Lyapunov exponent for products of Markovian random matrices

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We analyze the effect of finite memory on the Lyapunov exponent of products of random matrices by considering Markov trials. We study three different cases of physical interest: the one-dimensional Anderson model with correlated random potentials, light propagation in media with correlated random optical indices, and a mimic of the deterministic chaos appearing in dynamical systems with few degrees of freedom. In general the Lyapunov exponent is found to have the same qualitative shape as the inverse of the correlation length of the Markov process. We, however, observe that this rough proportionality fails in some relevant situations. We explain this unexpected behavior in the localization problem by using simple arguments.

I. INTRODUCTION

Chaotic and disordered systems can often be described by means of products of random matrices. For example, in the statistical mechanics of disordered chains, one obtains all the thermodynamic properties from the eigenvalues of the product of suitable transform matrices.[1] Random matrices have also been used to mimic strongly chaotic behavior arising in deterministic dynamics.[2-4] These matrices are usually assumed *uncorrelated*. This is a reasonable assumption in extreme cases of strong disorder or chaoticity, but in typical situations correlations are often present and may play an important role. The question arises of whether finite memory effects are relevant for the properties of products of random matrices. There are only a few studies on this subject, mainly dedicated to localization and electronic transport in media with correlated random potentials.[5-7] Naively, one could expect that the increasing of correlations leads to more coherent behaviors. In the electronic transport, e.g, this should mean that the localization length should be roughly proportional to the correlation length of the random potential. This is not always true. Indeed, it was found that the presence of correlation between impurities in the Anderson model leads to an enhancement of localization.[5] The introduction of a small amount of short-range disorder might thus be sufficient to switch the system from a metallic to an insulating regime.

The main purpose of this paper is the study of disordered and dynamical systems by means of product of random matrices, taking into account the memory effects by using Markovian processes. We shall focus our attention on the behavior of the maximal Lyapunov exponent of a product of Markovian random matrices involved in three different problems. In Sec. II we study the one-dimensional Anderson model with various correlated random potentials. We find that the naive scenario (i.e., localization length roughly proportional to the correlation length of the potential) does not hold near the band center and near the band edge of the pure system. In Sec. III we discuss the problem of light propagation in media with random optical indices. Section IV is dedicated to deterministic dynamical systems. We show that the maximum Lyapunov exponent of the Lozi map cannot be estimated by using a product of independent random matrices, while a Markovian product gives an excellent approximation. Section V is devoted to conclusions.

II. LOCALIZATION IN CORRELATED RANDOM POTENTIALS

Let us consider the one-dimensional discrete Schrödinger or nearest-neighbor tight-binding model,

$$\psi_{n+1}+\psi_{n-1}+V_n\psi_n=E\psi_n \ , \tag{2.1}$$

where ψ_n is the wave function on the nth site of a one-dimensional lattice, E is the energy, and V_n is the diagonal random potential. Equation (2.1) has been usually studied with uncorrelated potentials, i.e.,

$$\langle V_n \rangle = 0, \quad \langle V_n V_m \rangle = \sigma^2 \delta_{n,m} \ . \tag{2.2}$$

We want to consider here the effects of spatial correlations on the localization length of ψ_n. For this reason we will use a Markovian rule to generate the random potential.

By defining

$$\mathbf{z}_n = \begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix}, \quad A_n = \begin{bmatrix} E-V_n & -1 \\ 1 & 0 \end{bmatrix},$$

Eq. (2.1) can be written in matrix form as $\mathbf{z}_n = A_n \mathbf{z}_{n-1}$. The characteristic localization length ξ is then defined as the inverse of the Lyapunov exponent λ of the product of the transfer matrices A_n,

$$\xi^{-1} = \lambda = \lim_{N\to\infty} \frac{1}{N} \ln \frac{|\mathbf{z}_N|}{|\mathbf{z}_0|} \quad \text{with } \mathbf{z}_N = \prod_{n=1}^{N} A_n \mathbf{z}_0 . \tag{2.3}$$

The Oseledec's theorem,[8] applied to random matrices, ensures the existence and the uniqueness of the limit λ.

A. Two-state Markov process

As a first example, let us consider the case where the potential can take only two values, i.e., $V_n = \pm V$. For sake of simplicity we assume that the two values occur with the same probability, i.e., $p_+ = p_- = \frac{1}{2}$. With this assumption the Markov process is described by the 2×2 transition matrix,

$$W_{i,j} = \begin{bmatrix} W_{+,+} & W_{+,-} \\ W_{-,+} & W_{-,-} \end{bmatrix} = \begin{bmatrix} \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{bmatrix} , \tag{2.4}$$

where $W_{i,j}$ is the probability of jumping from the ith state to the jth (with "+" corresponding to $V_n = +V$ and and "−" to $V_n = -V$, respectively), and $0 < \epsilon < 1$. We will not consider the case of $p_+ \neq p_-$, although the extension is straightforward.

