kerr medium. The number of the plates M is set to be ∞ under the constraint of a constant total loss and the Kerr effect.

given in Eqs. (32) and (33) are larger than those for the loss-free ($\eta=\eta_p=0$) case, where $\Delta=1/F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle$. The increase is larger for case (b) than for case (a), as we have anticipated.

The loss-error function Δ in case (c) can be obtained as a limit to an infinite number of alternating loss plates and Kerr media as shown in Fig. 6. Let M be both the number of loss plates and the number of Kerr media, l_s (l_p) the loss of each plate for the signal (probe) beam, respectively, and η (η_p) the total insertion loss for the signal (probe) beam. Then, we have

$$(1-l_s)^M=1-\eta \tag{34}$$

and

$$(1-l_p)^M=1-\eta_p\ . \tag{35}$$

The losses in each plate and in each Kerr medium section lead to the recursive equations

$$\hat{a}_n=(\sqrt{1-l_s}\hat{a}_{n-1}+\sqrt{l_s}\hat{d}_n\exp\{i\kappa[(1-l_p)\hat{b}^\dagger_{n-1}\hat{b}_{n-1}+l_p\hat{c}^\dagger_n\hat{c}_n+\sqrt{l_p(1-l_p)}](\hat{b}^\dagger_{n-1}\hat{c}_n+\hat{c}^\dagger_n\hat{b}_{n-1})]\} \tag{36}$$

and

$$\hat{b}_n=(\sqrt{1-l_p}\hat{b}_{n-1}+\sqrt{l_p}\hat{c}_n)\exp\{i\kappa[(1-l_s)\hat{a}^\dagger_{n-1}\hat{a}_{n-1}+l_s\hat{d}^\dagger_n\hat{d}_n+\sqrt{l_s(1-l_s)}(\hat{a}^\dagger_{n-1}\hat{d}_n+\hat{d}^\dagger_n\hat{a}_{n-1})]\}\ . \tag{37}$$

Here, $\kappa\equiv\sqrt{F}/M$ is an effective value for the Kerr effect in each medium. Solving these recursive equations and letting $M\to\infty$, the loss-error function is obtained:

$$\Delta=\frac{1}{F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle}\left[\frac{\ln(1-\eta)}{\eta}\right]^2\frac{1}{(1-\eta_p)}+\frac{\eta(2-\eta)+2(1-\eta)\ln(1-\eta)}{\eta^2}\ . \tag{38}$$

A solution of the differential equations describing the fields propagating through a lossy Kerr medium gives the same result.[7]

We concentrate on case (c), as the analysis of case (c) gives a QND measurement criterion for the intrinsic medium characteristics, which are independent of input or output coupling loss. Then substituting the calculated value of Δ into Eq. (6), the QND measurement criterion for a Kerr medium can be obtained in terms of $\chi^{(3)}$, the total insertion loss η, and the product of the signal and probe beam powers. This criterion can be written as

$$F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle>\frac{[\ln(1-\eta)]^2}{[-2(1-\eta)\ln(1-\eta)-\eta](1-\eta_p)} \tag{39}$$

and is plotted in Fig. 7 as a function of total insertion loss where $\eta=\eta_p$ is assumed for simplicity. Here, the vertical axis is $F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle$, which can be written in terms of the beam power product as

$$F\langle\hat{n}_{\text{in}}\rangle\langle\hat{n}_p\rangle=\frac{\pi^2}{\epsilon_0^4c^2}\frac{(\chi^{(3)})^2L^2}{\lambda_s\lambda_p A_{\text{eff}}^2n^2}P_sP_p\ . \tag{40}$$

P_s is the signal beam power and P_p is the probe beam power.

Figure 7 also shows $F\langle n_s\rangle\langle n_p\rangle$ plotted for case (a) and case (b) and indicates that there is an upper limit on the loss value for each case. For example, an insertion loss of less than 0.5 is required to realize a QND measurement for case (a). This implies that a beamsplitter is better than the Kerr effect when the input coupling loss exceeds 0.5. In comparison, any loss value is allowed for case (b). This implies that when the photon number is correlated to the probe phase by a transparent Kerr medium, subsequent losses cannot destroy the quantum correlation completely. Note that the upper limit for case (c) is intermediate between those of case (a) and of case (b) with a value of 0.715.

