Dynamics of SU(1,1) coherent states driven by a damped harmonic oscillator

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We study the time evolution of SU(1,1) coherent states driven by a damped harmonic oscillator as described by the Kanai-Caldirola Hamiltonian. The coherence of these states is preserved and their time evolution is described by nonlinear "classical" equations. The trajectories of all SU(1,1) coherent states in phase space asymptotically approach a common point as $t \rightarrow \infty$. In all cases, this approach is accompanied by a damping of the associated energy expectation value to zero.

I. INTRODUCTION

There has been much recent interest in the construction of coherent states for time-dependent systems. Many reports involve some generalization of the usual harmonic-oscillator coherent states to time-dependent harmonic oscillators. For example, Hartley and Ray[1] have constructed oscillatorlike coherent states associated with a harmonic oscillator with a time-dependent frequency. The Lewis-Riesenfeld theory[2] of time-dependent invariants was used for the construction of these states. Such associated coherent states have most of the properties of the usual coherent states, as they follow the classical motion exactly. However, they do not minimize the uncertainty relations.

Very recently, Yeon *et al.*[3] constructed a set of coherent states for a damped harmonic oscillator governed by the Kanai-Caldirola Hamiltonian[4]

$$H=\tfrac{1}{2}(e^{-\lambda t}\pi^2+\omega^2 q^2 e^{\lambda t}) , \tag{1.1}$$

where π denotes the canonical momentum, and $[q,\pi]=i$. This Hamiltonian, and the general problem of the damped harmonic oscillator, have recently been reviewed by Dekker.[5] The coherent states of Yeon *et al.* are exact in the sense that they follow the classical motion of a damped harmonic oscillator: the phase-space trajectories spiral to point attractors. Other examples of coherent states for the system defined by Eq. (1.1) are given in Ref. 6. This Hamiltonian has also been shown to be relevant to quantum optics,[7] specifically in relation to field decay in a Fabry-Perot cavity.

On the other hand, it will be shown presently that the Hamiltonian of Eq. (1.1) also preserves, under time evolution, coherent states associated with the dynamical group SU(1,1). The coherent states (CS's) referred to are the Perelomov states.[8] The SU(1,1) CS's are known to possess certain nonclassical properties, being an example of the so-called *squeezed states* (actually squeezed vacuum states) familiar from quantum optics.[9–11] In a previous study,[10] the most general Hamiltonian which preserves an arbitrary SU(1,1) CS under time evolution was derived. Associated with such a Hamiltonian there also exists a time-dependent invariant of the Lewis-Riesenfeld type.[12] Other time-dependent oscillator-type systems, in particular the example of a time-dependent frequency $\omega(t)$, with no damping, have been discussed by other authors.[13–15]

Returning to Eq. (1.1), we find that the "classical" equations of motion that arise for the SU(1,1) CS's are nonlinear. The qualitative behavior of their solutions will be studied and illustrated with some numerical results. In some analogy to the case of the oscillator-like CS of Refs. 3 and 7, the asymptotic trajectory (i.e., as $t \rightarrow \infty$), for an initial SU(1,1) CS (a squeezed vacuum state), approaches a limit point, as would be expected for the classical damped oscillator. The time evolution of the variances of the quadratures

$$X_1=\tfrac{1}{2}(a+a^\dagger), \quad X_2=\frac{1}{2i}(a-a^\dagger) , \tag{1.2}$$

is also studied. We find that regardless of which quadrature is initially squeezed, an enhanced squeezing appears in the X_2 quadrature. In fact, the variance of that quadrature tends asymptotically to zeros as $t \rightarrow 0$, regardless of the initial condition. The other quadrature, may also become squeezed for a number of time intervals, before its variance increases without bound as $t \rightarrow \infty$.

The plan of the paper is as follows. In Sec. II we briefly review the dynamics of the SU(1,1) CS's and discuss the associated time-dependent invariant. In Sec. III SU(1,1) CS's are constructed for the Hamiltonians of Eq. (1.1). The actual time evolution of these coherent states is then discussed. Some concluding remarks are made in Sec. IV.

II. DYNAMICS OF SU(1,1) COHERENT STATES

Our discussion is kept brief here, and the reader is referred to Ref. 10 for a more detailed exposition. The Lie algebra of SU(1,1) consists of the generators K_0 and $K_\pm$ satisfying the commutation relations

$$[K_0,K_\pm]=\pm K_\pm, \quad [K_-,K_+]=2K_0 . \tag{2.1}$$

The Casimir operator is given by

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+) . \tag{2.2}$$

We consider here only the positive discrete representation $D^+(k)$ whose basis states are $|n,k\rangle$, which obey the relations $K_0|n,k\rangle=(n+k)|n,k\rangle$ and $C|n,k\rangle=k(k-1)|n,k\rangle$. The SU(1,1) CS's, according to Perelomov,[8] are given by

$$|\xi,k\rangle=S(\alpha)|0,k\rangle , \tag{2.3}$$

where

$$S(\alpha)=\exp(\alpha K_+-\alpha^*K_-) , \tag{2.4}$$

$\alpha=-\frac{1}{2}\theta e^{-i\phi}$, and $\xi=-\tanh(\theta/2)e^{-i\phi}$. The group parameters θ and ϕ have the ranges $-\infty<\theta<\infty$ and $0\le\phi\le 2\pi$.

