PHYSICAL REVIEW A VOLUME 39, NUMBER 12 JUNE 15, 1989

Quantum dissipation for the kicked particle

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(Received 3 November 1988)

The evolution of the density matrix is calculated in the Wigner representation for the periodically kicked particle $[\mathcal{H}=\frac{1}{2}p^2+\frac{1}{2}Kx^2\sum_n\delta(t-nT)]$ in the presence of dissipation. The dissipation is introduced via a linear coupling to a bath of harmonic oscillators. The behavior in various regions of the parameter space for the problem is analyzed in detail and compared with the Markovian approximation. The degree of agreement with the Markovian approximation is classified. In various regions in parameter space the model reduces to various examples that were studied in the past (damped particle, harmonic oscillator). Most regions of the parameter space were not explored in the past and are of potential interest. One of these regions exhibits a limit cycle.

I. INTRODUCTION

Chaotic systems were investigated extensively in recent years. Hamiltonian systems are characterized by diffusion in phase space while dissipative systems are characterized by strange attractors.[1–3] For both types of systems simple distributions of initial conditions develop complicated structures in phase space in the course of the evolution.[4] Introduction of external noise into such systems strongly affects their behavior.[5,6]

In quantum mechanics, dissipation results inevitably from the coupling of the system to many degrees of freedom. The coupling introduces noise into the system in addition to dissipation of energy. This noise can have a crucial effect on quantum interference. Some aspects of the coupling to the external bath were modeled by incomplete treatment of the quantum correlations.[7]

A prototype system for the investigation of the quantal behavior of classically chaotic systems is the periodically kicked rotor. In the absence of any coupling to external systems it exhibits localization in momentum that is similar to Anderson localization in disordered solids.[8,9] External noise destroys this localization in the same way that it is destroyed by phonons in real solids.[10,11] For this system, however, if noise arises from the coupling to an external bath, it leads to dissipation that finally suppresses the diffusion in momentum.[12] Hence the coupling to the bath has two competing effects.

A common assumption in most of the earlier work is that one may use the Markovian approximation for the calculation of the time evolution of such systems, i.e., it is assumed that one may use the same evolution law to propagate the system iteratively over and over again.[11,12] Physically dissipation arises from coupling of the system to degrees of freedom of some bath, thus the Markovian approximation is not self-evident, and one may expect manifestation of some memory effects.

It seems that the most elegant way to introduce dissipation into a system is by the formalism of Feynman and Vernon[13] which was followed by work of Cardeira and Leggett.[14] The formalism has been applied to investigate the relaxation problem of a damped harmonic oscillator,[14] the diffusion of a Brownian particle,[15] and the tunneling problem.[16] The formalism has not been applied yet, as far as we know, to time-dependent problems, in particular, to such that are chaotic. An attractive way to investigate the time evolution of a particle is by the Wigner representation.[17] It emerges, that for an isolated particle which is described by a quadratic Hamiltonian, the quantum propagator of Wigner's function is identical to the Liouville propagator of a classical distribution in phase space. For nonquadratic potentials the structure of the propagator is more complicated. This was demonstrated[4] for a particle that is periodically kicked by a quartic potential, which is a chaotic system.

In the present paper we will investigate the quantal behavior of a free particle that is periodically kicked by a quadratic potential, in the presence of dissipation. Without dissipation it is an exactly solvable problem.[18,19] It exhibits a transition between stable and unstable motion. When the system is embedded in a box the motion in the unstable regime becomes chaotic with the Lyapunov exponent of the unbounded system.[20] Therefore, the solution of this problem will hopefully enable us to make progress in the understanding of dissipation for simple quantum systems that are chaotic in the classical limit.

The outline of the paper is as follows: In Sec. II the general formalism is presented in the Wigner representation. The Markovian approximation and its implications are outlined in Sec. III. The formalism is applied to some simple examples in Sec. IV, namely the damped particle and the damped oscillator. The periodically kicked particle that is the main subject of the paper is investigated in Sec. V. Finally, the results are summarized and discussed in Sec. VI.

II. WIGNER REPRESENTATION OF THE QUANTUM DISSIPATION FORMALISM

In this section the propagator of a particle that is coupled to a bath of oscillators is calculated in the Wigner representation,[17] using results that were obtained by Feynman and Vernon[13] and by Caldeira and Leggett.[14]

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We are interested in the evolution of a system whose unperturbed Hamiltonian is

$$\mathcal{H}_0=\tfrac{1}{2}p^2+V_0(x;\tau)\ , \tag{2.1}$$

where x and p are conjugate coordinates. The bath is defined by the Hamiltonian

$$\mathcal{H}_B=\sum_\alpha \frac{p_\alpha^2}{2m_\alpha}+\tfrac{1}{2}m_\alpha\omega_\alpha^2 q_\alpha^2\ . \tag{2.2}$$

The coupling of the system to the bath is assumed to be linear, namely

$$\mathcal{H}_I=x\sum_\alpha C_\alpha q_\alpha\ , \tag{2.3}$$

where C_α are coupling constants. The total Hamiltonian which describes the system and the bath is

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I+\mathcal{H}_B\ . \tag{2.4}$$

The state of the system whose unperturbed Hamiltonian is (2.1) will be described by the reduced probability density matrix

$$\rho(R,r)\equiv\rho(x'',x')\equiv\langle x''|\rho|x'\rangle\ , \tag{2.5}$$

where we introduced the variables

$$R=\tfrac{1}{2}(x''+x')\ , \tag{2.6}$$

$$r=x''-x'\ . \tag{2.7}$$

In the Wigner representation the density matrix (2.5) is

$$\rho(R,P)\equiv\int dr\, e^{-iPr}\rho(R,r)\ . \tag{2.8}$$

Note that various functions are denoted by ρ, but they will be distinguished in what follows by their arguments.

It is assumed that initially (at $t=0$) the density matrix of the system and bath is factorized, namely

$$\rho_{t=0}(x'',q'';x';q')=\rho_{t=0}(x'',x')\rho_{\rm eq}(q'',q')\ , \tag{2.9}$$

where $\rho_{t=0}(x'',x')$ represents arbitrary preparation of the system and

$$\rho_{\rm eq}(q'',q')=\langle q''|e^{-\beta\mathcal{H}_B}|q'\rangle/{\rm tr}(e^{-\beta\mathcal{H}_B}) \tag{2.10}$$

represents the bath in canonical thermal equilibrium. The reciprocal temperature of the bath is β. Units where $\hbar=1$ will be used in this work. Using the formalism of Feynman and Vernon[13] the propagator $J(R,P;R_0,P_0)$ of the density matrix in the Wigner representation is computed. Thus the evolution of the system is given by

$$\rho_t(R,P)=\int J(R,P;R_0,P_0)\rho_{t=0}(R_0,P_0)dR_0dP_0\ . \tag{2.11}$$

In what follows the recipe for the calculation of the propagator $J(R,P;R_0,P_0)$ is outlined. It is based on work by Feynman and Vernon that was followed by Caldeira and Leggett.[14] Note that the notations are closer to those of the later reference (Eq. 3.18 there). The expression for the propagator is

$$J(R,r;R_0,r_0)=\int_{R_0}^{R}\int_{r_0}^{r}DR\,Dr\,e^{iS_{\rm eff}[R,r]-S_N[r,r]}\ , \tag{2.12}$$

where

$$S_{\rm eff}[R,r]\equiv S_0[x'']-S_0[x'] +\int_0^t\int_0^\tau d\tau\, d\tau' 2\alpha(\tau-\tau')r(\tau)R(\tau')\ , \tag{2.13}$$

$$S_N[r,r]\equiv\tfrac{1}{2}\int_0^t\int_0^t d\tau\, d\tau'\phi(\tau-\tau')r(\tau)r(\tau')\ , \tag{2.14}$$

with

$$\alpha(\tau-\tau')=\int_0^\infty d\omega\frac{1}{\pi}J(\omega)\sin[\omega(\tau-\tau')]\ , \tag{2.15}$$

and

$$\phi(\tau-\tau')=\int_0^\infty d\omega\frac{1}{\pi}J(\omega)\coth(\tfrac{1}{2}\beta\omega)\cos[\omega(\tau-\tau')]\ . \tag{2.16}$$

The distribution of the oscillators of the bath is chosen such that

$$J(\omega)\equiv\frac{\pi}{2}\sum_\alpha\frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha)\ . \tag{2.17}$$

The action of the unperturbed system defined by $\mathcal{H}_0$ is

$$S_0[x]=\int_0^t d\tau[\tfrac{1}{2}\dot{x}^2-V_0(x;\tau)] \tag{2.18}$$

finally, in the Wigner representation the propagator is

$$J(R,P;R_0,P_0) =\int\int dr\, dr_0 e^{-iPr}J(R,r;R_0,r_0)e^{iP_0r_0}\ . \tag{2.19}$$

In what follows it will be shown that $S_{\rm eff}$ determines the classical trajectory in presence of dissipation. Feynman and Vernon had shown that S_N of (2.14) can be interpreted as a noise term. It may arise from introduction of "Gaussian noise" with the correlation function $\phi(\tau-\tau')$ into the system. If $S_{\rm eff}$ is replaced by $S|_F$, namely

$$S|_F\equiv S_{\rm eff}+\int_0^t d\tau F(\tau)x(\tau)\ , \tag{2.20}$$

where $F(\tau)$ is a random force satisfying

$$\overline{F(\tau)}=0\ , \tag{2.21}$$

$$\overline{F(\tau)F(\tau')}=\phi(\tau-\tau')\ , \tag{2.22}$$

then the average of the propagator over realizations of this force is identical to (2.12). We shall see that in terms of the Wigner representation S_N is responsible for the diffusion of a wave packet in phase space, thus establishing correspondence with the Langevin formalism.