Assuming that the medium loss per unit length is constant, the total insertion loss is a function of the medium length. We can thus calculate the required beam power for a given medium length in a given Kerr medium using Eq. (39). The beam power required for a single-mode optical silica fiber is shown in Fig. 8 for various loss values. For example, the optimum fiber length is 8 km with 0.2 dB/km loss and requires 40 mW for both the signal and probe beams. This can be accomplished with existing fibers and lasers.

A figure of merit for a lossy Kerr medium can be defined using Eq. (39). Differentiating Eq. (39) with respect to distance, we obtain the beam power product

$$P_sP_p=7.17\left[\frac{\epsilon_0^2c}{\pi}\right]^2\frac{1}{M_0^2}\ , \tag{41}$$

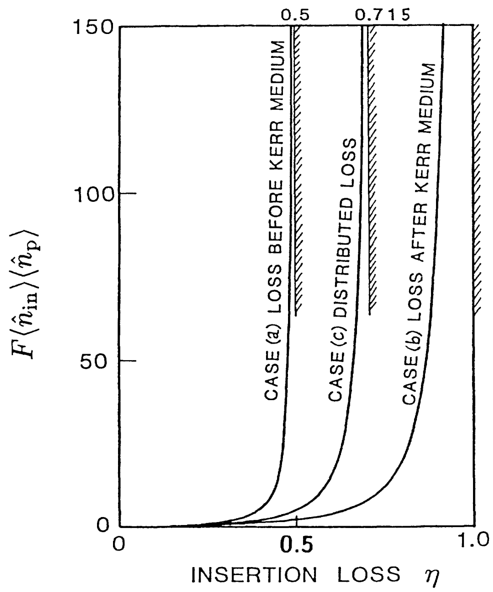

FIG. 7. Required values for $F\langle\hat{n}_{\rm in}\rangle\langle\hat{n}_p\rangle$ vs the total insertion loss in order to realize a QND measurement of the photon number. The total loss should be smaller than 0.5 for case (a), while any loss value is allowed for case (b). Case (c) sets the upper loss limit between them: 0.715.

where M_0 is defined as

$$M_0 \equiv \frac{\chi^{(3)}}{n^2\alpha\sqrt{\lambda_s\lambda_p}\,A_{\rm eff}} . \tag{42}$$

Here, n and α are the refractive index and linear loss of the Kerr medium, λ_s and λ_p are the signal and probe beam wavelengths, and $A_{\rm eff}$ is the effective cross-sectional area in which the beams copropagate. We can take M_0 as a figure of merit for QND measurements in a lossy Kerr medium. The larger the value of M_0, the smaller the value of the beam power product required to produce the QND measurement.

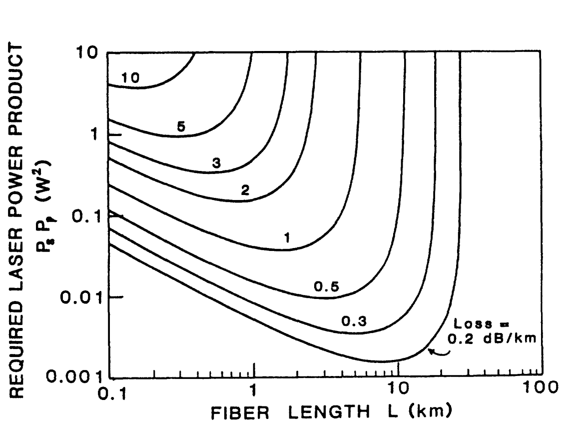

FIG. 8. Required laser power product vs medium length. A single-mode optical silica fiber is assumed as the Kerr medium. Parameters used are as follows: $\chi^{(3)}=3.6\times10^{-33}$ (MKS), $\lambda_s=1.55$ μm, $\lambda_p=1.32$ μm, $A_{\rm eff}=50$ $\mu{\rm m}^2$, and $n=1.45$.