The most general form for a Hamiltonian which preserves the coherence of an arbitrary initial SU(1,1) CS has been shown to be[10]

$$H(t)=A(t)K_0+F(t)K_++F^*(t)K_-+\beta(t) , \tag{2.5}$$

where $A(t)$ and $\beta(t)$ are arbitrary real functions of time and $F(t)$ is an arbitrary complex function of time. The "classical" equations of motion are[16]

$$\dot{\xi}=\{\xi,\mathcal{H}_k\}, \quad \dot{\xi}^*=\{\xi^*,\mathcal{H}_k\} , \tag{2.6}$$

where $\mathcal{H}_k=\langle\xi,k|H|\xi,k\rangle$ and

$$\{A,B\}=\frac{(1-|\xi|^2)^2}{2ik}\left[\frac{\partial A}{\partial\xi}\frac{\partial B}{\partial\xi^*}-\frac{\partial A}{\partial\xi^*}\frac{\partial B}{\partial\xi}\right] \tag{2.7}$$

defines a generalized Poisson bracket. The classical motion described by these equations takes place on a Lobachevski plane where $|\xi|<1$. For the Hamiltonians of Eq. (2.5) one arrives at the evolution equation[10]

$$\dot{\xi}=-iA(t)\xi-iF^*(t)\xi^2-iF(t) . \tag{2.8}$$

(Note that this equation, as it appears in the above-cited reference, is incorrect.)

Associated with the Hamiltonian of Eq. (2.5) is a time-dependent invariant operator[12] $I(t)$ which satisfies the equation

$$\frac{dI}{dt}\equiv\frac{\partial I}{\partial t}+i[H,I]=0 . \tag{2.9}$$

This invariant has the form

$$I(t)=K_0\cosh\theta+\tfrac{1}{2}K_+e^{-i\phi}\sinh\theta+\tfrac{1}{2}K_-e^{i\phi}\sinh\theta+\gamma , \tag{2.10}$$

where $\theta(t)$ and $\phi(t)$ are determined from the solutions of Eq. (2.8) and γ is an arbitrary real constant. The SU(1,1) CS's are eigenstates of this operator, with real eigenvalues according to the relation

$$I(t)|\xi,k\rangle=(k+\gamma)|\xi,k\rangle . \tag{2.11}$$

In order for the SU(1,1) CS's to be solutions of the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=H(t)\psi , \tag{2.12}$$

they must have the following form:[12]

$$|\xi(t),k\rangle_s=\exp[i\Phi(t)]|\xi(t),k\rangle . \tag{2.13}$$

For consistency, the phase $\Phi(t)$ may be fixed as a constant by removing the arbitrariness of $\beta(t)$ in Eq. (2.5), if we set[12]

$$\beta(t)=k\left[\frac{(|\xi|^2+5)}{2(1-|\xi|^2)}(F\xi^*+F^*\xi)-A(t)\right] . \tag{2.14}$$

It is always possible to add the $\beta(t)$ function to any Hamiltonian, since the dynamics is not affected. In what follows we assume that Eq. (2.14) is satisfied.

III. APPLICATION TO THE DAMPED HARMONIC OSCILLATOR

As is well known, the SU(1,1) Lie algebra may be realized in terms of annihilation and creation operators, $a=(q+i\pi)/\sqrt{2}$ and $a^\dagger=(q-i\pi)/\sqrt{2}$, respectively, as follows:

$$K_0=\tfrac{1}{4}(aa^\dagger+a^\dagger a)=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2}) , \quad K_+=\tfrac{1}{2}a^{\dagger 2}, \quad K_-=\tfrac{1}{2}a^2 . \tag{3.1}$$

Setting

$$K_1=\tfrac{1}{2}(K_++K_-) , \tag{3.2}$$

we have

$$\pi^2=2(K_0+K_1), \quad q^2=2(K_0-K_1) . \tag{3.3}$$

The Casimir operator, Eq. (2.2), is evaluated as $C=-\frac{3}{16}$, so that $k=\frac{1}{4},\frac{3}{4}$. The $k=\frac{1}{4}$ solution corresponds to states with even numbers of quanta (or photons). If we define the quadrature operators as

$$X_1=\tfrac{1}{2}(a+a^\dagger), \quad X_2=\frac{1}{2i}(a-a^\dagger) , \tag{3.4}$$

their respective variances may be written as

$$V(X_{1,2})=\langle K_0\rangle\pm\tfrac{1}{2}[\langle K_+\rangle+\langle K_-\rangle] . \tag{3.5}$$