In order to make further progress, one has to specify the distribution (2.17) of the oscillators. Following Caldeira and Leggett[14] we will use one that is linear for frequencies that are smaller than some cutoff ω_c and that vanishes above this cutoff. It is convenient to introduce an exponential cutoff

$$J(\omega)=\eta\omega e^{-\omega/\omega_c}\ . \tag{2.23}$$

It will become evident that this distribution results in a classical friction with the coefficient η. With this distribution one finds

$$\alpha(\tau)=-\eta\frac{\partial}{\partial\tau}\left[\frac{1}{\pi}\frac{\tau_c}{\tau_c^2+\tau^2}\right]\ , \tag{2.24}$$

and

$$\phi(\tau)=\frac{\eta}{\pi}\left[\frac{\tau_c^2-\tau^2}{(\tau_c^2+\tau^2)^2}\right]+2\frac{\eta}{\beta}\frac{1}{2\pi\beta}\left[\frac{1}{(\tau/\beta)^2}-\left[\frac{\pi}{\sinh(\pi\tau/\beta)}\right]^2\right], \quad (2.25)$$

where $\tau_c=1/\omega_c$. In the limit $\tau_c\to 0$, (2.24) takes the form

$$\alpha(\tau)=-\eta\frac{\partial}{\partial\tau}\delta(\tau), \quad (2.26)$$

consequently S_{eff} is local in time. $\phi(\tau)$ has two regimes of behavior, the short time where

$$\phi(\tau)\simeq-\frac{\eta}{\pi}\frac{1}{\tau^2}, \quad \tau_c\ll\tau\ll\beta \quad (2.27)$$

and the long-time regime

$$\phi(\tau)\simeq-2\frac{\eta}{\beta}\frac{2\pi}{\beta}e^{(-2\pi/\beta)\tau}, \quad \beta\ll\tau. \quad (2.28)$$

It satisfies the sum rule

$$\int_0^\infty d\tau\phi(\tau)=2\frac{\eta}{\beta}. \quad (2.29)$$

We shall see (Sec. III) that for high temperature it reduces to a δ function, i.e.,

$$\phi(\tau)=2\frac{\eta}{\beta}\delta(\tau), \quad (2.30)$$

while at zero temperature

$$\phi(\tau)=\frac{\eta}{\pi}\left[\frac{\tau_c^2-\tau^2}{(\tau_c^2+\tau^2)^2}\right]\equiv\frac{\eta}{\pi}\phi_0(\tau). \quad (2.31)$$

The function $\phi_0(\tau)$ will be studied in detail in Appendix A.

In what follows we assume ω_c to be large compared to the dynamical time scales that appear in S_{eff}. Substitution of (2.18) and (2.24) in the general expression (2.13) yields after integration by parts

$$S_{\text{eff}}=\int_0^t dt\,[\dot{R}\dot{r}-\eta\dot{R}r-V(R+\tfrac{1}{2}r)+V(R-\tfrac{1}{2}r)-\eta R_0 r_0], \quad (2.32)$$

where V is the renormalized potential

$$V(x;\tau)\equiv V_0(x;\tau)-\frac{1}{\pi}\eta\omega_c x^2. \quad (2.33)$$

The last term of (2.32) may be factored out of the propagator and has the effect of operating on the initial state with

$$J_{\text{switching}}(R_0,R_0;R_-,P_-)=2\pi\delta(P_0-(P_--\eta R_-))\delta(R_0-R_-). \quad (2.34)$$

In this work we confine ourselves to potentials of the form

$$V(x;\tau)=-f_1(\tau)x+\tfrac{1}{2}f(\tau)x^2. \quad (2.35)$$

Thus the final path-integral expression, dropping the switching term, takes the form (2.12) with (2.13) replaced by

$$S_{\text{eff}}=\int_0^t d\tau[\dot{R}\dot{r}-\eta\dot{R}r-f(\tau)Rr+f_1(\tau)r], \quad (2.36)$$

and where $\phi(\tau-\tau')$ in (2.14) is given by (2.25).

For the explicit calculation of the propagator, one has to find the classical path $[R^{\text{cl}},r^{\text{cl}}]$ for which S_{eff} is stationary. Using (2.36) the resulting equations for the classical path are

$$\ddot{R}+\eta\dot{R}+f(\tau)R=f_1(\tau), \quad (2.37)$$

$$\ddot{r}-\eta\dot{r}+f(\tau)r=0, \quad (2.38)$$

with the boundary conditions

$$R(0)=R_0, \quad R(t)=R, \quad (2.39)$$

$$r(0)=r_0, \quad r(t)=r. \quad (2.40)$$

One can prove (Appendix B) that for the quadratic potential (2.35) the path integral (2.12) leads to

$$J(R,r;R_0,r_0)=Ne^{iS_{\text{eff}}(R,r;R_0,r_0)-S_N(r,r_0)}, \quad (2.41)$$

with S_{eff} and S_N of the general form

$$S_{\text{eff}}=(a_{ff}R+a_{fi}R_0+a_f)r+(a_{if}R+a_{ii}R_0+a_i)r_0, \quad (2.42)$$

$$S_N=\tfrac{1}{2}(b_f r^2+2brr_0+b_i r_0^2), \quad (2.43)$$

and N is a constant independent of $(R,r;R_0,r_0)$. The a's will be called convection coefficients and the b's will be called diffusion coefficients. The motivation for this terminology will become evident below.

The Wigner representation of the propagator (2.41) is obtained using (2.19)

$$J(R,P;R_0,P_0)=\bar{N}\exp\left[-\frac{1}{2}\frac{1}{b_f b_i-b^2}\begin{bmatrix}U\\W\end{bmatrix}^t\begin{bmatrix}b_i & -b\\-b & b_f\end{bmatrix}\begin{bmatrix}U\\W\end{bmatrix}\right], \quad (2.44)$$

where

$$U=a_{ff}R+a_{fi}R_0-P+a_f, \quad (2.45)$$

$$W=a_{if}R+a_{ii}R_0+P_0+a_i. \quad (2.46)$$

The diffusion results from S_N. If the b's vanished, the propagator (2.44) would reduce to the form

$$J(R,P;R_0,P_0)=\tilde{N}\delta(U)\delta(W)=\delta(P-P_t(R_0,P_0))\delta(R-R_t(R_0,P_0)), \quad (2.47)$$

where $R_t(R_0,P_0)$ and $P_t(R_0,P_0)$ may be found by solving the coupled equations $U=0$ and $W=0$ for R,P. In the presence of the noise term S_N, i.e., when the b's do not vanish, the propagator is a Gaussian with finite width that is peaked at R_t,P_t. Formally, one may write the propagator (2.44) in terms of the deviation from R_t,P_t

$$J(R,P;R_0,P_0)$$
$$=N\exp\left[-\frac{1}{2}\begin{bmatrix}R-R_t\\P-P_t\end{bmatrix}^t\begin{bmatrix}\Delta_R & -\Delta\\-\Delta & \Delta_p\end{bmatrix}\begin{bmatrix}R-R_t\\P-P_t\end{bmatrix}\right], \quad (2.48)$$

with

$$\Delta_R=\frac{1}{b_ib_i-b^2}(a_{ff}^2b_i-2a_{ff}a_{if}b+a_{if}^2b_f), \quad (2.49)$$

$$\Delta_P=\frac{1}{b_fb_i-b^2}(b_i), \quad (2.50)$$

$$\Delta=\frac{1}{b_fb_i-b^2}(a_{ff}b_i-a_{if}b), \quad (2.51)$$

and note

$$\Delta_R\Delta_P-\Delta^2=\frac{1}{b_fb_i-b^2}(a_{if}^2). \quad (2.52)$$

Consider the nonphysical preparation

$$\rho_{t=0}(R,P)=2\pi\delta(P-P_0)\delta(R-R_0). \quad (2.53)$$

Operating on this distribution with the propagator (2.48), the resulting distribution is a Gaussian centered at R_t,P_t with a spread in R and P given by

$$\sigma_R^2\equiv\langle(R-R_t)^2\rangle=\frac{\Delta_P}{\Delta_R\Delta_P-\Delta^2}=\left[\frac{1}{a_{if}}\right]^2b_i, \quad (2.54)$$

$$\sigma_P^2\equiv\langle(P-P_t)^2\rangle$$
$$=\frac{\Delta_R}{\Delta_R\Delta_P-\Delta^2}$$
$$=\left[\frac{a_{ff}}{a_{if}}\right]^2b_i-2\left[\frac{a_{ff}}{a_{if}}\right]b+b_f. \quad (2.55)$$

If one takes a physical Gaussian preparation, operating on it with the propagator (2.48), the resulting state is a convolution of Gaussians, leading to a spread

$$\sigma_x^2\equiv\langle(x-\langle x\rangle)^2\rangle=\sigma_{\bar R}^2+\sigma_R^2, \quad (2.56)$$

$$\sigma_p^2\equiv\langle(p-\langle p\rangle)^2\rangle=\sigma_{\bar P}^2+\sigma_P^2, \quad (2.57)$$

where $\sigma_{\bar R}^2,\sigma_{\bar P}^2$ are defined to be σ_x^2,σ_p^2 for the problem with no diffusion. In particular $\sigma_{\bar R}^2=\sigma_{\bar P}^2=0$ for the unphysical preparation (2.53) that was discussed previously.