The parameter ϵ gives the amount of "memory" in the process. It is, in fact, easy to see that

$$\langle V_n V_m \rangle = V^2 (2\epsilon - 1)^{|n-m|} .$$

It is thus possible to introduce a correlation length $l(\epsilon)$ for the potential as

$$\frac{1}{l(\epsilon)} = -\lim_{|n-m|\to\infty} \frac{1}{|n-m|} \ln |\langle V_n V_m \rangle| = -\ln|2\epsilon - 1| .$$

The canonical measure of the degree of memory is, however, given by the Shannon extropy,[9]

$$H = -\sum_j p_j \sum_i W_{i,j} \ln W_{i,j} . \tag{2.5}$$

In our case, since the probabilities of the + and − states are equal to $\frac{1}{2}$, from Eq. (2.4) one gets

$$H(\epsilon) = -[\epsilon \ln\epsilon + (1-\epsilon)\ln(1-\epsilon)] .$$

We stress that the behaviors of $l(\epsilon)$ and $H^{-1}(\epsilon)$ as function of ϵ are qualitatively the same. In particular, they have a minimum at $\epsilon = \frac{1}{2}$ (corresponding to a uncorrelated situation) and a divergence as $\epsilon \to 0, 1$.

Naively one would expect that the Lyapunov exponent $\lambda(\epsilon)$ and the Shannon entropy $H(\epsilon)$ behave, as function of ϵ, in a similar way. This expectation is actually confirmed by numerical simulations, see Fig. 1, but *only* for energies "far" from the band edge ($E = +2$) and the band center ($E = 0$) of the pure system. It is worth noting that the Lyapunov exponent does not reach its maximum value (maximal localization) at $\epsilon = \frac{1}{2}$, but at a value which depends on E.

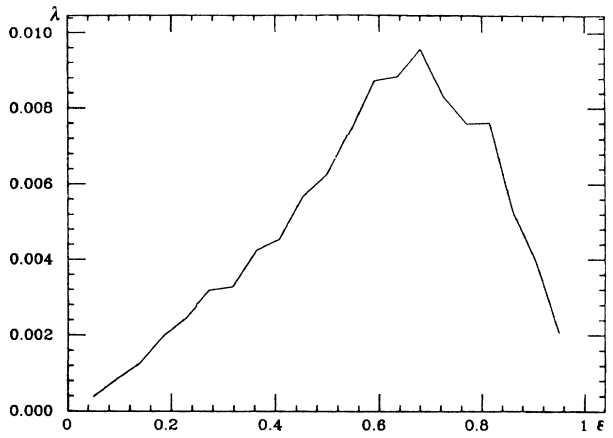

FIG. 1. λ vs ϵ for the two-state Markov process with $E = 1.8$ and $V = 0.1$.

At the center and at the edge of the band of the pure system a different behavior appears. In Fig. 2 we can observe that for $E \approx \pm 2$, λ increases with ϵ. This implies that for $\epsilon > \frac{1}{2}$ the localization length λ^{-1} decreases as the correlation length l of the potential increases. A similar surprising behavior appears also for $E \approx 0$ and $\epsilon < \frac{1}{2}$ (see Fig. 3). The behavior near the center of the band has been analyzed by Johnston and Kramer[7] by means of a perturbative approach. Here, we want to give a simple argument for it. Let us now discuss first the case of energies close to the band edge and ϵ close to 1. For small values of $1-\epsilon$, a typical sequence of V_n is made by many subsequent $+V$'s [for $\approx 1/(1-\epsilon)$ subsequent sites on the chain] followed by approximately the same number of $-V$'s, and so on, in an almost periodic way. Thus a rough estimate of the Lyapunov exponent can be obtained by considering the product

$$(A_-)^{1/(1-\epsilon)} (A_+)^{1/(1-\epsilon)} , \tag{2.6a}$$

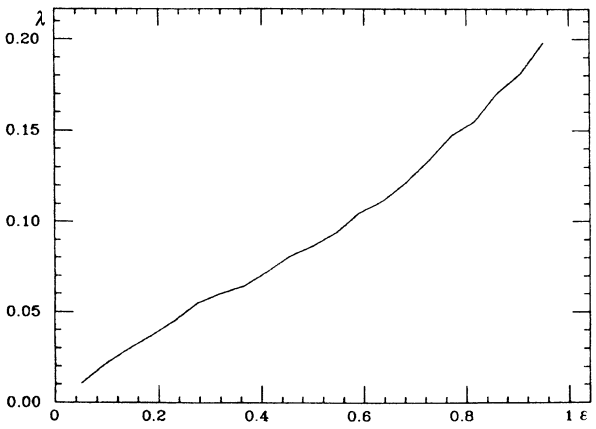

FIG. 2. λ vs ϵ for the two-state Markov process with $E = 1.99$ and $V = 0.2$.