For an SU(1,1) CS they are given by [cf. Eqs. (2.23) in Ref. 10 for the relevant matrix elements]

$$V(X_{1,2})=k\left[\frac{1+|\xi|^2}{1-|\xi|^2}\right]\pm\frac{2kRe(\xi)}{1-|\xi|^2} . \tag{3.6}$$

For the squeezed vacuum, we take $k=\frac{1}{4}$. (A squeezed state exists when one of the variances obeys the inequality $V(X_{1,2})<\frac{1}{4}$). The squeezing is in the X_1 quadrature when the phase $\phi=0$, and in the X_2 quadrature when $\phi=\pi$. For $\phi=\pi/2$ there is no squeezing.[9,11]

Using Eq. (3.3), the Kanai-Caldirola Hamiltonian of Eq. (1.1) may now be written in terms of the SU(1,1) generators as

$$H=(e^{-\lambda t}+e^{\lambda t}\omega^2)K_0 + \tfrac{1}{2}(e^{-\lambda t}-\omega^2 e^{\lambda t})(K_+ + K_-)+\beta(t) , \quad (3.7)$$

where an arbitrary $\beta(t)$ has been added. A comparison with Eq. (2.5) then gives the association

$$A(t)=e^{-\lambda t}+e^{\lambda t}\omega^2, \quad F(t)=\tfrac{1}{2}(e^{-\lambda t}-\omega^2 e^{\lambda t}) . \quad (3.8)$$

Without loss of generality we let $\omega^2=1$ so that

$$A(t)=2\cosh(\lambda t), \quad F(t)=-\sinh(\lambda t) . \quad (3.9)$$

From Eq. (2.8), the time evolution of the SU(1,1) CS associated with the Hamiltonian in Eq. (3.7) is then given by the nonlinear equation

$$\dot{\xi}=-2i\cosh(\lambda t)\xi+i\sinh(\lambda t)\xi^2+i\sinh(\lambda t) , \quad (3.10)$$

subject to the initial condition

$$\xi(0)=\xi_0=-\tanh\left[\frac{\theta_0}{2}\right]e^{-i\phi_0} . \quad (3.11)$$

We now proceed with an analysis of the qualitative behavior of the solutions to Eq. (3.10) in the phase plane defined by their real and imaginary parts. By letting $\xi(t)=x(t)+iy(t)$, Eq. (3.10) is equivalent to the nonlinear nonautonomous first-order system