Let us now elaborate on the calculation of the convection coefficients. Formally, one should take the classical path (2.37)–(2.40) and substitute it in the functional $S_{\rm eff}[R,r]$ to obtain $S_{\rm eff}(R,r;R_0,r_0)$ and then to identify the a's. Alternatively, exploiting the insight obtained from the Wigner representation, one may use the following prescription. One solves Eq. (2.37) for given initial conditions and writes the solution in the form

$$\begin{bmatrix}R_t\\P_t\end{bmatrix}=\begin{bmatrix}a_{rr} & a_{rp}\\a_{pr} & a_{pp}\end{bmatrix}\begin{bmatrix}R_0\\P_0\end{bmatrix}+\begin{bmatrix}a_r\\a_p\end{bmatrix}. \quad (2.58)$$

Note that only a_r and a_p depend on the force $f_1(\tau)$. Now, one casts the solution (2.58) into the forms [compare with (2.45)–(2.46)]

$$U\equiv a_{ff}R_t+a_{fi}R_0-P_t+a_f=0, \quad (2.59)$$

$$W\equiv a_{if}R_t+a_{ii}R_0+P_0+a_i=0. \quad (2.60)$$

Identification of the coefficients enables one to express the a's of (2.59) and (2.60) in terms of the a's of (2.58). In particular,

$$a_{ff}=\left[\frac{a_{pp}}{a_{rp}}\right],\quad a_{if}=-\left[\frac{1}{a_{rp}}\right]. \quad (2.61)$$

Let us now turn to find the diffusion coefficients. One takes the classical path (2.38) and (2.40), substitutes it in the functional $S_N[r,r]$ to obtain $S_N(r,r_0)$, and then identifies the b's. It is convenient to write the solution of (2.38) and (2.40) in the form

$$r(\tau)=r_0C_i(\tau)+rC_f(\tau), \quad (2.62)$$

where $C(\tau)$ are solutions of (2.38) satisfying the boundary conditions

$$C_i(0)=1,\quad C_i(t)=0, \quad (2.63)$$

$$C_f(0)=0,\quad C_f(t)=1. \quad (2.64)$$

Thus one obtains (2.43) with the diffusion coefficients

$$b_i=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C_i(\tau)C_i(\tau'), \quad (2.65)$$

$$b=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C_i(\tau)C_f(\tau'), \quad (2.66)$$

$$b_f=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C_f(\tau)C_f(\tau'). \quad (2.67)$$

Substituting (2.61) and (2.65)–(2.67) in the formulas (2.54) and (2.55) for σ_R^2 and σ_P^2, one obtains

$$\sigma_R^2=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C_R(\tau)C_R(\tau'), \quad (2.68)$$

$$\sigma_P^2=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C_P(\tau)C_P(\tau'), \quad (2.69)$$

where

$$C_R(\tau)\equiv a_{rp}C_i(\tau) \quad (2.70)$$

and

$$C_P(\tau)\equiv a_{pp}C_i(\tau)+C_f(\tau). \quad (2.71)$$

Note that $C_R(\tau)$ and $C_P(\tau)$ are solutions of (2.38) satisfying the boundary conditions

$$C_R(0)=a_{rp},\quad C_R(t)=0, \quad (2.72)$$

$$C_P(0)=a_{pp},\quad C_P(t)=1. \quad (2.73)$$

We conclude this section noting a simplification which emerges in cases where $f(\tau)$ is symmetric in time.

Let $a(t)$ be a solution of

$$\ddot R+\eta\dot R+f(\tau)R=0 \quad (2.74)$$

with initial conditions

$$a(0)=0,\quad \dot a(0)=1. \quad (2.75)$$

Then, we claim that definition (2.58) of the a's implies

$$a_{rp}=a(t)\ , \tag{2.76}$$

$$a_{pp}=\dot{a}(t)\ . \tag{2.77}$$

Furthermore, from the latter definitions of $C_R(\tau)$ it emerges that if $f(t-\tau)=f(\tau)$, as is the case in the following applications, then

$$C_R(\tau)=a(t-\tau)\ . \tag{2.78}$$

For constant $f(\tau)$

$$C_P(\tau)=\dot{a}(t-\tau) \tag{2.79}$$

holds as well. Substitution of (2.78) in (2.68) yields

$$\sigma_R^2(t)=\int_0^t\int_0^t d\tau\, d\tau'\phi(\tau-\tau')a(\tau)a(\tau')\ . \tag{2.80}$$

The advantage of this formula, compared to (2.68), is that $a(\tau)$ does not depend implicitly on the elapsed time t.

III. MEMORY EFFECTS AND THE MARKOVIAN APPROXIMATION

In this section memory effects on the propagator will be analyzed. We start the analysis with the investigation of S_{eff}, which is a local functional and therefore does not lead to any memory effects, and proceed with the analysis of S_N, which may lead to memory effects due to its nonlocal character.

The action S_{eff} is a local additive functional of the path, i.e.,

$$S_{\text{eff}}[\text{path};t_2,t_0]=S_{\text{eff}}[\text{path};t_2,t_1]+S_{\text{eff}}[\text{path};t_1,t_0]\ ,\quad t_0<t_1<t_2\ . \tag{3.1}$$

This is a special feature of the choice $J(\omega)=\eta\omega$ (Ohmic dissipation). Therefore, had the term S_N been absent from the action, the propagator would have had the group property

$$\begin{aligned}J(R_2,P_2;R_0,P_0;t_2,t_0)&=\int\int dR_1dP_1J(R_2,P_2;R_1,P_1;t_2,t_1)\\&\quad\times J(R_1,P_1;R_0,P_0;t_1,t_0)\ .\end{aligned} \tag{3.2}$$

The term S_N is responsible for the fact that the propagator does not satisfy (3.2). Physically this is due to the fact that at $t=t_1$ the state of the system and bath is no longer factorized as in (2.9).

In some problems the time interval $[0,t]$ is divided into subintervals of length T in a natural way, i.e., $t=NT$. If one assumes that the group property holds, one obtains the Markovian approximation. It neglects correlations between the various subintervals. In this approximation

$$\tilde{J}(N)=J(1)*\cdots*J(1)*J(1)\ , \tag{3.3}$$

where $J(1)$ is a propagator for a subinterval of time T, and $*$ denotes the convolution of (3.2). The resulting path-integral expression for the propagator of N steps is

$$\tilde{J}(R,r;R_0,r_0)=\int_{R_0}^{R}\int_{r_0}^{r}DR\,Dr\,e^{iS_{\text{eff}}[R,r]-\tilde{S}_N[r,r]}\ , \tag{3.4}$$

with

$$\tilde{S}_N[r,r]=\sum_{n=1}^{N}\tfrac{1}{2}\int_{(n-1)T}^{nT}\int_{(n-1)T}^{nT}d\tau\,d\tau'\phi(\tau-\tau')r(\tau)r(\tau')\ , \tag{3.5}$$

and thus for the spread one gets the following formula,

$$\tilde{\sigma}^2(t)=\sum_{n=1}^{N}\int_{(n-1)T}^{nT}\int_{(n-1)T}^{nT}d\tau\,d\tau'\phi(\tau-\tau')C(\tau)C(\tau')\ , \tag{3.6}$$

where $C(\tau)$ is either C_R or C_P of (2.70) and (2.71). It should be compared with the exact results,

$$\sigma^2(t)=\int_0^t\int_0^t d\tau\,d\tau'\phi(\tau-\tau')C(\tau)C(\tau')\ , \tag{3.7}$$

and thus it enables one to check quantitatively the validity of the Markovian approximation.

We turn now to a general discussion concerning the expected behavior of $\sigma^2(t)$ and $\tilde{\sigma}^2(t)$. For this purpose we define the following time scales: τ^*, the typical time scale over which variation in $C(\tau)$ takes place; and β, the reciprocal temperature of the bath. We denote by t the total time during which the evolution takes place. A general assumption is $\tau^*\ll t$; otherwise the definition of τ^* is meaningless. In what follows some of the typical cases are discussed.