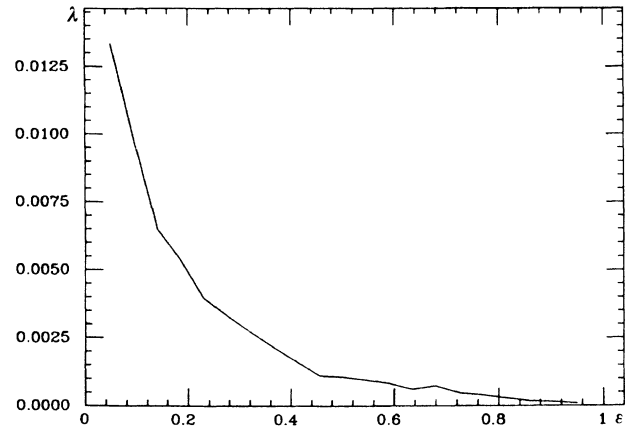

FIG. 3. λ vs ϵ for the two-state Markov process with $E=0.12$ and $V=0.1$.

where

$$A_+=\begin{bmatrix} E-V & -1 \\ 1 & 0 \end{bmatrix}, \quad A_-=\begin{bmatrix} E+V & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.6b}$$

are the transfer matrices for $V_n=+V$ and $V_n=-V$, respectively. This leads to

$$\lambda=\tfrac{1}{2}\ln|l'| , \tag{2.7}$$

where l' is the largest eigenvalue of the matrix A_-, if $E\approx 2$, or of the matrix A_+, if $E\approx -2$. The Lyapunov exponent is thus determined by the positive contribution of the eigenvalues of matrices "outside the band." We note that the density of states is nonzero, being related to the imaginary part[1] of the eigenvalues of the matrices "inside the band." One sees that as ϵ decreases below one, $\lambda(\epsilon)$ should decrease because the "typical" form of $\prod_n A_n$ will contain more and more matrices inside the band. We then conclude that in the range $2-V<|E|<2+V$, $d\lambda(\epsilon)/d\epsilon>0$.

A similar argument can be used to understand the behavior near the band center. In the limit $\epsilon=0$ the potential becomes periodic (e.g., $V_i=+V$, $V_{i+1}=-V$, $V_{i+2}=+V\cdots$). From the Bloch's theorem we know that for periodic potentials all the states are extended and organized in a band structure. Using the transfer matrix method, it is easy to realize that for our potential, the localization length and the density of states are related to the eigenvalue of product

$$A_+\cdot A_- . \tag{2.8}$$

A straightforward calculation shows that there are indeed two bands of extended states for $V<|E|<(4+V^2)^{1/2}$. Therefore, since $|E|<V$ is outside the bands, we have a positive Lyapunov exponent given by

$$\lambda=\tfrac{1}{2}\ln|l| , \tag{2.9}$$

where l is the largest eigenvalue of the matrix (2.8).

As ϵ becomes nonzero, blocks of matrices A_+ (A_-) start to appear in the product $\prod_n A_n$. For $|E|<V$, both matrices A_+ and A_- have complex conjugate eigenvalues of module one. We can thus argue that for $|E|<V$ each block do not contribute to the Lyapunov exponent of the product. Since the length of these blocks increases as ϵ grows to one, we have that $d\lambda(\epsilon)/d\epsilon<0$, for $|E|<V$.

Note that an argument similar to that used to explain the behavior near the band edge and ϵ close to 1, shows that Eq. (2.9) is a good estimation of λ also for $|E|\approx V$.