$$\dot{x}=A(t)y+2F(t)xy ,$$
$$\dot{y}=-A(t)x-F(t)(x^2-y^2+1) , \quad (3.12)$$

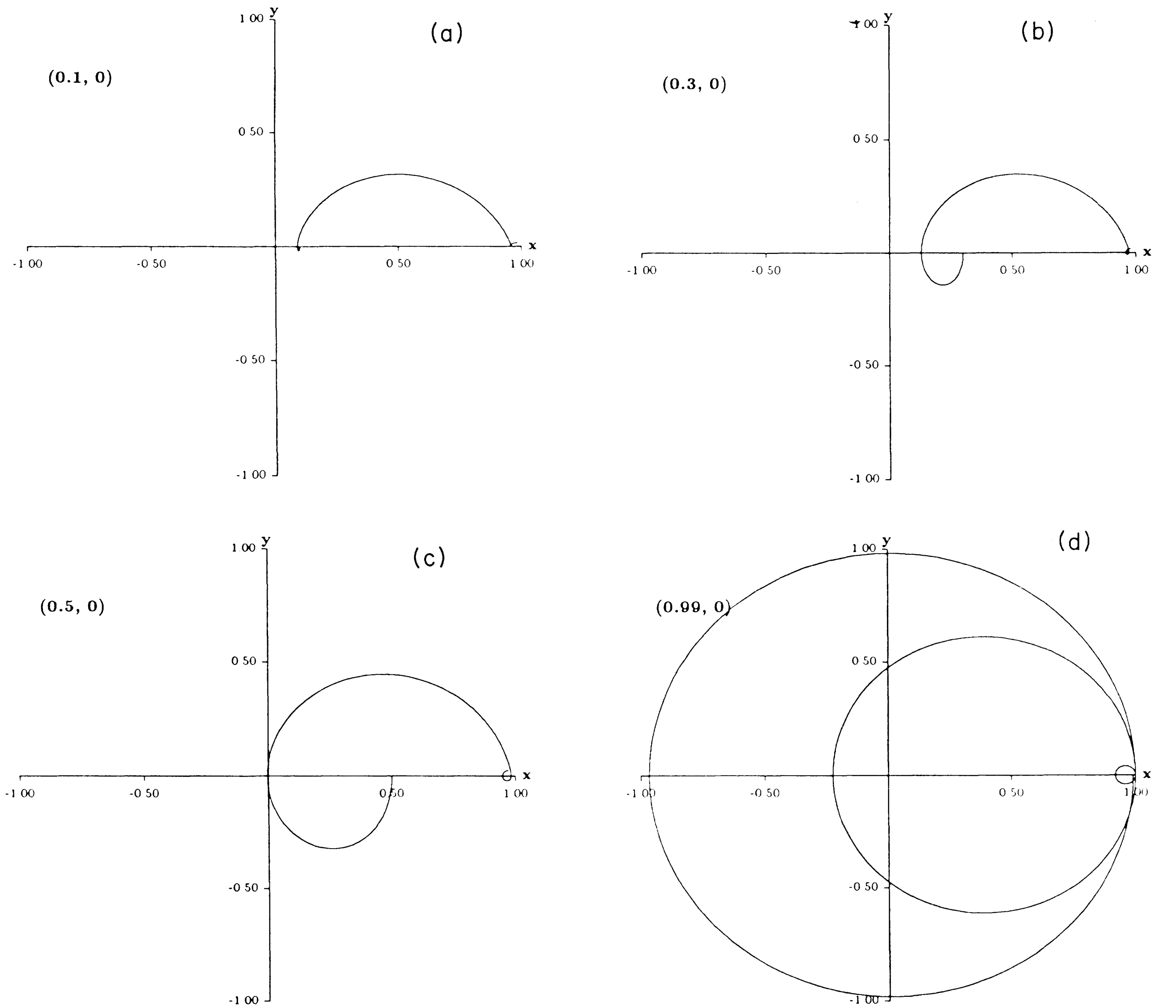

FIG. 1. Evolution of the complex SU(1,1) CS parameter $\xi(t)=x(t)+iy(t)$ as given by Eq. (3.10), corresponding to the following initial conditions $(x(0),y(0))$, (a) (0.1,0),(b) (0.3,0), (c) (0.5,0), (d) (0.99,0). In all cases, the $\xi(t)$ spiral asymptotically toward the point (1,0).

where, for simplicity, we have resubstituted for the hyperbolic functions according to Eq. (3.9). If we now let $R(t)^2 = x(t)^2 + y(t)^2$, then from Eq. (3.12)

$$x\dot{x} + y\dot{y} = R\dot{R} = F(t)y(R^2 - 1) . \quad (3.13)$$

Thus for *any* $A(t)$ and $F(t)$ the curve $R = 1$ is a closed orbit, and no solutions can escape from within the unit circle, which is consistent with the requirement that $|\xi| < 1$.

An understanding of the qualitative nature of solutions to the system in Eq. (3.12) can be had if we consider separately the time-dependent vector fields associated with the first and second terms on the right-hand sides of each equation. For example, setting $F(t) = 0$, the resulting vector field generates a family of circular orbits concentric about the origin. Furthermore, since $A(t)$ increases without bound, so do the oscillation frequencies of the orbiting trajectories. If we now set $A(t) = 0$, and note that $F(t) < 0$, the resulting vector field generates a family of trajectories which emanate from the point $(0, -1)$ and travel upward along "lines of longitude" toward the point (0,1) as $t \rightarrow \infty$. The system in (3.12) is a superposition of these two phase portraits, from which we conclude the following qualitative features: $\dot{y} > 0$ for $x < 0$ (i.e., solution curves point upward), (ii) $\dot{x} > 0$ for $y > 0$, $\dot{x} < 0$ for $y < 0$, and $\dot{x} = 0$ for $y = 0$. Thus, in the left half-disk,