At high temperatures, i.e., when $\beta\ll\tau^*$, $\phi(\tau-\tau')$ may be approximated by the δ function of (2.30). Since S_N is local in this case, the propagator satisfies the group property (3.2). Consequently, for high temperatures (HT) the Markovian approximation is exact and formulas (3.6) and (3.7) take the form

$$\sigma^2(t)|_{\text{HT}}=\tilde{\sigma}^2(t)|_{\text{HT}}=2\frac{\eta}{\beta}\int_0^t|C(\tau)|^2d\tau\ . \tag{3.8}$$

At low temperatures, i.e., when $\tau^*\ll\beta$, one has to distinguish between the short-time regime $\tau^*\ll t<\beta$ and the long-time regime $\beta\leq t$. For short times the bath may be considered as if it was at zero temperature (ZT), because the tail of $\phi(\tau)$ does not manifest itself. The formula for the spread takes the form

$$\sigma^2(t)|_{\text{ZT}}=\frac{\eta}{\pi}\int_0^t\int_0^t d\tau\,d\tau'\phi_0(\tau-\tau')C(\tau)C(\tau')\ . \tag{3.9}$$

As for the behavior of the Markovian approximation in the zero-temperature case, let us assume that $T\ll\tau^*$; then $C(\tau)$ is approximately constant within each of the subintervals of length T and one finds

$$\tilde{\sigma}^2(t)|_{\text{ZT}}=\frac{\eta}{\pi}2\ln(\omega_cT)\int_0^t|C(\tau)|^2d\tau\ , \tag{3.10}$$

where we used (A20). Thus the Markovian approximation yields in this case a high-temperature-like behavior with an incorrect prefactor.

Finally, let us note the cutoff dependence of the diffusion. It is evident from (3.8) that the diffusion is cutoff independent at high temperatures. At zero temperature, Eq. (3.9) applies and the cutoff dependence enters as an edge effect which is investigated in Appendix A. Hence for zero temperature one finds

$$\sigma^2(t)|_{\rm ZT}=\frac{\eta}{\pi}[|C(0)|^2+|C(t)|^2]\ln\omega_c+\Delta\sigma^2\ , \qquad (3.11)$$

but the Markovian approximation yields

$$\tilde{\sigma}^{\,2}(t)|_{\rm ZT}=\frac{\eta}{\pi}\sum_{n=1}^{N}[|C((n-1)T)|^2+|C(nT)|^2]\times\ln\omega_c+\Delta\tilde{\sigma}^{\,2}\ , \qquad (3.12)$$

where $\Delta\sigma^2$ and $\Delta\tilde{\sigma}^{\,2}$ are cutoff-independent terms. The spread $\tilde{\sigma}^{\,2}(t)$ is typically dominated by the cutoff-dependent term [note (3.10) as an extreme example]; thus within the framework of the Markovian approximation the cutoff dependence is much more pronounced then for the exact results.

IV. SIMPLE EXAMPLES

In this section the formalism that was presented in Sec. II is applied to the damped particle and damped harmonic oscillator. These problems were already solved in the past. Their solution is presented here mainly in order to demonstrate the formalism and to compare their behavior to the one of the kicked particle that will be studied in Sec. V.

A. Damped particle

For the damped particle the renormalized potential $V(x;\tau)$ of (2.33) vanishes, i.e., $f(\tau)=f_1(\tau)=0$. The solution (2.58) is

$$R=R_0+\frac{1}{\eta}(1-e^{-\eta t})P_0\ , \qquad (4.1)$$

$$P=e^{-\eta t}P_0\ . \qquad (4.2)$$

Identification of the convection coefficients yields

$$a_{rp}=\frac{1}{\eta}(1-e^{-\eta t})\ , \qquad (4.3)$$

$$a_{pp}=e^{-\eta t}\ . \qquad (4.4)$$

The functions $C_R(\tau)$ and $C_P(\tau)$ are found using Eq. (2.38) with boundary conditions (2.72) and (2.73), leading to

$$C_R(\tau)=\frac{1}{\eta}(1-e^{-\eta(t-\tau)})\ , \qquad (4.5)$$

$$C_P(\tau)=e^{-\eta(t-\tau)}\ . \qquad (4.6)$$

The spread $\sigma_R^2(t)$ and $\sigma_P^2(t)$ is readily calculated using formulas (2.68) and (2.69).

In the high-temperature limit one finds for $1\ll\eta t$ [see (A25) and (A28)]

$$\sigma_R^2(t)|_{\rm HT}=2\frac{\eta}{\beta}\left[\frac{1}{\eta}\right]^2 t \qquad (4.7)$$

and

$$\sigma_P^2(t)|_{\rm HT}=2\frac{\eta}{\beta}\left[\frac{1}{2\eta}\right]\ . \qquad (4.8)$$

In the zero-temperature limit one finds [see (A24) and (A27)]

$$\sigma_R^2(t)|_{\rm ZT}=\frac{\eta}{\pi}\left[\frac{1}{\eta}\right]^2[\ln(\omega_c t)+\ln(\eta t)]\ , \qquad (4.9)$$

$$\sigma_P^2(t)|_{\rm ZT}=\frac{\eta}{\pi}\ln(\omega_c/\eta)\ . \qquad (4.10)$$

These results should be compared with those of Ref. 15.

In Fig. 1 the Wigner function of a Gaussian wave packet is shown for $1\ll\eta t$. Figure 1(a) demonstrates this function, if S_N vanishes, while in Fig. 1(b) the diffusion is taken into account. Note that

$$\sigma_x^2=\sigma_0^2+\sigma_R^2(t) \qquad (4.11)$$

and

$$\sigma_p^2=\sigma_P^2(t)\ , \qquad (4.12)$$

where the spread σ_0^2 is σ_R^2 of (2.56) for $1\ll\eta t$ and is illustrated in the figure, while asymptotically $\sigma_P^2=0$.

B. Damped harmonic oscillator

The renormalized potential is taken to be

$$V(x)=\tfrac{1}{2}\Omega^2x^2\ ; \qquad (4.13)$$

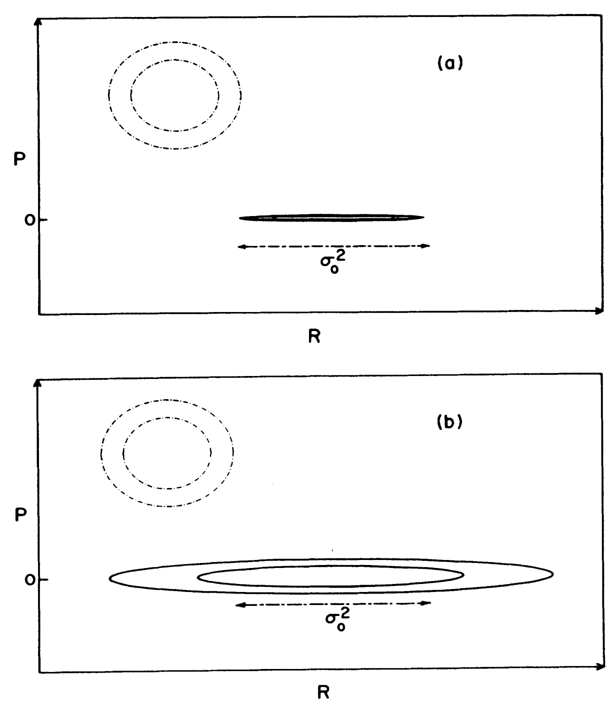

FIG. 1. The spread in time for the damped particle. (a) Without diffusion. (b) In the presence of diffusion. The contours are of constant ρ. The dashed contours represent the initial preparation.

thus $f_1(\tau)=0$, $f(\tau)=\Omega^2$. Here we shall use the shortcut outlined in (2.74)–(2.80) to obtain results for the spread. The convection coefficient $a(t)$ is

$$a(t)=\frac{1}{\nu}e^{-\eta t/2}\sinh(\nu t) , \qquad (4.14)$$

where

$$\nu^2=\left[\frac{\eta}{2}\right]^2-\Omega^2 . \qquad (4.15)$$

The different regions in the parameter space (η,Ω^2) of the problem are illustrated in Fig. 2(a). Alternatively, one can use the (η,ν) parameter space as demonstrated in Fig. 2(b). In different regions the motion is qualitatively different. The various regions are E^+, inverted well; M^+, damped particle; R^+, overdamped oscillator; R_c^+, critical damping; and R_*^+, damped oscillator. The equations of the boundaries between regions in parameter space are

$$\nu=\frac{\eta}{2},\quad \Omega^2=0 \quad \text{for } M^+ , \qquad (4.16)$$

while

$$\nu=0,\quad \Omega^2=\left[\frac{\eta}{2}\right]^2 \quad \text{for } R_c. \qquad (4.17)$$