B. Markovian process with a continuum of states

Similar behaviors have been observed also for the Gaussian Markovian process,

$$V_n=aV_{n-1}+b\eta_n , \tag{2.10}$$

where a and b are constants (with $|a|<1$), and η_n Gaussian uncorrelated variables,

$$\langle\eta_n\rangle=0, \quad \langle\eta_n\eta_m\rangle=\delta_{n,m} .$$

It is straightforward to show that

$$\langle V_n V_{n+m}\rangle=\left[\frac{a}{|a|}\right]^m\sigma^2 e^{-|m|/l} , \tag{2.11}$$

where σ and l are related to a and b by

$$\sigma^2=\frac{b^2}{1-a^2}, \quad l=-\frac{1}{\ln|a|} . \tag{2.12}$$

Our numerical results show a behavior similar to the one observed for the two-state process, see, e.g., Figs. 4 and 5. In particular for energies far from the band edge, one has $\lambda^{-1}\propto l$. On the contrary, for energies close to the band edge and positive a, we see again a decreasing and eventually a saturation of the localization length as l increases. The anomalous behavior observed near the band center for the two-state process is exhibited, in this case, in the limit $a\to -1$, $b\to 0$ with σ finite, where the potential V_n becomes periodic. The origin of these behaviors is the same discussed in Sec. II A. The estimation of the

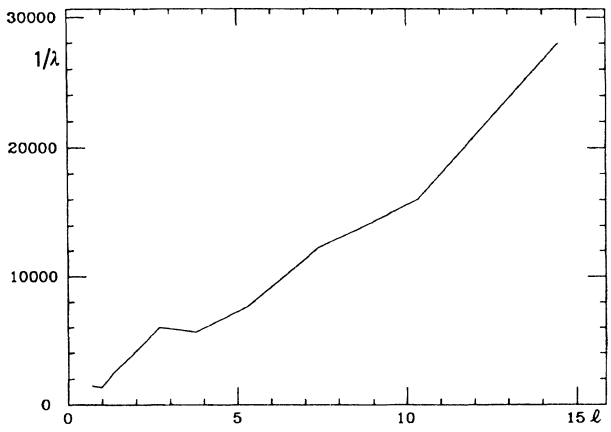

FIG. 4. λ^{-1} vs l for the process (2.10) with $E=0$ and $\sigma=0.1$.

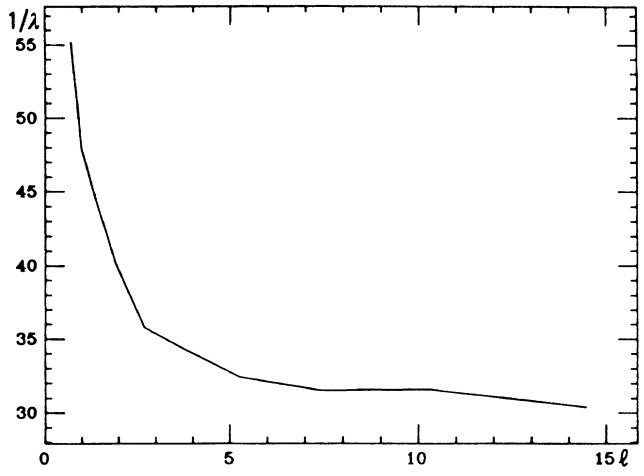

FIG. 5. λ^{-1} vs l for the process (2.10) with $E=1.9$ and $\sigma=0.1$.

limit values of λ needs a more accurate calculation to handle the limits $N\to\infty$ and $a\to\pm 1$, since in general they do not commute. In the limit $a\to 1$, $\lambda(a)$ goes to $\langle \ln|l(V)|\rangle$, where $l(V)$ is the largest eigenvalues of

$$A=\begin{bmatrix} E-V & -1 \\ 1 & 0 \end{bmatrix},$$

and the average is taken over a Gaussian distribution of V with zero mean and variance σ^2. A similar form can be obtained in the limit $a\to -1$. Note that this result is equivalent to the (2.7), where there are only two possible choices for the potential, one of which gives an eigenvalue of modulus one. The numerical results are in excellent agreement with these theoretical estimations.

III. A PROBLEM OF OPTICS

In Sec. II we have seen how the use of products of random matrices yields a quite simple way of understanding some properties of the one-dimensional Schrödinger equation with random correlated potentials. However, their use is not limited to these problems but can be extended to any problem which may be described by products of suitable transfer matrices. Here we consider the problem of the propagation of a light beam in a succession of $N\gg 1$ layers of thickness a with random correlated optical indices. The case of uncorrelated optical index has been studied by Bouchaud and Le Doussal.[10]