IV. LIMITATIONS IMPOSED BY NONLINEAR ABSORPTION

In Sec. III we assumed that the loss of the Kerr medium is constant and linear. A linear loss can be expressed as the imaginary part of $\chi^{(1)}$, with the real part of $\chi^{(1)}$ corresponding to the square of the refractive index. The optical Kerr effect can be represented by the real part of $\chi^{(3)}$. The imaginary part of $\chi^{(3)}$ corresponds to nonlinear loss, which is not negligible for a resonantly enhanced nonlinear material. Such a nonlinear loss can spoil a QND measurement because it causes absorption of photons.

In this section we examine the effects of $\mathrm{Im}(\chi^{(3)})$ on QND measurements, assuming that linear losses are negligible. We will not consider self-transparency, where losses decrease with increasing power. Rather, we consider two-photon absorption, where the losses of the signal and probe waves are proportional to their intensities, i.e.,

$$\alpha_s=\alpha_2 I_p \tag{43}$$

and

$$\alpha_p=\alpha_2 I_s . \tag{44}$$

Here, I_s and I_p are optical intensities of the signal and probe beams, and α_s and α_p are the losses corresponding. The nonlinear loss coefficient α_2 is written in terms of $\mathrm{Im}(\chi^{(3)})$ as

$$\alpha_2=\frac{2\pi}{\lambda c\epsilon_0^2}\,\mathrm{Im}(\chi^{(3)}) . \tag{45}$$

The total losses for the signal and probe beams are η and η_p and can be expressed as

$$\eta=1-e^{-\alpha_2 I_p L} \tag{46}$$

and

$$\eta_p=1-e^{-\alpha_2 I_s L} , \tag{47}$$

where L is the medium length. Although these losses are nonlinear, we can regard them as linear when I_p and I_s are approximately constant. Then, we can use the QND measurement criterion for a Kerr medium with a linear loss. Substituting Eqs. (46) and (47) into (39), we can obtain the QND measurement criterion for a medium with $\mathrm{Im}(\chi^{(3)})$. Assuming that $\alpha_2 I_s L, \alpha_2 I_p L \ll 1$, this criterion becomes

$$\mathrm{Im}(\chi^{(3)}) < \frac{2\pi L}{cn^2\lambda\epsilon_0^2}[\mathrm{Re}(\chi^{(3)})]^2 , \tag{48}$$

where we assumed that $\lambda_s \cong \lambda_p \equiv \lambda$.

The reason that inequality (48) is linear with respect to $\mathrm{Im}(\chi^{(3)})$ but quadratic with respect to $\mathrm{Re}(\chi^{(3)})$ follows from the QND criterion. For a beamsplitter, Δ is approximately inversely proportional to η (when $\eta \ll 1$). But for a Kerr medium, Δ is inversely proportional to $\mathrm{Re}(\chi^{(3)})$ squares while η is proportional to $\mathrm{Im}(\chi^{(3)})$.

The QND criterion (48) is satisfied for nonresonant material, such as the silica in optical fibers. However, it

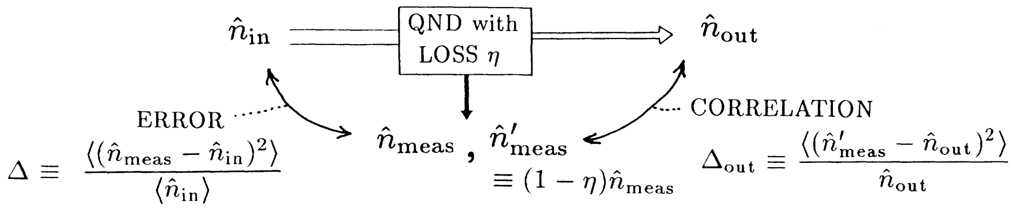

FIG. 9. Measurement error and output correlation of a lossy QND measurement scheme. Δ represents the difference between the input and readout photon numbers, while Δ_{out} is the difference between the output and readout photon numbers.