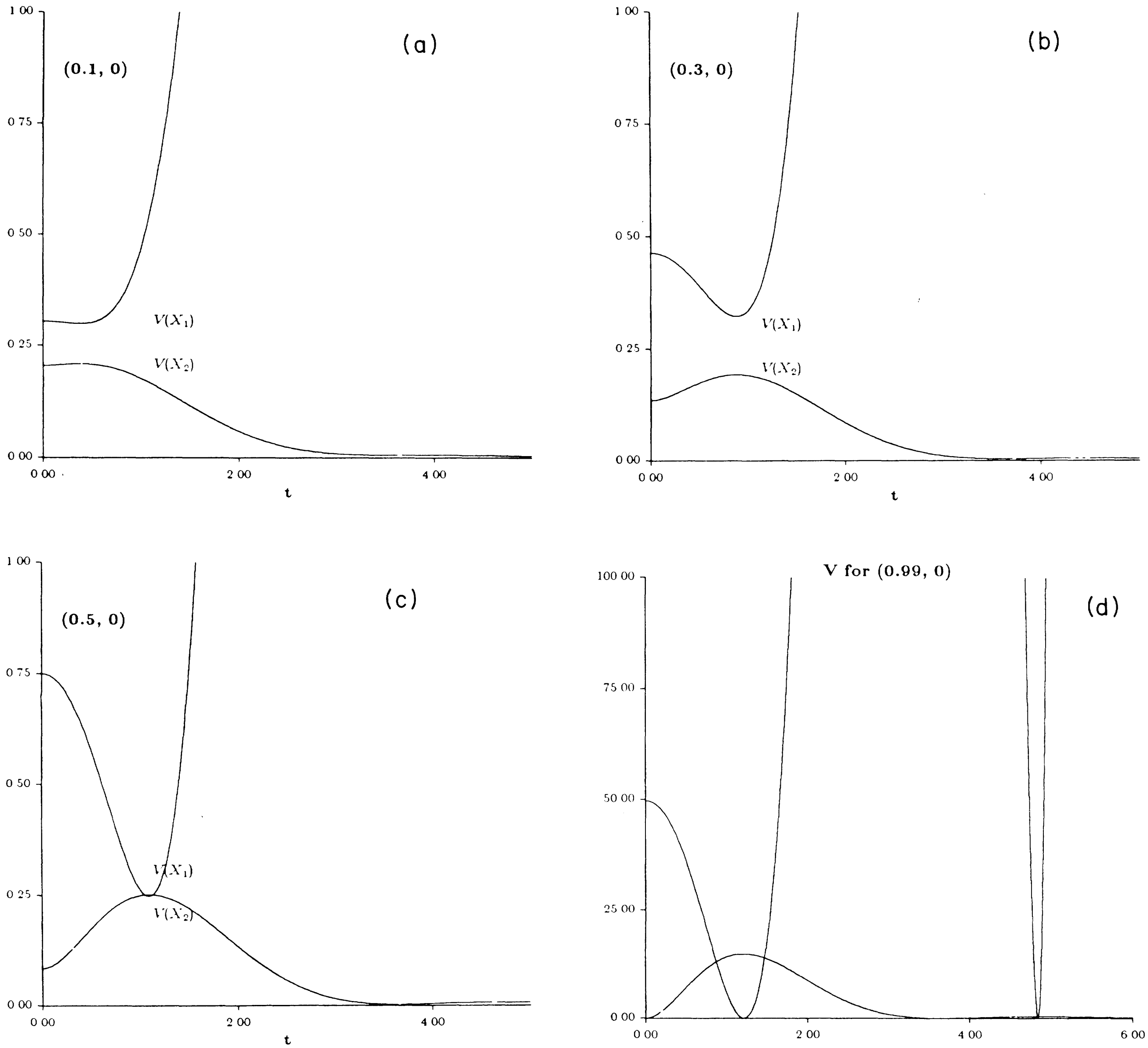

FIG. 2. Variances $V(X_1)$ and $V(X_2)$ associated with the SU(1,1) coherent states depicted in Figs. 1(a)–1(d). Initial conditions $(x(0), y(0))$: (a) (0.1,0), (b) (0.3,0), (c) (0.5,0), (d) (0.99,0). Although not evident from the gross scales the Heisenberg uncertainty principle is always obeyed, i.e., $V(X_1)V(X_2) \geq \frac{1}{16}$. Variances are calculated from Eq. (3.6).

solution curves travel in a clockwise manner. Note also that the unit circle $R=1$ cannot be an attractive limit cycle for any trajectories where $|\xi(t)|<1$ since, from Eq. (3.13), $\dot{R}<0$ for $y<0$ and $\dot{R}>0$ for $y>0$.

In the Appendix, we show that all trajectories starting inside the unit circle asymptotically approach the point $(x,y)=(1,0)$ in the limit $t\rightarrow\infty$. However, this approach is somewhat different from the usual approach of a trajectory to an equilibrium point, since (i) the point (1,0) is not an equilibrium point and (ii) solution curves can not cross the unit circle. Solution curves spiral arbitrarily closely to this point from within the unit circle. As a result, the distance from the point $(x(t),y(t))$ to (1,0) exhibits a damped oscillatory behavior, rather than the usual monotonic exponential decrease as the equilibrium point is approached. The behavior described above is corroborated by numerical calculations of the phase-plane trajectories (Runge-Kutta fourth-order integrations), some results of which are presented in Fig. 1, corresponding to various initial conditions, $x(0)=x_0$, $y(0)=0$, with $\omega=1$ and $\lambda=1$. The "classical" motion of the $\xi(t)$ trajectories is, in a sense, quite analogous to that of generalized oscillator states discussed in Refs. 3 and 6. The latter follow the true motion of a damped oscillator, and approach a point attractor as $t\rightarrow\infty$. For purposes of comparison, the evolution of the SU(1,1) CS with the initial condition of Fig.