Let us consider a light beam, sent from an embedding medium with optical index n_0, falling on a "one-dimensional random medium" made by $N\gg 1$ layers of thickness a and optical index n_i. It is a straightforward exercise to show that the (complex) transmission and reflection coefficients t_N and r_N of the N layers are related by

$$\begin{bmatrix} t_N e^{ik_0 aN} \\ ik_0 t_N e^{ik_0 aN} \end{bmatrix} = \left[\prod_{i=1}^{N+1} Q_i\right] \begin{bmatrix} 1+r_1 \\ ik_0(1-r_1) \end{bmatrix}, \tag{3.1}$$

where

$$Q_i=\begin{bmatrix} \cos(ak_i) & \dfrac{1}{ak_i}\sin(ak_i) \\ -ak_i\sin(ak_i) & \cos(ak_i) \end{bmatrix}, \tag{3.2}$$

k_0 is the wave number of the light beam in the embedding medium, and

$$k_i^2=\left[\frac{\omega}{c}\right]^2[n_i^2-n_0^2\sin^2(\theta_0)] \tag{3.3}$$

is the wave number in the ith layer. Here ω is the light beam frequency, c the light speed in vacuum, and θ_0 the incidence angle (which is assumed to be in the interval $[0,\pi/2]$).

We are interested in studying how the penetration length ξ_p, defined as

$$\xi_p^{-1}=-\lim_{N\to\infty}\frac{1}{N}\ln|t_N|, \tag{3.4}$$

is modified by introducing a spatial correlation among the optical indices of the layers, i.e., assuming $\langle n_i n_j\rangle\neq\langle n_i\rangle^2$. Here, as in Sec. II $\langle\cdots\rangle$ indicates the average over the spatial disorder.

From the theory of products of random matrices we have that

$$|t_N|\sim e^{-N\lambda},\quad N\gg 1 \tag{3.5}$$

where λ is the maximal Lyapunov exponent of the product $\prod_i Q_i$, so that $\xi_p=1/\lambda$.

Let us assume for the moment that n_i does not fluctuate, namely, $n_i=\bar{n}<n_0$. It is well known that in this case there exists a critical incidence angle, given by $\sin\theta_c=\bar{n}/n_0$, such that the transmission coefficient, as function of the penetration depth, changes from an oscillatory behavior ($\xi_p^{-1}=\lambda=0$) for $\theta_0<\theta_c$, to an exponential decay ($\xi_p^{-1}=\lambda>0$) for $\theta_0>\theta_c$.

When n_i is a random variable, it is useful to define an "average critical incidence angle" θ_c as

$$\sin\theta_c=\frac{\langle n\rangle}{n_0}, \tag{3.6}$$

which plays a role similar to θ_c of the pure system.

We will consider n_i of the form

$$n_i=\langle n\rangle(1+\sigma x_i), \tag{3.7}$$

where x_i is a Gaussian variable with zero mean and

$$\langle x_i x_j\rangle=\exp\left[-\frac{|i-j|}{l}\right]. \tag{3.8}$$

The random process x_i has been generated according to the discretized Langevin equation (2.10). For simplicity, and without losing in generality, we may assume that $\sigma\ll 1$.

We study ξ_p as function of the optical index correlation length l. The two cases of interest are [case (1)]

$$\frac{\sin^2\theta_c-\sin^2\theta_0}{\sigma\sin^2\theta_c}\gg 1 \tag{3.9}$$

and [case (2)]

$$\frac{\sin^2\theta_c-\sin^2\theta_0}{\sigma\sin^2\theta_c}<1\ . \tag{3.10}$$

In case (1) k_i^2 defined by Eq. (3.3), is almost always positive. This implies that the matrices Q_i have complex conjugate eigenvalues with modulus one. This is a situation similar to that found in Sec. II for the Schrödinger equation with the energy "well" inside the band. We thus expect an increasing of ξ_p as function of l. In Fig. 6 we show ξ_p versus l obtained from numerical simulations for the case (1). From this figure we see that ξ_p increases almost linearly with l.

Case (2) corresponds to the case of positive and negative k_i^2. Therefore, there is a succession of transmission, in the layers where $k_i^2>0$, and suppression, where $k_i^2<0$, of the light beam. In terms of matrices Q_i this means that there is a succession of matrices with complex conjugate eigenvalues with modulus one, and matrices with real eigenvalues with the largest one of modulus larger than one. This is similar to the one-dimensional Schrödinger equation near the band edge, so that one expects that ξ_p goes to a limit value and does not increase with the correlation length l.

