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Fractal dimensionality for the η model

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We present simulation results for the fractal dimensionality of the η model for a wide range of values of η. We check the theoretical predictions for the curve $d_f(\eta)$ proposed by Halsey [Phys. Rev. Lett. **59**, 2067 (1987)] against the values we get from calculated quantities and find good agreement between theory and the computer experiments.

In spite of the enormous amount of attention received in the past few years, diffusive models for growth phenomena are still far from being completely understood. Since 1981, when Witten and Sander introduced the "diffusion-limited aggregation" model[1] (DLA), many sophisticated numerical studies have been devoted to identifying and characterizing the properties of the structures generated by diffusive processes. The theoretical developments have not been quite as good: Although the essential physics underlying these phenomena has been discussed starting with the earliest papers,[1,2] in the following years there has been little analytical progress. Quite recently, Halsey[3] has proposed an interesting relation between the fractal dimensionality and the corresponding "multifractal" spectrum of exponents for the η model; in this Rapid Communication we present computer simulation results for the fractal dimensionality of the η model for a wide range of values of η, and check the theoretical predictions against the values we measure in the simulations.

The η model is a growth model introduced by Niemeyer, Pietronero, and Wiesmann[4] in 1984 to account for the patterns formed in the dielectric breakdown of insulating media in a planar geometry. It is defined by a growth rule that assigns at any point $\mathbf{x}$ of the surface S of the cluster a growth probability $P_\eta(\mathbf{x})$ proportional to the local normal electric field raised to the power η

$$P_\eta \propto [\mathbf{n}\cdot\nabla\Phi(\mathbf{x})]^\eta, \quad \mathbf{x}\in S, \tag{1}$$

where we have indicated by $\Phi(\mathbf{x})$ the electric potential at the point $\mathbf{x}$; the cluster itself is assumed to be at potential $\Phi=1$, and far from it there is an enclosing circular electrode kept at potential $\Phi=0$. This rule defines a different model for every value of η, and in all these models it is observed that the radius of gyration R and the number of particles in the cluster N are related through a power law with a nontrivial exponent $d_f(\eta)$, known as fractal dimensionality of the cluster

$$R^{d_f(\eta)} \sim N. \tag{2}$$

For $\eta=1$, one recovers the Witten-Sander model, whose fractal dimensionality in a 2D space is $d_f \simeq 1.7$; for $\eta=0$ the model is equivalent to the Eden model[5] and therefore generates compact clusters, for which $d_f=d=2$; in general, d_f appears to monotonically decrease as η increases.[4,6,7] From the study of the growth probability distribution $\{P_\eta(\mathbf{x})\}_{\mathbf{x}\in S}$ one finds a multifractal spectrum of exponents,[6,7] i.e., the different moments of the growth probability distribution scale with independent exponents $\tau_\eta(q)$

$$\int_S [P_\eta(\mathbf{x})]^q dS \sim R^{-\tau_\eta(q)}. \tag{3}$$

Besides the growth probability distribution, it is possible to consider the "harmonic measure" of the surface of the η-model clusters. The harmonic measure is the set $\{P(\mathbf{x})\}_{\mathbf{x}\in S}$ where

$$P \propto [\mathbf{n}\cdot\nabla\Phi(\mathbf{x})], \quad \mathbf{x}\in S, \tag{4}$$

and coincides with the growth probability distribution for $\eta=1$. Therefore, if we are interested in the spectrum of the harmonic measure of a generic η model, we still have to grow the clusters with rule (1), but at each stage of the growth we study the distribution of probabilities given by (4). Quite clearly the results obtained by studying the growth probability distribution and the harmonic measure are analytically related by simple transformations. In fact, if for the harmonic measure we define the scaling of the moments through $\tau_1(q)$

$$\int_S [P(\mathbf{x})]^q dS \sim R^{-\tau_1(q)}, \tag{5}$$

from the definitions, taking into account the normalization, we find[8]

$$\tau_\eta(q) = \tau_1(\eta q) - q\tau_1(\eta). \tag{6}$$

However, from a numerical point of view, it is simpler to study the harmonic measure, since it only involves the exponentiation to powers q, and not to powers ηq like the growth probability distribution.

In a recent article[3] Halsey proposed an equation of motion for the surface harmonic measure of the η model, from which, assuming for the moments of the harmonic measure the scaling form (5) in R (apart from possible logarithmic corrections), he derives the following relation:

$$d_f(\eta) = \tau_1(q)\,|_{q=\eta+2} - \tau_1(q)\,|_{q=\eta}. \tag{7}$$

Relation (7) for any η connects the fractal dimensionality $d_f(\eta)$ to the scaling exponents of the $(\eta+2)$th and the ηth moments of its harmonic measure; in its derivation it is assumed that the multifractal spectrum $\tau_1(q)$ does not vary too widely between different realizations of the aggregates.