must be taken into account when using resonantly enhanced $\chi^{(3)}$ nonlinear material for QND measurements of photon number. The Kramers-Kronig relationship between $\mathrm{Re}(\chi^{(3)})$ and $\mathrm{Im}(\chi^{(3)})$ is an integral over frequency and does not impose a relationship between $\mathrm{Re}(\chi^{(3)})$ and $\mathrm{Im}(\chi^{(3)})$ at any given frequency. Therefore, one can always choose a frequency or a frequency domain where inequality (48) holds. In this paper, we assume measurements at single frequencies, so measurement times are long. However, for measurements with short optical pulses, the situation becomes more complicated, and a QND measurement criterion must take the response time of $\chi^{(3)}$ and the probe and signal pulse widths into account.

V. QND MEASUREMENT CRITERION FOR A CORRELATION EXPERIMENT

In this section we consider for an experimental system that meets the requirements of the QND measurement criterion. Our discussion of the QND measurement criterion was based on loss-error function. This function, however, cannot be obtained experimentally unless we use either a perfect QND measurement scheme or an ideal number-state generator to test the system, neither of which is available. However, it is possible experimentally to obtain a correlation between the measured photon number and the number of photons that exit the device. For this reason, we consider the relationship between the QND measurement criterion and this correlation.

The correlation between the measured photon number and the exiting photons can be evaluated using the variance Δ_{out} of the difference between the measured photon number and the transmitted photon number. This variance is $\langle(\hat{n}'_{meas}-\hat{n}_{out})^2\rangle/\langle\hat{n}_{out}\rangle$, where $\hat{n}'_{meas}$ is defined as $\hat{n}'_{meas}\equiv(1-\eta)\hat{n}_{meas}$, which normalizes the photon number to facilitate comparison with $\hat{n}_{out}$. The variance Δ_{out} does not directly evaluate the measurement error Δ, because the measurement error is the difference between the input and estimated photon numbers. Rather, Δ_{out} is the difference between the estimated and transmitted photon numbers, as shown in Fig. 9.

When the measurement error satisfies the QND measurement criterion, we can calculate the variance Δ_{out} as follows. Using a similar analysis to that illustrated in Fig. 6, we obtain

$$\Delta_{out}=\frac{1}{F\langle\hat{n}_{in}\rangle\langle\hat{n}_p\rangle}\left[\frac{\ln(1-\eta)}{\eta}\right]^2+\frac{\eta(2-\eta)+2(1-\eta)\ln(1-\eta)}{\eta^2}\ , \tag{49}$$

where we assume that $\eta_p=\eta$. Equations (39) and (49) then lead to an inequality

$$\Delta_{out}<\frac{1+2(1-\eta)\ln(1-\eta)}{\eta}\ . \tag{50}$$

This inequality establishes the QND measurement criterion in terms of Δ_{out} for case (c) and is shown graphically in Fig. 10.

It should be noted that there seems to be no upper loss limit in Fig. 10. It may seem that any loss value is possible, if Δ_{out} is made small enough, which contradicts the existence of the loss limit as indicated in Fig. 7. That this is not so can be seen by considering the minimum value of Δ_{out}. By letting the magnitude of the Kerr effect go to infinity in Eq. (49), the minimum value of Δ_{out} is seen to be

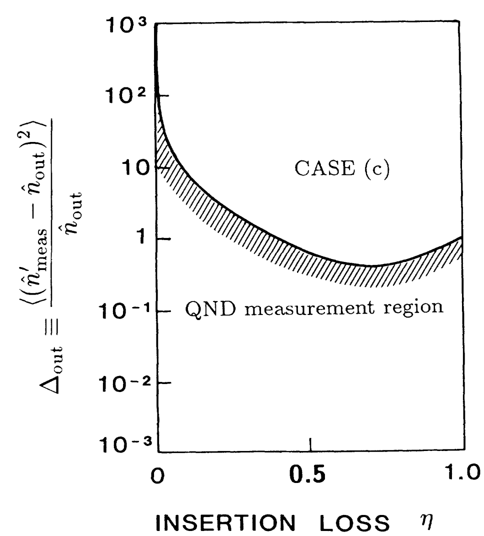

FIG. 10. Criterion for Δ_{out} for QND measurement [case (c)]. A smaller Δ_{out} should be obtained if a Kerr medium satisfies the QND measurement criterion.