The spread $\sigma_R^2(t)$ is obtained using (2.80). The asymptotic behavior of $a(\tau)$ is

$$|a(\tau)|\sim e^{\gamma\tau} , \qquad (4.18)$$

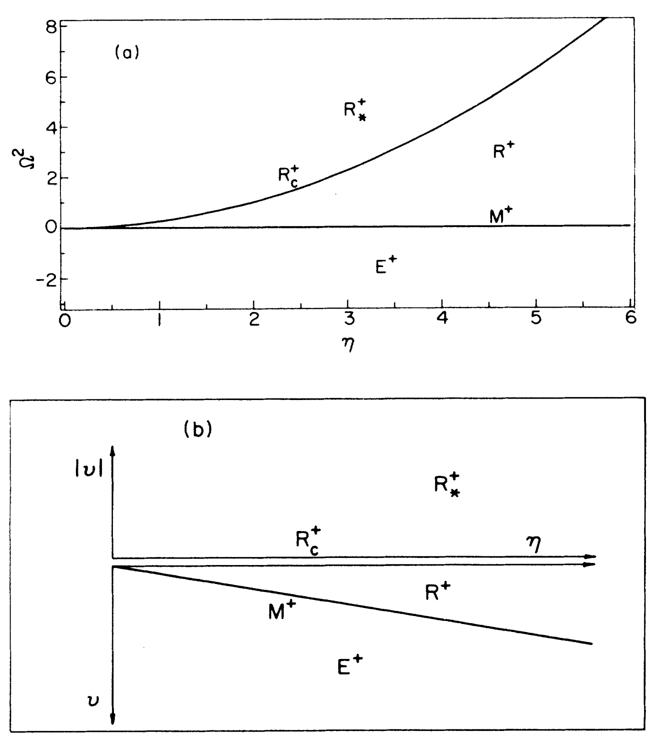

FIG. 2. The parameter space for the damped harmonic oscillator (see text).

with $\gamma=\mathrm{Re}(-\eta/2+\nu)$; therefore $\sigma_R^2(t)$ behaves asymptotically, for $|1/\gamma|\ll t$, as a constant in the R regions (where $\gamma<0$), and grows exponentially with time in the E regions of the parameter space (where $0<\gamma$). For high temperatures (2.80) reduces to

$$\sigma_R^2(t)|_{\mathrm{HT}}=2\frac{\eta}{\beta}\int_0^t|a(\tau)|^2d\tau . \qquad (4.19)$$

Asymptotically in time [see (A31), (A34), and (A28)]

$$\sigma_R^2(t)=\frac{1}{\Omega^2}\left[\frac{1}{\beta}\right] \quad \text{for } R \text{ regions} \qquad (4.20)$$

and

$$\sigma_R^2(t)=2\frac{\eta}{\beta}\left[\frac{1}{\nu}e^{(\nu-\eta/2)t}\right]^2\frac{1}{(\nu-\eta/2)} \quad \text{for } E^+ \text{ region} . \qquad (4.21)$$

At zero temperature (2.80) takes the form

$$\sigma_R^2(t)|_{\mathrm{ZT}}=\frac{\eta}{\pi}\int_0^t\int_0^t d\tau\, d\tau'\phi_0(\tau-\tau')a(\tau)a(\tau') . \qquad (4.22)$$

Asymptotically in time it yields [see (A33), (A30), and (A27)]

$$\sigma_R^2(t)=\begin{cases}\dfrac{2}{\pi}\arctan\left[\dfrac{2|\nu|}{\eta}\right]\dfrac{1}{2|\nu|} \quad \text{for } R_*^+ & (4.23)\\[2ex] \dfrac{2}{\pi}\operatorname{arctanh}\left[\dfrac{2\nu}{\eta}\right]\dfrac{1}{2\nu} \quad \text{for } R^+ & (4.24)\\[2ex] \dfrac{\eta}{\pi}\left[\dfrac{1}{\nu}e^{(\nu-\eta/2)t}\right]^2\ln\left[\dfrac{\omega_c}{\nu-\eta/2}\right] \quad \text{for } E^+ . & (4.25)\end{cases}$$

Finally, in the R regions, where the motion is relaxational, the final state of a wave packet for $1\ll\eta t$ is a Gaussian centered at the origin with a spread

$$\sigma_x^2=\sigma_R^2 , \qquad (4.26)$$

$$\sigma_p^2=\sigma_P^2 . \qquad (4.27)$$

V. KICKED PARTICLE

We study a damped particle that is periodically kicked with a period T. The renormalized potential is

$$V(x;\tau)=\tfrac{1}{2}Kx^2\sum_n\delta(t-(nT-0^+)) . \qquad (5.1)$$

This is just the potential (2.35) with $f_1(\tau)=0$ and $f(\tau)=K\sum_n\delta(\tau-nT)$. The motion will be followed for a time t. For simplicity it will be assumed that it consists of an integer number $N=t/T$ of "steps." The solution (2.58) is here

$$\begin{bmatrix}R\\P\end{bmatrix}=A^N\begin{bmatrix}R_0\\P_0\end{bmatrix} , \qquad (5.2)$$

where the one-step matrix is

$$A=\begin{bmatrix}1 & 0\\ -K & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{\eta}(1-e^{-\eta T})\\ 0 & e^{-\eta T}\end{bmatrix}$$

$$=\begin{bmatrix}1 & \frac{1}{\eta}(1-e^{-\eta T})\\ -K & e^{-\eta T}-\frac{K}{\eta}(1-e^{-\eta T})\end{bmatrix}. \quad (5.3)$$

In order to find the convection coefficients, one has to diagonalize the matrix A. Its eigenvalues satisfy

$$\lambda_+\lambda_-=\det A\ , \quad (5.4)$$

$$\lambda_+ + \lambda_-=\mathrm{tr}A\ , \quad (5.5)$$

where

$$\det A=e^{-\eta T}\ , \quad (5.6)$$

$$\mathrm{tr}A=(1+e^{-\eta T})-\frac{K}{\eta}(1-e^{-\eta T})\ . \quad (5.7)$$

Note that $\mathrm{tr}A$ depends on ηT and KT, while $\det A$ is independent of KT. The eigenvalues are

$$\lambda_\pm=\mathrm{sgn}(\mathrm{tr}A)e^{-\eta T/2\pm\nu T}\ , \quad (5.8)$$

where ν satisfies

$$\left|\frac{\mathrm{tr}A}{2}\right|=e^{-\eta T/2}\cosh(\nu T) \quad (5.9)$$

and can be real or imaginary, in the latter case $0\le|\nu|T\le\pi/2$.

Before proceeding with calculations, let us investigate in some detail the various regions in parameter space $(\eta T, KT)$ of the problem. Equations (5.4) and (5.5) for $\lambda_\pm$ are presented graphically in Fig. 3. Varying KT for fixed nT enables one to classify the solutions of (5.4) and (5.5) as follows. For $0<\mathrm{tr}A$ there are the following regimes: E^+, escape, $0<\lambda_-<1<\lambda_+$; M^+, damped particle, $0<\lambda_-<\lambda_+<1$; R^+, relaxation, $0<\lambda_-<\lambda_+<1$; R_c^+, relaxation $0<\lambda_-=\lambda_+<1$; and R_*^+, relaxation $\lambda_-=(\lambda_+)^*$. For $\mathrm{tr}A<0$ there is a similar classification of regimes that are denoted by E^-, M^-, R^-, R_c^-, R_*^-, and defined as above, but with the change of sign $\lambda_\pm\to-\lambda_\pm$. The parameter space is exhibited in Fig. 4(a). It is more illuminating to parametrize each of the regions using $(\eta T,\nu T)$, as shown in Fig. 4(b), and to consider ν/η and T as independent parameters. The equations of the boundaries between the various regimes are

$$\nu T=\eta T/2,\quad KT=0,\quad \text{for } M^+\ , \quad (5.10)$$

$$\nu T=0,\quad KT=\eta T\tanh(\tfrac{1}{4}\eta T)\quad \text{for } R_c^+\ , \quad (5.11)$$

$$\nu T=0,\quad KT=\eta T\coth(\tfrac{1}{4}\eta T)\quad \text{for } R_c^-\ , \quad (5.12)$$

$$\nu T=\eta T/2,\quad KT=2\eta T\coth(\tfrac{1}{2}\eta T)\quad \text{for } M^-\ . \quad (5.13)$$

At this point it is appropriate to comment about various regions in the parameter space.

(a) The region M^+ (damped particle) has been investigated in Sec. IV A.

(b) The limit $T\to 0$ at the regions E^+, R^+, R_*^+, and R_c^+ yields the problem of the damped harmonic oscillator discussed previously (see Sec. IV B). All the other regions do not reduce to problems that were studied in the past and therefore are potentially interesting. It turns out that the results are tractable in particular in the following regions.

(c) The marginal region M^- that describes a limit cycle.