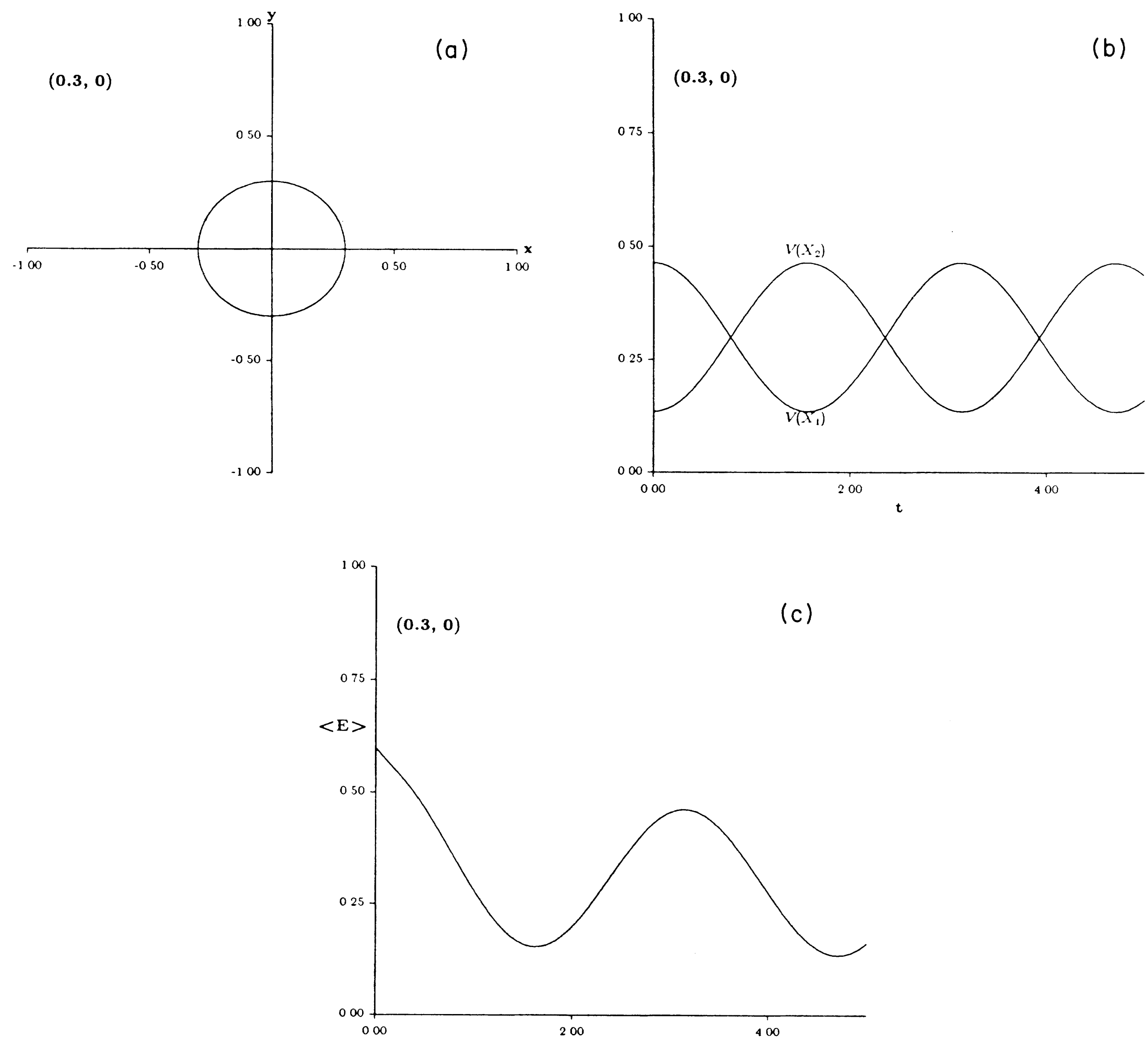

FIG. 3. Evolution of an SU(1,1) CS state in the case of no interaction with the damped harmonic oscillator, i.e., $\lambda=0$, with initial condition $(x(0),y(0))=(0.3,0.0)$. (a) Plot of $\xi(t)=x(t)+iy(t)$. (b) Variances $V(X_1)$ and $V(X_2)$ corresponding to trajectory in (a). (c) Evolution of the expectation value $\langle E\rangle$ for the noninteracting state in (a).

1(b), but with no damping interaction, i.e., $\lambda=0$, is shown in Fig. 3(a). The actual trajectory is given by $\xi(t)=\xi_0\exp(-2i\omega t)$. The angular frequency $\chi=\omega$ is constant, as is the radius, $\rho=\xi_0$. The variances and energy associated with this case of noninteraction are plotted in Figs. 3(b) and 3(c) and will be discussed below.