In the special case $\theta_0=\theta_c$, the problem becomes formally equivalent to one-dimensional localization. In fact, for small σ, we can expand the elements of Q_i. Taking only the first terms in σ one gets

$$Q_i=\begin{bmatrix}1+\frac{1}{2}\eta_i & 1+\frac{1}{6}\eta_i\\ \eta_i & 1+\frac{1}{2}\eta_i\end{bmatrix}, \tag{3.11}$$

where $\eta_i=-2\sigma(a\bar{k})^2x_i$, $\bar{k}=\langle n\rangle\omega/c$. On the other hand, Eq. (2.1) can be written as

$$\begin{bmatrix}\psi_{i+1}-\psi_i\\ \frac{1}{2}(\psi_{i+1}+\psi_i)\end{bmatrix}=\begin{bmatrix}1+\frac{1}{2}\nu_i & \nu_i\\ 1+\frac{1}{4}\nu_i & 1+\frac{1}{2}\nu_i\end{bmatrix}\begin{bmatrix}\psi_i-\psi_{i-1}\\ \frac{1}{2}(\psi_i+\psi_{i-1})\end{bmatrix}, \tag{3.12}$$

where $\nu_i=V_i-E-2$. However, in order to have a complete correspondence we have to have $\langle\nu_i\rangle=0$, i.e., $E=-2$ (remember that $\langle V_i\rangle=0$). In other words, this means that the propagation of a light beam at $\theta_0=\theta_c$ is formally equivalent to the one-dimensional Schrödinger equation at the band edge $E=-2$. In Fig. 7, we show ξ_p obtained from numerical simulations for the case (2). Let us stress that the different features of λ as function of l are not sensitive to the details of the probabilistic rules used to generate x_i. Indeed, the same qualitative results have been obtained by a two-state Markov process.

IV. MIMIC OF DYNAMICAL SYSTEMS WITH FEW DEGREES OF FREEDOM

The chaotic behavior of deterministic dynamical systems can be often analyzed in terms of suitable random processes. In particular, products of independent random matrices have been used to mimic some features of chaotic mappings. Such an approach is, in fact, quite effective to estimate the maximum Lyapunov exponent of a map by simple calculations. Chirikov[11] has used it in the standard map, when the nonlinear parameter K is large enough, and Benettin[2] has found the scaling of the maximum Lyapunov exponent in terms of a perturbation parameter in billiards. Moreover, we have shown that the product of independent random matrices are able to provide a satisfactory result for the whole spectrum of Lyapunov exponents in Hamiltonian systems made of N nearest-neighbor coupled oscillators.[3,4]

Time correlations are known to be small in all these cases. In general, however, the mimic of a system with

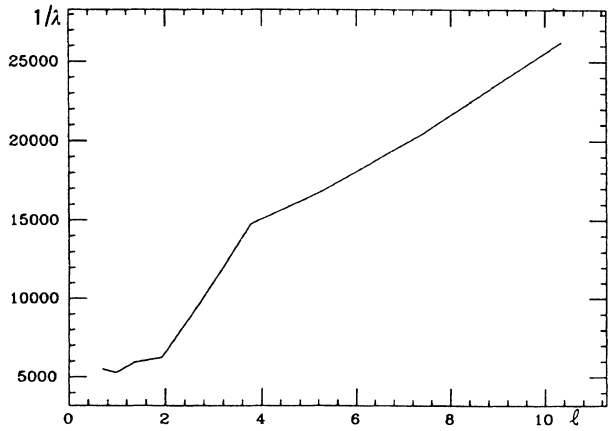

FIG. 6. λ^{-1} vs l with $\theta_c-\theta_0=0.2$ rad, $\sigma=0.1$, $n_0=1.2$, and $\langle n\rangle=1$.

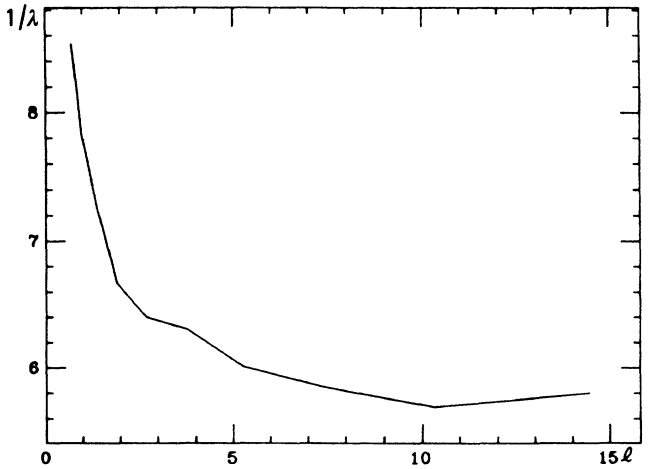

FIG. 7. λ^{-1} vs l with $\theta_c-\theta_0=0.001$ rad, $\sigma=0.1$, $n_0=1.2$, and $\langle n\rangle=1$.