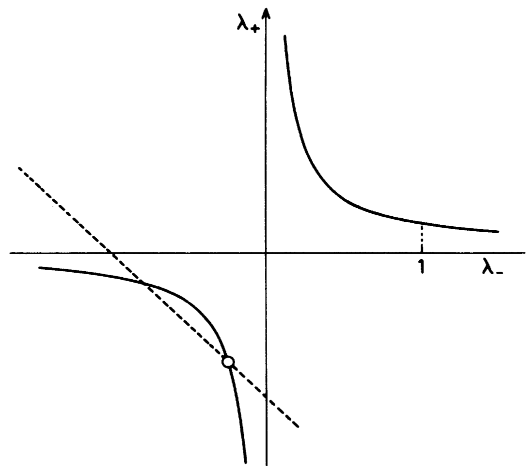

FIG. 3. Illustration of Eqs. (5.4) and (5.5). The solid lines represent (5.4) while the dashed one represents (5.5). Note that only one solution with $|\lambda_-|<|\lambda_+|$ is circled. The other solution is obtained via a permutation of λ_+ and λ_-.

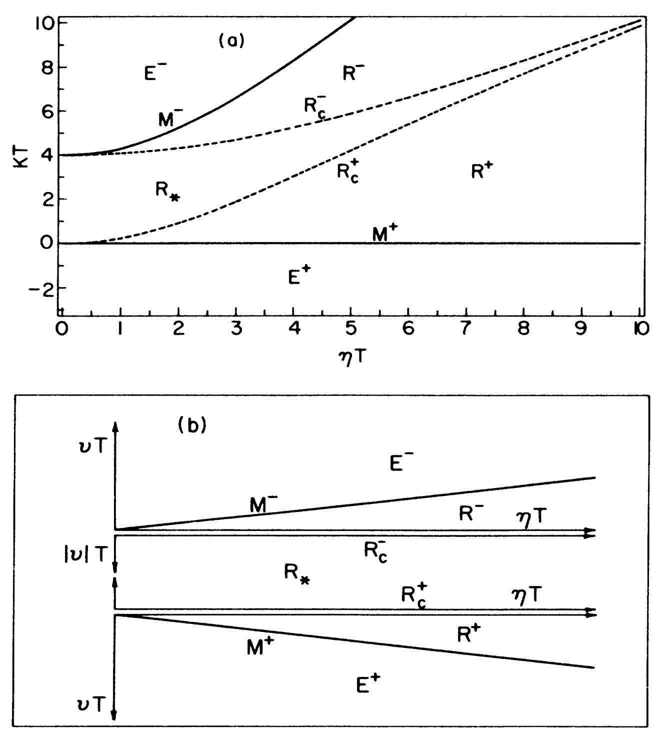

FIG. 4. The parameter space of the kicked particle (see text).

(d) The limit $T \to 0$, i.e., when $(\eta T, \nu T) \ll 1$, in the regions E^-, R^-, R_c^-, and R_*^-.

(e) The limit opposite to $T \to 0$, i.e., when $1 \ll (\eta T, \nu T)$, in the regions $E^\pm$ and $R^\pm$.

The convection coefficients of (2.58) can be calculated. In particular,

$$a_{rp} = a(NT) = \frac{2}{\eta}\left[\frac{\sinh(\frac{1}{2}\eta T)}{\sinh(\nu T)}\right] e^{-\eta NT/2}\sinh(\nu NT) \tag{5.14}$$

for $0 < \mathrm{tr} A$, or else it should be multiplied by $-(-1)^N$. In the interval $[(n-1)T, nT]$ the function $a(\tau)$ is

$$a(\tau) = \left[\frac{a_n - e^{-\eta T}a_{n-1}}{1 - e^{-\eta T}}\right] + \left[\frac{a_{n-1} - a_n}{1 - e^{-\eta T}}\right] e^{-\eta(\tau - (n-1)T)} , \tag{5.15}$$

where $a_n \equiv a(nT)$. The spread $\sigma_R^2(t)$ is given now by formula (2.80), while in the Markovian approximation it takes the form

$$\tilde{\sigma}_R^2(t) = \sum_{n=1}^{N} \int_{(n-1)T}^{nT}\int_{(n-1)T}^{nT} d\tau\, d\tau' \phi(\tau - \tau') a(\tau) a(\tau') . \tag{5.16}$$

Using the same argumentation as in Sec. IV B, we conclude that in the R regions $\sigma_R^2(t)$ and $\tilde{\sigma}_R^2(t)$ behave asymptotically as constants, while in the E regions of parameter space these functions grow exponentially with time. We shall denote the time within which the asymptotic behavior in these regions is achieved by t^*, namely $t^* = |1/\mathrm{Re}(-\eta/2 + \nu)|$.

We now turn to a more detailed study of $\sigma_R^2(t)$ and $\tilde{\sigma}_R^2(t)$, referring to the various regions (a)–(e) mentioned above.

(a) In the marginal region M^+ (damped particle) $a(\tau)$ reduces to (4.3), namely

$$a(\tau) \equiv a(\tau)|_{M^+} , \tag{5.17}$$

and thus expressions (4.7) and (4.9) for $\sigma_R^2(t)$ hold. The Markovian approximation (5.16) at zero temperature [see (3.10)] is

$$\tilde{\sigma}_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi}\left[\frac{1}{\eta}\right]^2 2\ln(\omega_c T)\frac{t}{T} . \tag{5.18}$$

The Markovian approximation gives nonphysical high-temperature-like behavior, whereas the actual behavior is logarithmic.

(b) In the limit $T \to 0$ at regions E^+, R^+, R_c^+, and R_*^+, $a(\tau)$ reduces to that of (4.14), namely

$$a(\tau) \equiv a(\tau)|_{(T\to 0)^+} . \tag{5.19}$$

The Markovian approximation at zero temperature is

$$\tilde{\sigma}_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi}\left[\frac{1}{\eta}\right]^2 2\ln(\omega_c T)\frac{1}{T}\int_0^t [a(\tau)|_{T\to 0^+}]^2 d\tau , \tag{5.20}$$

which is different from (4.22), e.g., we expect considerable qualitative difference in the vicinity of M^+ for $t \le t^*$.

(c) In the marginal region M^- (limit cycle) $a(\tau)$ reduces to the form

$$a(\tau) = a(\tau)|_{M^+} + \varphi(\tau) , \tag{5.21}$$

where $\varphi(\tau)$ is a periodic function of period $2T$ satisfying $\varphi(nT) = -(-1)^n$, and $a(\tau)|_{M^+}$ is defined in (5.17). In the limit $\eta T \ll 1$, $a(\tau)$ is piecewise linear and one may use results (A47) and (A48) to obtain for high temperatures

$$\sigma_R^2(t)|_{\mathrm{HT}} = \tilde{\sigma}_R^2(t)|_{\mathrm{HT}} = 2\frac{\eta}{\beta}\left[\frac{1}{\eta}\right]^2 \tfrac{1}{3}t , \tag{5.22}$$

and for zero temperature

$$\sigma_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi}\left[\frac{1}{\eta}\right]^2 \frac{32}{\pi^2}\frac{t}{T} , \tag{5.23}$$

while the Markovian approximation is

$$\tilde{\sigma}_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi}\left[\frac{1}{\eta}\right]^2 2\ln(\omega_c T)\frac{t}{T} . \tag{5.24}$$

The prefactors of these formulas are slightly modified for general ηT. One observes that qualitatively the Markovian approximation predicts the right behavior, but with a wrong prefactor. This observation is generalized in (d).

(d) In the limit $T \to 0$, and in the regions E^-, R^-, R_c^-, and R_*^-, $a(\tau)$ reduces to

$$a(\tau) = a(\tau)|_{(T\to 0)^+} + \varphi(\tau) , \tag{5.25}$$

where $\varphi(\tau)$ was encountered in (e). Again, we use results (A47) and (A48) to obtain

$$\sigma_R^2(t)|_{\mathrm{HT}} = \tilde{\sigma}_R^2(t)|_{\mathrm{HT}} = 2\frac{\eta}{\beta}\tfrac{1}{3}\int_0^t [a(\tau)|_{(T\to 0)^+}]^2 d\tau \tag{5.26}$$

for high temperatures, while for zero temperature

$$\sigma_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi}\frac{32}{\pi^2}\frac{1}{T}\int_0^t [a(\tau)|_{(T\to 0)^+}]^2 d\tau , \tag{5.27}$$

omitting edge-term effects. The Markovian approximation at zero temperature is

$$\tilde{\sigma}_R^2(t)|_{\mathrm{ZT}} = \frac{\eta}{\pi} 2\ln(\omega_c T)\frac{1}{T}\int_0^t [a(\tau)|_{(T\to 0)^+}]^2 d\tau . \tag{5.28}$$

Thus we may generalize the observation of (e) and state that the Markovian approximation qualitatively predicts the right behavior, but the prefactor is incorrect.