An alternative expression for $d_f(\eta)$ is given by[8]

$$d_f(\eta) = 1 + \eta\alpha_1^{\min}(\eta) - \tau_1(q)\,|_{q=\eta}, \tag{8}$$

where, for any η, $\alpha_1^{\min}$ is the exponent that gives the scaling of the harmonic measure at the tips of the aggregate $P_{\max}$ with the linear dimension of the aggregate R

$$P_{\max} \sim R^{-\alpha_1^{\min}} . \quad (9)$$

The meaning of the relation (8) becomes more evident if we recast it in the form

$$d_f(\eta) = 1 + \alpha_\eta^{\min}(\eta) , \quad (10)$$

where $\alpha_\eta^{\min}$ is defined in terms of the growth probability distribution

$$P_{\eta,\max} \sim R^{-\alpha_\eta^{\min}} . \quad (11)$$

Equation (10) tells us that the fractal dimensionality of an aggregate grown with rule (1) is determined by the scaling exponent $\alpha_\eta^{\min}$ of the tips, the portion of it that grows with the fastest rate, as first proposed by Turkevich and Scher.[9]

To perform the simulation we exploit the electrostatic formulation of diffusive growth[4] in a form suited for the lattice version of the model. If we regard the surface of the cluster as the surface of a conductor kept at potential $\Phi =$cost, the potential in the space will be given by

$$\Phi(\mathbf{x}) = \int_S \sigma(\mathbf{x}') G(\mathbf{x},\mathbf{x}') da' , \quad (12)$$

where $\sigma(\mathbf{x}')$ is the electric charge at point $\mathbf{x}'$, $G(\mathbf{x},\mathbf{x}')$ is the Green's function appropriate to the boundary conditions of the problem, and the integration is performed on all the points belonging to the surface S. Applying the operator $\mathbf{n}\cdot\nabla$ to both sides of (12), we find that

$$\sigma(\mathbf{x}) = \mathbf{n}_x \cdot \nabla_x \Phi(\mathbf{x}) = \int_S \sigma(\mathbf{x}) \mathbf{n}_x \cdot \nabla_x G(\mathbf{x},\mathbf{x}') da' . \quad (13)$$

Equation (13) is an integral relation that immediately gives us $\sigma(\mathbf{x})$ $[\propto P(\mathbf{x})]$ once we know $G(\mathbf{x},\mathbf{x}')$. The discretized version of (13) is

$$P_i = \sum_{j=1}^{N_S} A_{i,j} P_j , \quad (14)$$

where

$$A_{i,j} = G(\mathbf{r}_i,\mathbf{r}_j) - G(\mathbf{r}_{n_i},\mathbf{r}_j) . \quad (15)$$

$G(\mathbf{r}_i,\mathbf{r}_j) = G(|\mathbf{r}_i - \mathbf{r}_j|)$ is the lattice Green's function for the square lattice,[10] $\mathbf{r}_i$ and $\mathbf{r}_j$ are the position of the sites i and j on the surface of the cluster, respectively, $\mathbf{r}_{n_i}$ is the position of the site internal to the cluster on the normal to the site i; the sum is over the N_S sites that constitute the surface S of the cluster.

At each state of the growth, that starts from a single seed site, we solve the system (14) of N_S equations in N_S unknowns to get the harmonic measure and the growth probability distribution for our cluster; the growth probability distribution is then used to add another particle to the aggregate.

The precision in the determination of the probabilities that we reach with this method is quite good (order e^{-11}); but since at each step of the process we have to invert a $N_S \times N_S$ matrix, our clusters are rather small, $N \simeq 150$. Typically we average the harmonic measure moments on 1000 different realizations of the aggregate; then we perform normal scaling analysis to find the exponents $\tau_1(q)$.[11]

The procedure just described allows us to measure all the quantities involved in relations (7) and (8). We estimate the statistical error on most of our measures to be $\sim 2\%-3\%$. It should be pointed out that $\alpha_i^{\min}$ is systematically low if compared to the value obtained by using the scaling relation

$$\eta \alpha_1^{\min} = \alpha_\eta^{\min} + \tau_1(\eta) = [d_f(\eta) - 1] + \tau_1(\eta) ,$$

and the measured values for $d_f(\eta)$ and $\tau_1(\eta)$; we do not know if this is a consequence of an error in the scaling relation, or of systematic errors in the computation. For $q \to 0$ the small probabilities count more and more in the calculation of the q moments and therefore in the determination of the $q \sim 0$ part of the $\tau_1(q)$ spectrum: since no matter how precisely we determine the growth probabilities we still have to cut the ones smaller than the smallest value we consider to be accurately calculated, we expect that the error bars in $\tau_1(q)$ for $q < 0.5$ will be somewhat larger than the ones on the rest of the curve and become worse as $q \to 0$.

It is also important to notice that the underlying lattice might introduce anisotropy effects. While these effects should be irrelevant at such small sizes for $\eta \leq 2$, they might become more important at large values of η. On the other hand, the smallness of the clusters used in our analysis does not seem to introduce systematic errors, as can be seen by comparing our results with the ones found in Refs. 7 and 11 for $\eta = 1$, and in Ref. 7, for $\eta = 2$, obtained by completely different methods for much larger aggregates.