non-negligible correlation effects requires a Markovian product of random matrices. To be explicit, let us consider the case of a two-dimensional map

$$\mathbf{x}(i+1)=\mathbf{F}(\mathbf{x}(i)) \quad \text{with } \mathbf{x}\in\mathbb{R}^2 , \tag{4.1}$$

with an F-invariant set Ω. The maximum Lyapunov exponent λ can be computed in terms of the linearized map,

$$A_{ij}(\mathbf{x}(n))=\frac{\partial F_i(\mathbf{x}(n))}{\partial x_j} . \tag{4.2}$$

Following the dynamics on Ω, one defines

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\ln\frac{\left\|\prod_{i=1}^{N}A(\mathbf{x}(i))z\right\|}{\|z\|} , \tag{4.3}$$

where $\mathbf{z}$ is a generic tangent vector $\in\mathbb{R}^2$ and $\mathbf{x}(i)$ belongs to Ω, for all i's. In general, the invariant set Ω is called strange if λ is positive. Assuming good ergodic properties of the systems, the limit λ exists and is the same[8] for almost all initial conditions $\mathbf{x}(1)$.

In order to illustrate the role of the correlations, we will consider the two following maps.

(i) The Baker's transformation of the square $[0,1]\times[0,1]$ into itself,

$$\left.\begin{aligned} x(n+1)&=\gamma_a x(n)\\ y(n+1)&=\frac{1}{\alpha}y(n)\end{aligned}\right\} \quad \text{if } y(n)\le\alpha , \tag{4.4a}$$

$$\left.\begin{aligned} x(n+1)&=\gamma_b x(n)\\ y(n+1)&=\frac{1}{1-\alpha}[y(n)-\alpha]\end{aligned}\right\} \quad \text{if } y(n)>\alpha , \tag{4.4b}$$

where $0\le x(n), y(n)\le 1$, $\gamma_a\le\gamma_b<0.5$, and $\alpha<0.5$.

(ii) The Lozi map,[12]

$$\begin{aligned} x(n+1)&=a|x(n)|+y(n)+1 ,\\ y(n+1)&=bx(n) .\end{aligned} \tag{4.5}$$

In both cases, there are just two possible forms for the linearized tangent maps, respectively,

$$A_+=\begin{bmatrix}\frac{1}{\alpha} & 0\\ 0 & \gamma_a\end{bmatrix}, \quad A_-=\begin{bmatrix}\frac{1}{1-\alpha} & 0\\ 0 & \gamma_b\end{bmatrix} \tag{4.6}$$

for the Baker's transformation and

$$A_+\begin{bmatrix}a & 1\\ b & 0\end{bmatrix}, \quad A_-=\begin{bmatrix}-a & 1\\ b & 0\end{bmatrix} \tag{4.7}$$

for the Lozi map. We can thus try to estimate the Lyapunov exponent by describing the dynamics of the system in terms of a sequence of the two symbols $+$ and $-$. To have a measure of the degree of randomness of $\mathbf{F}$ one can compute the topological entropy,[13] $h_t(\mathbf{F})$ which gives the estimate of the number of different trajectories of the map $\mathbf{F}$,

$$h_t=\lim_{n\to\infty}\frac{\ln\tilde{N}(n)}{n} , \tag{4.8}$$

where $\tilde{N}(n)$ is the number of possible sequences of $+$ and $-$ states of length n, generated by the map.

For the Baker's transformation $\tilde{N}(n)=2^n$ (all the possible sequences are admissible) and $h_t=\ln 2$. On the contrary, it is well known that the Lozi map has h_t smaller than $\ln 2$. This means that for n large enough, some sequences of $+$ and $-$ are now allowed. For instance, when $b=0.5$ and $a=1.7$ in (4.5), we have estimated $h_t=0.502\pm0.002$. Indeed, already for $n=5$, there are 5 forbidden sequences among the 32 possible ones (e.g., the five-minus sequence) which do not appear in the actual dynamics. Indeed, the smaller the topological entropy, the larger the expected correlation effects.