Finally, let us take the opposite limit $t^* \ll T$ in the regions $E^\pm$ and $R^\pm$. Here the expression (2.80) as well as (5.16) for $\sigma_R^2(t)$ and $\tilde{\sigma}_R^2(t)$, respectively, are dominated by $a(\tau)$ within one step, i.e., for $R^\pm$

$$\sigma_R^2(t) \simeq \tilde{\sigma}_R^2(t) = \int_0^T\int_0^T d\tau\, d\tau' \phi(\tau - \tau') a(\tau) a(\tau') + O(e^{-\eta T}) , \tag{5.29}$$

while for $E^\pm$

$$\sigma_R^2(t)\simeq\tilde{\sigma}_R^2(t) = \int_{(N-1)T}^{NT}\int_{(N-1)T}^{NT} d\tau\, d\tau' \phi(\tau-\tau')a(\tau)a(\tau') + O(e^{-\eta T}) . \tag{5.30}$$

Thus in this region the Markovian approximation gives the correct answer. Physically, the asymptotic behavior is achieved within one step, whereas in the previous examples it is achieved over a period of the order η^{-1} and ν^{-1}, which is much longer than T. Recalling that $a(\tau)$ within one step is a solution of the damped particle equation, one obtains at $R^{\pm}$ for high temperatures

$$\sigma_R^2(t)|_{\rm HT}=2\frac{\eta}{\beta}[a(T)]^2T , \tag{5.31}$$

where we used again $1\ll\eta T$, and for low temperatures

$$\sigma_R^2(t)|_{\rm ZT}=\frac{\eta}{\pi}[a(T)]^2[\ln(\omega_c T)+\ln(\eta T)] . \tag{5.32}$$

Note that $[a(T)]^2$ is a constant. For the $E^{\pm}$ regions (5.31) and (5.32) hold but with $[a(T)]^2$ replaced by $[a(t)]^2$. In this case note that $[a(t)]^2$ grows exponentially with t, thus giving the expected behavior.

VI. SUMMARY

In this paper the evolution of the density matrix of the kicked particle was calculated in presence of dissipation. It was found that for high temperatures the Markovian approximation is exact. Even at zero temperature it predicts qualitatively correct asymptotic behavior in time whenever either an instability (E regions) or relaxation (R regions) takes place. The behavior is qualitatively similar to the one that is found for high temperatures. The prefactor, however, is different and it exhibits pronounced cutoff dependence. Therefore the Markovian approximation tends to introduce unphysical dependence on high frequencies.

When the period T is longer compared to the typical time scales of the classical trajectories, the Markovian approximation is accurate. The reason is that the asymptotic behavior of the spread as function of time is achieved within one period.

We turn now to the case where the period T is much shorter compared to the natural time scales of the problem. The Markovian approximation may qualitatively agree with the accurate behavior (with an incorrect cutoff-dependent prefactor) simply because the exact solution is proved to be high-temperature-like and thus similar to the Markovian one. Otherwise the Markovian approximation predicts qualitatively wrong behavior. An extreme example is the damped particle case where it predicts linear behavior of the spread where the correct behavior is logarithmic in time.

ACKNOWLEDGMENTS

It is our pleasure to thank A. J. Leggett, M. Revzen, A. Ron, and U. Smilansky for many useful discussions. We thank R. Krieselman for help with the use of computers. The work was supported in part by the U.S.–Israel Binational Science Foundation (BSF) and by the Fund for Promotion of Research at the Technion. D.C. acknowledges the hospitality of A. J. Leggett at the University of Illinois and S.F. acknowledges the hospitality of R. E. Prange at the University of Maryland where the work was completed.

APPENDIX A

In this appendix various integrals that involve the correlation function $\phi_0(\tau-\tau')$ are calculated. $\phi_0(\tau)$ is a generalized Fourier transform of

$$\phi_0(\omega)\equiv\pi|\omega| \quad (|\omega|\le\omega_c) . \tag{A1}$$

$\phi_0(\omega)$ vanishes for $\omega_c\le|\omega|$; it is assumed that the physical results do not depend significantly on the nature of the cutoff. The definition of $\phi_0(\tau)$ takes the form

$$\phi_0(\tau)\equiv-\frac{1}{\tau^2} \quad (\tau_c\le\tau) , \tag{A2}$$

where $\tau_c\equiv1/\omega_c$, with the further definition

$$\int_0^\infty\phi_0(\tau)d\tau\equiv0 , \tag{A3}$$

thus one obtains

$$\int_0^t\phi_0(\tau)d\tau=\frac{1}{t} \quad (\tau_c\le t) . \tag{A4}$$

We will be interested, in particular, in the correlation between functions $C_{\rm I}$ and $C_{\rm II}$ defined as

$$C_{\rm I}(*)C_{\rm II}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\tau\, d\tau'\phi_0(\tau-\tau')C_{\rm I}(\tau)C_{\rm II}(\tau') = \int_0^{\omega_c}d\omega\,\omega\,{\rm Re}[C_{\rm I}^*(\omega)C_{\rm II}(\omega)] . \tag{A5}$$

In particular,

$$C(*)C=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\tau\, d\tau'\phi_0(\tau-\tau')C(\tau)C(\tau') = \int_0^{\omega_c}d\omega\,\omega|C(\omega)|^2 . \tag{A6}$$

We want to gain some insight into (A6) and, in particular, we want to investigate the dependence of $C(*)C$ on the cutoff ω_c. For this purpose we rewrite (A6) in the form

$$C(*)C=\int_{-\infty}^{\infty}d\tau\, C(\tau)[\delta_\tau(*)C] , \tag{A7}$$

where

$$\delta_\tau(\tau')\equiv\delta(\tau'-\tau) , \tag{A8}$$

and

$$\delta_\tau(*)C=\int_{-\infty}^{\infty}\phi(\tau-\tau')C(\tau')d\tau' . \tag{A9}$$

We shall investigate the behavior of $\delta_\tau(*)C$ for some simple examples and then make some more general statements.

The behavior of $\delta_\tau(*)C$ near a discontinuity is illustrated using a simple example. For

$$C(\tau)=\Theta(\tau) \tag{A10}$$

one finds

$$\delta_\tau(*)C=\frac{1}{\tau} \ . \tag{A11}$$

If the sharp discontinuity is smeared linearly over an interval $[0,T]$, namely

$$C(\tau)=\tau(\Theta(\tau)-\Theta(\tau-T))+\Theta(\tau-T) \ , \tag{A12}$$

one finds

$$\delta_\tau(*)C=-\frac{1}{T}\ln\left|1-\frac{T}{\tau}\right| \ . \tag{A13}$$

Graphical illustration of (A11) and (A13) is given in Fig. 5. The asymptotic behavior of (A13) is

$$\delta_\tau(*)C\approx\begin{cases}\dfrac{1}{\tau}, & T\ll|\tau| \quad \text{(A14)}\\ \dfrac{1}{T}\ln|\tau/T|, & |\tau|\ll T \ . \quad \text{(A15)}\end{cases}$$

One observes that for $T\ll|\tau|$ the behavior is the same as in (A11), but in the vicinity of the step the behavior is less singular, and the smearing (A12) acts effectively like a cutoff for the $1/\tau$ behavior where $1/T$ replaces ω_c.

In this work we confine ourselves to the case where $C(\tau)$ is a piecewise analytic function, namely a function that in each interval coincides with a function that is analytic in a domain that covers the interval. For such functions cutoff dependence arises only due to discontinuities. If $C(\tau)$ is discontinuous only at $\tau=0$, then, as one can see from (A11) and (A7),

$$C(*)C=[C(0^+)-C(0^-)]^2\ln\omega_c+\mathcal{C} \ , \tag{A16}$$

where $\mathcal{C}$ represents cutoff-independent term. In the present work $C(\tau)$ is continuous within the interval $[0,t]$; thus the ω_c dependence enters as an edge effect, namely

$$C(*)C=\{[C(0)]^2+[C(t)]^2\}\ln\omega_c+\mathcal{C} \ . \tag{A17}$$

In what follows we list some formulas that are useful for the calculations of this paper. For

$$C(\tau)=\Theta(\tau)-\Theta(\tau-T) \ , \tag{A18}$$

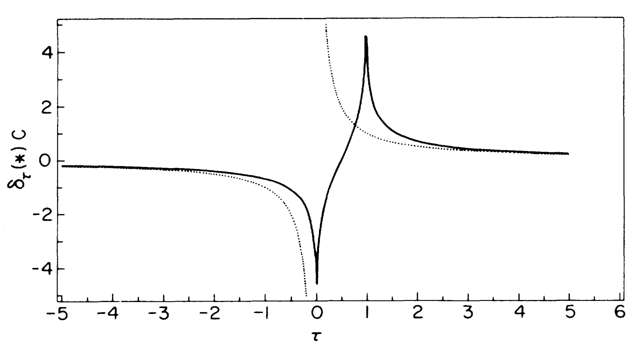

FIG. 5. The correlation $\delta_\tau(*)C$ near a step. The dashed line is for the case of a sharp discontinuity (A10) and the solid line for the linear smearing (A12). The units are of $T=1$.