In Fig. 1 we present our numerical results for the curve $d_f(\eta)$: as expected for $\eta \simeq 0$, $d_f \simeq 2$; then d_f monotonically decreases, touching the DLA value $d_f \simeq 1.7$ when $\eta = 1$. For $\eta > 1$, the slope of the curve decreases and d_f approaches more slowly the limiting value $d_f = 1$.

In Fig. 2 we compare the experimental curve $d_f(\eta)$ with the theoretical predictions (7) and (8), for which the

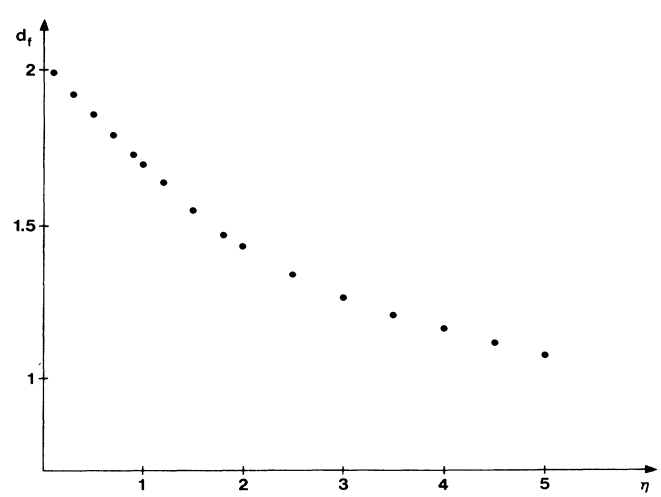

FIG. 1. $d_f(\eta)$.

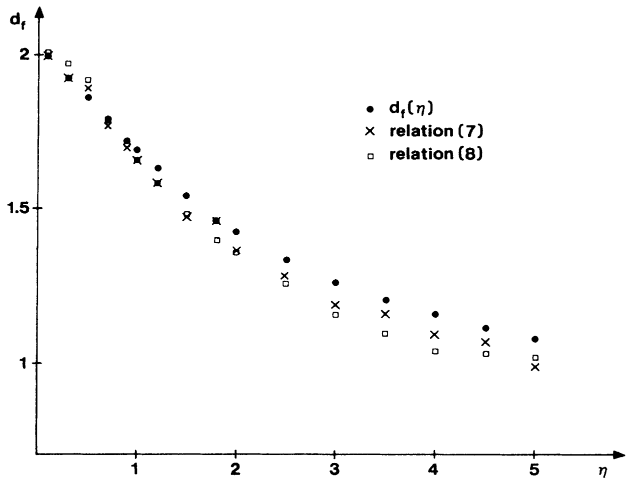

FIG. 2. Comparison between $d_f(\eta)$ and the two theoretical predictions (7) and (8).

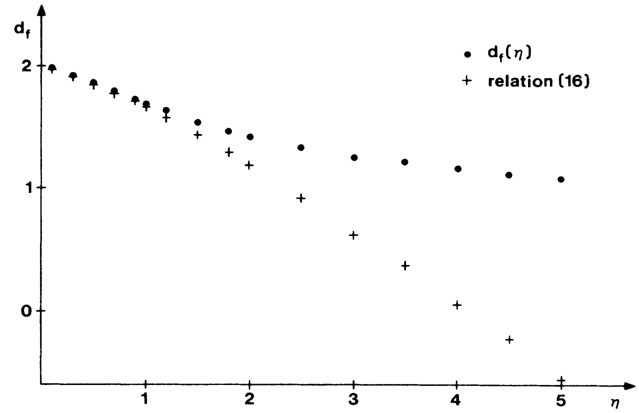

FIG. 3. Comparison between $d_f(\eta)$ and the theoretical prediction (16).

values $\tau_1(q)|_{q=\eta+2}$, $\tau_1(q)|_{q=\eta}$, and $\alpha_1^{\min}(\eta)$ have been determined by simulations. As one can directly see from the graphs, the agreement is quite good on the entire curve, always at most in a 5% error range; within the quality of the data we consider the two relations verified.

A different prediction for $d_f(\eta)$ was given by Turkevich and Scher:[9]

$$d_f(\eta)=2+\eta[\alpha_1^{\min}(\eta)-1]\,; \tag{16}$$

the comparison with measured values is offered in Fig. 3, and as forseen by Halsey,[3] the curve (16) is not consistent with the experimental d_f's for large values of η. In fact, we observe a reasonable agreement for $\eta < 1.5$, but from there on, the two graphs diverge more and more, and at $\eta = 2.5$ the fractal dimensionality given by (16) crosses the lower bound for d_f, $d_f = 1$.

In conclusion, we have presented the curve $d_f(\eta)$ for a large range of values of η; moreover, from the direct measurements of the scaling exponents of the moments of the growth probability distribution and the harmonic measure taken in computer experiments, we have been able to compare the experimental $d_f(\eta)$ with some theoretical predictions, finding good agreement with those proposed by Halsey.

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