We now focus on the variances $V(X_{1,2})$ associated with the SU(1,1) coherent states, as given by Eq. (3.5). It is again assumed that a function $\beta(t)$ of the form in Eq. (2.14) is chosen so that Eq. (2.13) is satisfied with a constant phase $\Phi(t)=\Phi$. The variances in the x and y coordinates for the trajectories of Fig. 1 are plotted in Fig. 2. There are some noteworthy features. In Figs. 2(a)–2(c) the variance of the initially squeezed quadrature increases, attains a maximum, and then tends asymptotically to zero as $t\to\infty$. The greater the degree of initial squeezing, the greater the value of the variance at the hump maximum, so that, for sufficient initial squeezing, the quadrature may be unsqueezed for a definite time interval, before the eventual decay to zero. During this time interval, the other (initially unsqueezed) quadrature becomes squeezed, before its variance eventually tends to infinity. In all cases, although not immediately evident from the graphs, the Heisenberg uncertainty principle is obeyed, i.e., $V(X_1)V(X_2)\geq\frac{1}{16}$. In Fig. 2(d), with initial condition $x_0=0.99$, we see that there can exist more than one time interval in which squeezing-nonsqueezing characteristics are exchanged. Presumably, with appropriate initial conditions, such exchanging could be induced over an arbitrary number of time intervals. [A little investigation shows that these periods of exchange occur when the trajectory $\xi(t)$ is farthest from the point (1,0).] Figure 3(b) shows the variances in the case of no interaction, i.e., $\lambda=0$ for the trajectory of Fig. 3(a). A comparison of the plots in Figs. 3(b) and 2(b) shows that the interaction between the squeezed SU(1,1) CS's and the damped harmonic oscillator produces even greater squeezing than for the undamped case, in other words *enhanced squeezing*.

Finally, we consider the expectation value of the mechanical energy, given by the operator

$$\begin{aligned} E &= e^{-2\lambda t}\tfrac{1}{2}\pi^2+\tfrac{1}{2}\omega^2x^2 \\ &= e^{-2\lambda t}(K_0+K_1)+\omega^2(K_0-K_1) \ . \end{aligned} \tag{3.14}$$

The calculation of the expectation value $\langle E\rangle$ is straightforward using $\langle K_0\rangle$ and $\langle K_1\rangle$: the relevant matrix elements are given by Eqs. (2.23) in Ref. 10. Rather than being concerned with the actual calculation of $\langle E\rangle$ in time, we focus on its fundamental asymptotic behavior. Note that, from Eqs. (3.2) and (3.14), we may write

$$\langle E\rangle=\langle K_0\rangle(1+e^{-2\lambda t})+\tfrac{1}{2}[\langle K_+\rangle+\langle K_-\rangle](e^{-2\lambda t}-1) \ . \tag{3.15}$$

From Eq. (3.6), it then follows that $\langle E\rangle\to V(X_2)$ as $t\to\infty$. From the behavior of $V(X_2)$ observed above, it is seen that the expectation values of the mechanical energy follows a classical damping to zero. Again, this is analogous to the situation for ordinary coherent states described in Refs. 3 and 6. The oscillatory behavior of $\langle E\rangle$ in the undamped case, i.e., $\lambda=0$, for the trajectory of Fig. 3(a), is shown in Fig. 3(c).

IV. CONCLUSIONS

In this work, we have considered the interaction of a dissipative system, namely, the damped harmonic oscillator, with SU(1,1) coherent states which exhibit the nonclassical property of squeezing. Furthermore, the Hamiltonian in Eq. (1.1) describing the damped oscillator belongs to the general class of Hamiltonians which preserves SU(1,1) CS under time evolution. All trajectories approach the limit point $\xi=1$ in an oscillatory fashion. Also, the variances of the initially squeezed quadratures, as well as the expectation values of the mechanical energy, become damped to zero as $t\to\infty$. It is of natural interest to study the interaction of these states with other energy nonconserving systems to see whether a similar behavior is obtained. Work in that direction is in progress and will be reported elsewhere. We have also been studying the nonlinear aspects of the time evolution equation (2.8) for $\xi(t)$, in particular, Poincaré-type discrete evolution mappings associated with periodic and quasiperiodic "kicking." The results of these studies will be reported elsewhere.

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APPENDIX

Here we sketch the ideas behind a derivation of the asymptotic behavior exhibited by solutions to the system in Eq. (3.12). This nonautonomous system may be considered as the limit $\alpha\to1$ of the following family of systems:

$$\begin{aligned} \dot{x} &= A(t)y+2\alpha F(t)xy \ , \\ \dot{y} &= -A(t)x-\alpha F(t)(x^2-y^2+1), \quad 0\leq\alpha\leq1 \ . \end{aligned} \tag{A1}$$

Note that, in the limit $t\to\infty$, the point

$$x_e=\alpha^{-1}-(\alpha^{-2}-1)^{1/2} \tag{A2}$$

is an equilibrium point. (The nature of this point is undetermined from a linearization analysis, since the eigenvalues of the Jacobian matrix evaluated at this point are purely imaginary.) Note also, that in the limit $\alpha\to1$, $x_e\to1$.