one finds

$$\delta_\tau(*)C=\frac{1}{\tau}+\frac{1}{T-\tau} \ . \tag{A19}$$

Hence (A7) implies

$$C(*)C=2\ln\omega_c T \ , \tag{A20}$$

while

$$\int_{-\infty}^{\infty}|C(\tau)|^2d\tau=T \ . \tag{A21}$$

For the exponentially smeared step [compare with (A12)]

$$C(\tau)=[\Theta(\tau)-\Theta(\tau-T)](1-e^{-\gamma\tau}) \tag{A22}$$

one finds, for $1\ll\gamma T$,

$$\delta_\tau(*)C=\gamma e^{-\gamma\tau}[\Gamma+\ln|\gamma\tau|+\text{Ein}(\gamma\tau)] \ , \tag{A23}$$

where $\Gamma=0.58\ldots$ is Euler's constant and $\text{Ein}(z)$ is an entire function.[21] Note that for $\tau\ll T$ up to this constant the behavior of (A15) with $T\to\gamma^{-1}$ holds:

$$C(*)C\approx\ln(\omega_c T)+\ln(\gamma T) \tag{A24}$$

and

$$\int_{-\infty}^{\infty}|C(\tau)|^2d\tau\approx T \ . \tag{A25}$$

For the exponential function

$$C(\tau)=\Theta(\tau)e^{-\gamma\tau} \ , \tag{A26}$$

one finds

$$C(*)C=\ln\left[\frac{\omega_c}{\gamma}\right] \tag{A27}$$

and

$$\int_{-\infty}^{\infty}|C(\tau)|^2d\tau=\frac{1}{2\gamma} \ . \tag{A28}$$

For the function

$$C(\tau)=\Theta(\tau)e^{-\gamma\tau}\sinh(\nu\tau)\gamma \tag{A29}$$

with $\nu<\gamma$, integration yields

$$C(*)C=\frac{1}{2}\left[\frac{\nu}{\gamma}\right]\text{arctanh}\left[\frac{\nu}{\gamma}\right] \tag{A30}$$

and

$$\int_{-\infty}^{\infty}|C(\tau)|^2d\tau=\frac{1}{4\gamma}\frac{\nu^2}{\gamma^2-\nu^2} \ . \tag{A31}$$

For

$$C(\tau)=\Theta(\tau)e^{-\gamma\tau}\sin(\nu\tau) \ , \tag{A32}$$

one obtains

$$C(*)C=\frac{1}{2}\left[\frac{\nu}{\gamma}\right]\arctan\left[\frac{\nu}{\gamma}\right] \tag{A33}$$

and

$$\int_{-\infty}^{\infty}|C(\tau)|^2d\tau=\frac{1}{4\gamma}\frac{\nu^2}{\gamma^2+\nu^2} \ . \tag{A34}$$

We turn now to study autocorrelations of more complicated functions where approximations are required. Consider

$$C(\tau)=f(\tau)\sin(\nu\tau) \ , \tag{A35}$$

where $f(\tau)$ is slowly varying over time scales of the order $2\pi/\nu$. Using the fact that $|C(\omega)|^2$ is strongly peaked at ν, one finds

$$\begin{aligned} C(*)C &= \int_0^\infty d\omega\,\omega|C(\omega)|^2 \simeq \nu\int_0^\infty d\omega|C(\omega)|^2 \\ &= \pi\nu\int_{-\infty}^{\infty}|C(\tau)|^2 d\tau \simeq \frac{\pi}{2}\nu\int_{-\infty}^{\infty}|f(\tau)|^2 d\tau \ . \end{aligned} \tag{A36}$$

If $\sin(\nu\tau)$ in (A35) is replaced by a periodic function

$$C(\tau)=f(\tau)\left[a_0+\sum_{n=1}^{\infty}a_n\cos(n\nu\tau)+b_n\sin(n\nu\tau)\right] , \tag{A37}$$

it is found that

$$C(*)C\approx(a_0)^2 f(*)f+\frac{\pi}{2}\bar{\nu}\int_{-\infty}^{\infty}|f(\tau)|^2 d\tau \ , \tag{A38}$$

where

$$\bar{\nu}=\left[\sum_{n=1}^{\infty}(a_n^2+b_n^2)n\right]\nu \ . \tag{A39}$$

Edge effects and interference terms were not taken into account in the approximation (A38). In order to get an idea of the resulting error, we calculate it for two simple examples. For

$$C(\tau)=[\Theta(\tau)-\Theta(\tau-t)][a+b\sin(\nu\tau)] \tag{A40}$$

with $1\ll\nu t$, one finds

$$C(*)C=a^2 2\ln\omega_c t+b^2\frac{\pi}{2}\nu t+ab\pi \ . \tag{A41}$$

The interference term in (A40) is not taken into account in the approximation (A38), which introduces an error $O(t^0)$. For

$$C(\tau)=\Theta(\tau)e^{-\gamma\tau}\cos(\nu\tau) \tag{A42}$$

with $\gamma/\nu\ll 1$, one finds

$$C(*)C\simeq\frac{\pi}{2}\nu\frac{1}{2\gamma}+\ln\left[\frac{\omega_c}{\nu}\right] . \tag{A43}$$

the edge-effect term in (A42) is not taken into account by approximation (A37). However, practically, such a term may be guessed and added, since it results from a discontinuity. Finally, let us apply (A38) to find the autocorrelation of the periodic function that is defined as

$$C(\tau)=2\left|\frac{\tau}{T}\right|-1 \tag{A44}$$

in the interval $-T\le\tau\le T$ and is continued with the period of $2T$; hence $\nu=\pi/T$. The Fourier representation of $C(\tau)$ is

$$C(\tau)=-\frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)\nu\tau]}{(2m-1)^2} \ , \tag{A45}$$

$$\bar{\nu}=\left[\left[\frac{8}{\pi^2}\right]^2\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\right]\nu\approx\left[\frac{8}{\pi^2}\right]^2\nu \ , \tag{A46}$$

leading to

$$C(*)C\approx\frac{32}{\pi^2}\frac{1}{T}\int_{-\infty}^{\infty}|f(\tau)|^2 d\tau \ . \tag{A47}$$

Finally

$$\int_{-\infty}^{\infty}[C(\tau)]^2 d\tau=\tfrac{1}{3}\int_{-\infty}^{\infty}|f(\tau)|^2 d\tau \ . \tag{A48}$$

APPENDIX B

In this appendix it is shown that if the action S_{eff} of (2.12) is a functional of an order that does not exceed 2, then the propagator is given by (2.41) with

$$N\equiv\int_0^0\int_0^0 D\delta R\,D\delta r e^{i\delta^2 S_{\text{eff}}[\delta R,\delta r]-S_N[\delta r,\delta r]} \ , \tag{B1}$$

and where

$$S_{\text{eff}}(R,r;R_0,r_0)\equiv S_{\text{eff}}[R^{\text{cl}},r^{\text{cl}}] \ , \tag{B2}$$

$$S_N(r,r_0)\equiv S_N[r^{\text{cl}},r^{\text{cl}}] \ , \tag{B3}$$

while $\delta^2 S_{\text{eff}}$ is the quadratic part of S_{eff}. The classical path $[R^{\text{cl}},r^{\text{cl}}]$ is defined in (2.37)–(2.40). Note that N is *independent* of $(R,r;R_0,r_0)$.

For the proof one introduces a change of the integration variables in (2.12),

$$R(\tau)\to\delta R(\tau)\equiv R(\tau)-R^{\text{cl}}(\tau) \ , \tag{B4}$$

$$r(\tau)\to\delta r(\tau)\equiv r(\tau)-r^{\text{cl}}(\tau) \ , \tag{B5}$$

and gets (2.41) with

$$N'\equiv\int_0^0\int_0^0 D\delta R\,D\delta r \times e^{i\delta^2 S_{\text{eff}}[\delta R,\delta r]-S_N[\delta r,\delta r]-2S_N[r^{\text{cl}},\delta r]} \ , \tag{B6}$$

instead of N. At first glance N' appears to depend on r^{cl} and hence on $(R,r;R_0,r_0)$. Actually, this is not the case. Let us write (B5) in a matrix notation

$$N'=\int\int D\delta R\,D\delta r\exp\left[-\frac{1}{2}\begin{bmatrix}\delta R\\ \delta r\end{bmatrix}^t A\begin{bmatrix}\delta R\\ \delta r\end{bmatrix}+B^t\begin{bmatrix}\delta R\\ \delta r\end{bmatrix}\right] , \tag{B7}$$

where A and B are infinite dimension and have the general form

$$A=\begin{bmatrix}0 & A_{Rr}\\ A_{Rr} & A_{rr}\end{bmatrix},\quad B=\begin{bmatrix}0\\ B_r\end{bmatrix} . \tag{B8}$$

The Gaussian integral (B6) is

$$N'=Ne^{B^t A^{-1}B/2}=N \ . \tag{B9}$$

We made use of

$$A^{-1}=\begin{bmatrix}-A_{Rr}^{-1}A_{rr}A_{Rr}^{-1} & A_{Rr}^{-1}\\ A_{Rr}^{-1} & 0\end{bmatrix} . \tag{B10}$$

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