It has been proven that the Baker's transformation can be described by a Bernoulli trial where the probability of extracting a matrix A_+ is $p_+=\alpha$. Moreover, since the matrices A_+ and A_- commute, it is easy to see that the Lyapunov exponent of the Baker's transformation is

$$\begin{aligned}\lambda&=-[p_+\ln(\alpha)+p_-\ln(1-\alpha)]\\ &=-[\alpha\ln(\alpha)+(1-\alpha)\ln(1-\alpha)] .\end{aligned} \tag{4.9}$$

On the other hand, the linearized matrices of the Lozi map do not commute and the Bernoulli trials of matrices has to be done numerically, even if, in principle, good analytical approximations are available.[14] We have computed the probability p_+ from the frequency of A_+ in the dynamics of the Lozi map, as function of the parameter a, at fixed dissipation rate $b=0.5$. In Fig. 8, one sees

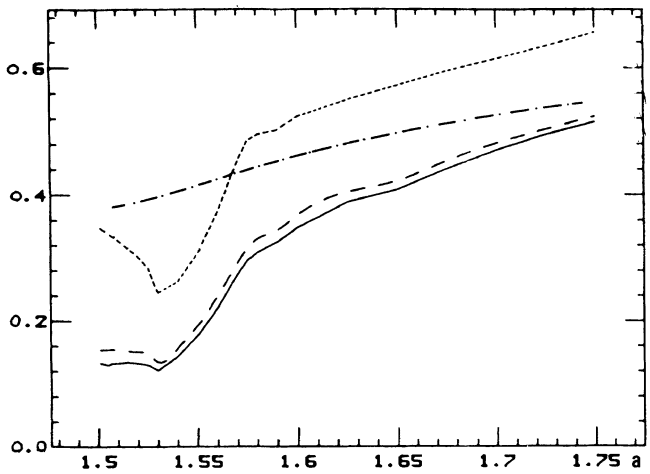

FIG. 8. Numerical computation of the Lyapunov exponent of the Lozi map (4.5) (solid line), compared with the Lyapunov exponent of a product of the two matrices A_+ and A_- given by (4.7) extracted according to a Bernoulli distribution (dot-dashed line) and according to a one-step Markov rule (dashed line) vs parameter a. The dotted line is the Shannon entropy of the one-step Markov process. The top of the vertical axis is given by $\ln 2$, the entropy of a Bernoulian process with two states of equal probability.

that the Lyapunov exponent of the product of independent random matrices cannot give an acceptable approximation of the Lyapunov exponent of the Lozi map. This difference becomes very significant for a close to the lower threshold $a \approx 1.51$, below which there is no strange attractor, since in this case the number of forbidden symbol sequences increases, e.g., for $a = 1.51$ one has $h_t \approx 0.16$. To reproduce by a random process the time correlations, we have numerically computed the transition matrix

$$W_{i,j} = \begin{bmatrix} W_{+,+} & W_{+,-} \\ W_{-,+} & W_{-,-} \end{bmatrix}, \tag{4.10}$$

where $W_{+,+}$ is the probability that A_+ follows A_+ and $W_{-,+}$ is the probability that A_+ follows A_-. It is worth stressing that for $a \leq 1.54$ one has $W_{-,-} = 0$. Indeed, Fig. 8 shows that the Lyapunov exponent of the Markovian random product is an excellent approximation of the Lyapunov exponent of the Lozi map. Note that the Shannon entropy of the corresponding one-step Markov process has the same qualitative shape of the Lyapunov exponent, as found for localization inside the band (see Fig. 1).

It is remarkable that one needs just a one-step Markov process to obtain the main features of the deterministic chaos exhibited by the Lozi map on the whole range of the permitted a values. The Bernoulli trial is practically equivalent to the Markov trial only for values of a where the Shannon entropy is close to ln2, i.e., where the elements of the one-step transition matrix $W_{i,j}$ become close to p_j. Unfortunately, it is not trivial to extend this approach to systems where the tangent map does not assume a finite number of possibilities, as, e.g., the Henon map. In these cases, one has to find a suitable partition of the attractor of the map and assign a particular tangent matrix to each element of the partition. The numerical calculation of the transition matrix $W_{i,j}$ thus becomes rather complicated and tedious, even if feasible. Moreover, the choice of the "good" partition can become crucial. In fact, it is necessary to find a suitable partition which gives a good description with a reasonable number of elements so that the implementation of the Markovian random process is not impossible.

V. CONCLUSIONS

We have studied the behavior of the maximum Lyapunov exponent of products of Markovian random matrices as function of the memory of the process, e.g., the correlation length or the Shannon entropy. Usually the Lyapunov exponent is proportional of to the inverse of the correlation length, as naively expected. However, in some relevant cases an opposite behavior is observed. The Lyapunov exponent increases with the correlation length of the process, and eventually saturates to an asymptotic value. These results, in general, do not depend on the details of the probabilistic rule used.

We stress that it is possible to build a suitable process such that $\lambda = 0$. For example, in one-dimensional localization, one can construct a correlated random potential with $\lambda = 0$ for a particular value of energy.[15] However, this is a very peculiar case which disappears under slight modifications either of the energy values or of the probability rule used to generate the potential.

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