For $\alpha<1$, each system possesses an attractive limit cycle Ω_α with center on the positive real axis. Substituting $\omega(t)=x(t)-\bar{x}_\alpha, |\bar{x}_\alpha|<1$, into Eq. (A1) to obtain, after some manipulation,

$$w\dot{w}+y\dot{y}=\alpha F(t)y[w^2+y^2-1-\bar{x}_\alpha^2]-A(t)\bar{x}_\alpha y \ . \tag{A3}$$

[The fact that $\bar{y}_{\bar{\alpha}}=0$ follows from the symmetry in Eq. (A1).] Setting $\rho(t)^2=\omega(t)^2+y(t)^2$ and noting that $A(t)/F(t)\to-2$ as $t\to\infty$, we have

$$\rho\dot{\rho} \sim \alpha F(t) y[\rho^2-(1+\bar{x}_\alpha^2-2\alpha^{-1}\bar{x}_\alpha^2)], \quad t\rightarrow\infty \ . \tag{A4}$$

The circle Ω_α given by

$$w^2+y^2=\bar{x}_\alpha^2-2\alpha^{-1}\bar{x}_\alpha+1=\rho_\alpha^2 \tag{A5}$$

is seen to be an invariant closed orbit, in the limit $t\rightarrow\infty$. The condition $0\leq\rho_\alpha\leq 1$, coupled with the restriction $|\bar{x}_\alpha|\leq 1$, implies that

$$0\leq\bar{x}_\alpha<x_e \ . \tag{A6}$$

The strict inequality on the right follows from the fact that x_e is a root of the quadratic equation in $\bar{x}_\alpha$ in Eq. (A5). The equality would imply that $\rho_\alpha=0$, i.e., that $\bar{x}_\alpha$ is an asymptotic equilibrium point, which is not possible for $\alpha<1$, from Eq. (A1). Thus, there exists a limit cycle Ω_α, of nonzero radius ρ_α, centered on the x interval [0,1]. In a fashion analogous to the Poincaré-Bendixon theorem for autonomous systems, the limit cycle must contain the asymptotic equilibrium point x_e. The parameters $\bar{x}_\alpha$ and ρ_α are not fixed, and depend upon the initial conditions of the trajectory. A sample trajectory, with important features labeled, is presented in Fig. 4.

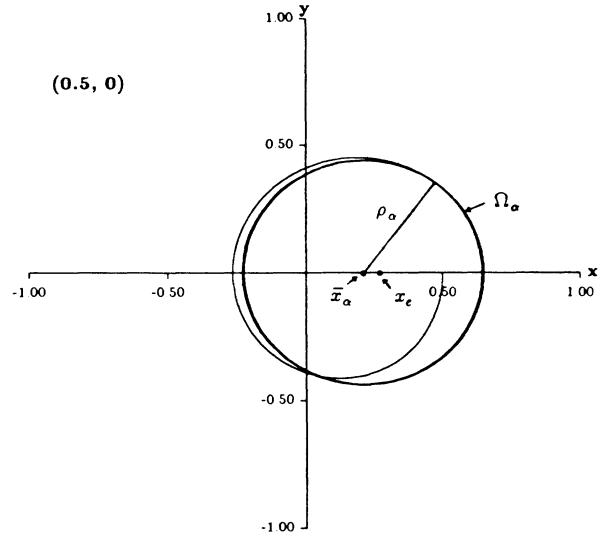

FIG. 4. Trajectory of a solution curve to the system in Eq. (A1), for the particular case $\alpha=0.5$, with initial condition $x(0)=0.5, y(0)=0$. Labeled are the limit cycle Ω_α, with radius ρ_α and center $\bar{x}_\alpha$, and the asymptotic equilibrium point x_e.

We now show that for $\alpha<1$, the angular velocity of the trajectory $\xi(t)$, as it approaches the circular limit cycle Ω_α, is unbounded as $t\rightarrow\infty$. Assuming that t is sufficiently large so that $\xi(t)$ is "asymptotically close" to Ω_α (see Ref. 17), we let $w(t)\equiv\rho\cos\chi$ and $y(t)=\rho\sin\chi$, so that $w\dot{y}-\dot{w}y=\rho^2\dot{\chi}$ (angular momentum). From Eq. (A1), a small calculation shows that, for $\rho>0$,

$$\dot{\chi}=2F(t)\{1-\alpha[w(t)+\bar{x}_\alpha]\} \ . \tag{A7}$$

For $\alpha<1$, the term in curly brackets is strictly positive for all time. Since $F(t)=-\sinh(\lambda t)$, it follows that $\dot{\chi}<0$ and $\dot{\chi}\rightarrow-\infty$ as $t\rightarrow\infty$, i.e., as $\xi(t)\rightarrow\Omega_\alpha$. The special case $\alpha=1$ deserves special attention, however, and the angular frequency remains bounded as $t\rightarrow\infty$. This effect is also seen in numerical calculations.

Recall that the asymptotic equilibrium point x_e, which lies in the interior region bounded by Ω_α, travels to the point $(x,y)=(1,0)$ as $\alpha\rightarrow 1$. In this limit, there is no limit cycle Ω_α, and trajectories spiral arbitrarily closely to the point $(1,0)$ as $t\rightarrow\infty$.

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