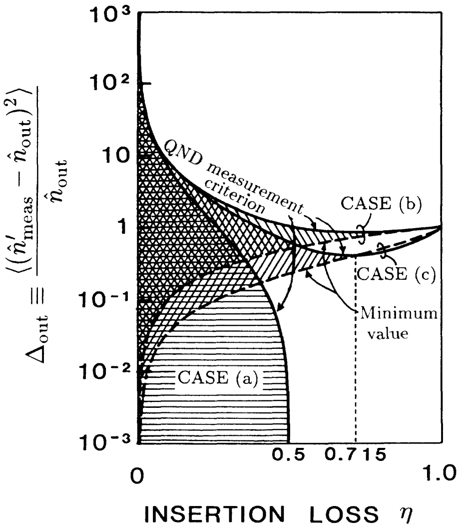

FIG. 11. Required and attainable $\Delta_{\rm out}$ for the three cases illustrated in Fig. 5. The loss region for the QND measurement is $0 \le \eta < 0.5$ for case (a), $0 \le \eta < 1$ for case (b), and $0 \le \eta < 0.715$ for case (c), which are consistent with Fig. 7.

$$\min(\Delta_{\rm out}) = \frac{\eta(2-\eta)+2(1-\eta)\ln(1-\eta)}{\eta^2} . \tag{51}$$

Hence the QND regime for the $\Delta_{\rm out}$ is indicated by the values greater than indicated in but below those indicated by Eq. (50). Figure 11 shows the QND regime of $\Delta_{\rm out}$ for cases (a), (b), and (c). This figure shows an upper loss limit of 0.5 for case (a), 1.0 for case (b), and 0.715 for case (c). This is consistent with Fig. 7.

VI. CONCLUSION

We have presented a photon-number QND measurement theory that takes dissipation into account. A QND measurement for a system with loss η is defined to have a smaller measurement error than that of a measurement which uses a beamsplitter that introduces a loss η. We take the measurement error (which is a function of the loss) as a loss-error function for a photon-number measurement scheme and obtain such a loss-error function for a lossy Kerr medium. This allows us to determine the QND measurement criterion in terms of loss, $\chi^{(3)}$, and the beam powers. We show that there is both an upper limit on the device length and an optimum device length for a medium with a given loss. We further show that the required signal and probe beam power product for a measurement is determined for a medium with a given Kerr constant. If we assume a single-mode optical silica fiber as the Kerr medium, the upper limit and the optimum lengths, respectively, are 27 and 8 km, and the required optical power is about 40 mW for both the signal and probe beams.

A QND measurement criterion for a medium with nonlinear absorption was also obtained. Furthermore, we show how these QND measurement criterion relates to experimental observables. We conclude that a QND measurement of photon number is possible using existing Kerr media and light sources.

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[1]V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, Science **209**, 547 (1980).

[2]C. M. Caves, K. S. Thorne, R. M. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).

[3]G. J. Milburn and D. F. Walls, Phys. Rev. A **28**, 2065 (1983).

[4]N. Imoto, H. A. Haus, and Y. Yamamoto, Phys. Rev. A **32**, 2287 (1985).

[5]M. D. Levenson, R. M. Shelby, M. Reid, and D. F. Walls, Phys. Rev. Lett. **57**, 2473 (1986).

[6]N. Imoto, S. Watkins, and Y. Sasaki, Opt. Commun. **61**, 159 (1987).

[7]N. Imoto (unpublished).