Quantum simulations of nonlinear optical damping: An exact solution for the stochastic differential equations and an interpretation of "spiking"

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This paper considers the "positive-P" description of nonlinear optical damping where one-photon and two-photon loss mechanisms are allowed. An exact solution of the corresponding stochastic differential equations is presented that graphically shows the breakdown of the positive-P representation found in earlier numerical-simulation work. The problem of "spiking" in simulations of nonlinear optical processes is then addressed, using the nonlinear damping process as an example. It is pointed out that although "spiking" can be exacerbated at high levels of quantum noise by numerical inaccuracy, the tendency to spike is in fact an analytic property of the stochastic variables.

I. INTRODUCTION

The success of the generalized P representation of Drummond and Gardiner[1] in describing the steady-state phase-space behavior of a number of nonlinear-optical systems has led to its application in the study of the dynamical behavior of such systems.[2–6] This growing body of work suggests that numerical simulations of phase-space behavior must be interpreted rather carefully, and that in some cases, the Fokker-Planck equation for the positive-P representation itself may even be physically suspect.

In this paper we use the positive-P representation to study the phase-space behavior of nonlinear optical damping. Recent numerical simulation work on this process has yielded physically implausible results which, nonetheless, were backed up by exploratory analytic work in the same study.[6] In Secs. II and III we present the positive-P Fokker-Planck equation (PPFP) for the nonlinear damping problem and derive an exact analytic solution for the equivalent system of stochastic differential equations (SDE). This solution makes explicit the conditions under which the PPFP may lead to a plausible physical description of the optical-damping problem. In addition, it allows a check on the accuracy of the Euler simulation of the SDE for this system.

Previous workers have also noted the presence of spiking behavior in the numerical simulation of the nonlinear damping process. This behavior, which is exascerbated when the numerical time step is large, appears to be a general feature of simulations which possess high "quantum-noise" levels. In Sec. IV we use the explicit analytic solution to show that spiking is governed analytically by stochastic phase factors that link the dependent variables of the system.

II. NONLINEAR OPTICAL DAMPING

Nonlinear optical damping is a problem that has attracted recurrent interest over the last two decades.[7–9] Here we shall consider damping of a single mode of the light field through a two-photon loss mechanism, allowing also the standard one-photon loss path. The effective Hamiltonian for this process is

$$H=\Gamma_2^\dagger a^2+\Gamma_1^\dagger a+\text{H.c.} \ , \tag{2.1}$$

where a is the boson-annihilation operator for the light field, and Γ_1 and Γ_2 are collective boson-annihilation operators for thermal reservoirs representing one-photon and two-photon losses, respectively.

To obtain a phase-space description of the dynamics of this system we introduce the positive-P representation of Drummond and Gardiner for the density operator ρ,

$$\rho=\int\int d^2\alpha\, d^2\beta \frac{P(\alpha,\beta)}{\langle\alpha|\beta^*\rangle}|\alpha\rangle\langle\beta^*| \ . \tag{2.2}$$

Here the $|\alpha\rangle$ and $|\beta\rangle$ are coherent states, with α and β independent complex variables, and we use the notation that $d^2z\equiv d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$.

We proceed by substituting Eq. (2.2) into the Liouville equation of motion for the density operator and then use standard techniques to eliminate the thermal reservoir variables.[10] This gives the following Fokker-Planck equation governing the evolution of the generalized P function $P(\alpha,\beta)$:

$$\begin{aligned}\dot P(\alpha,\beta)=\Bigg[&-\frac{\partial}{\partial\alpha}(-\tfrac{1}{2}\gamma_1\alpha-\tfrac{1}{2}\gamma_2\alpha^2\beta)\\&-\frac{\partial}{\partial\beta}(-\tfrac{1}{2}\gamma_1\beta-\tfrac{1}{2}\gamma_2\beta^2\alpha)\\&-\frac{1}{2}\frac{\partial^2}{\partial^2\alpha}\gamma_2\alpha^2-\frac{1}{2}\frac{\partial^2}{\partial^2\beta}\gamma_2\beta^2\Bigg]P(\alpha,\beta) \ ,\end{aligned} \tag{2.3}$$

where γ_1 and γ_2 are the one-photon and two-photon damping rates, respectively. We have assumed here that the thermal reservoirs are at zero temperature, and thus contribute no noise to the system.

The Ito stochastic equations corresponding to this Fokker-Planck equation when interpreted in its PPFP sense are

$$\Delta\alpha = -(\gamma\alpha + \alpha^2\beta)\Delta t + i\alpha\Delta w_1 \ ,$$
$$\Delta\beta = -(\gamma\beta + \beta^2\alpha)\Delta t - i\beta\Delta w_2 \ , \qquad (2.4)$$

where time has been scaled by $\frac{1}{2}\gamma_2$ and $\gamma = \gamma_1/\gamma_2$. Each $\Delta w_i(t)$ is a Weiner process of mean zero and variance 1 that is scaled as $\Delta t^{1/2}$.[10]

III. SOLUTION OF THE STOCHASTIC EQUATIONS

Equations (2.4) possess the following exact solution:

$$\alpha = \frac{\alpha_0 e^{-(\gamma - \frac{1}{2}q)t + iqw_1}}{\left[1 + 2\alpha_0\beta_0 \int_0^t dt \, e^{-2(\gamma - \frac{1}{2}q)t + iq(w_1 - w_2)}\right]^{1/2}} \ ,$$
$$\beta = \frac{\beta_0}{\alpha_0}\alpha e^{-iq(w_1 + w_2)} \ , \qquad (3.1)$$

where $w_i = \int_0^t dw_i$ and the index q allows recovery of the classical solution ($q = 0$) from the full quantum solution ($q = 1$).

This solution is most conveniently derived by recasting the Ito equations into Stratonovich form,

$$\Delta\alpha = -(\gamma - \tfrac{1}{2}q)\alpha\Delta t - \alpha^2\beta\Delta t + iq\alpha\Delta w_1 \ ,$$
$$\Delta\beta = -(\gamma - \tfrac{1}{2}q)\beta\Delta t - \beta^2\alpha\Delta t - iq\beta\Delta w_2 \ . \qquad (3.2)$$

Noting that the linear terms in α and β admit exponential time dependencies we assume trial solutions of the form

$$\alpha = A(t)e^{-(\gamma - \frac{1}{2}q)t + iqw_1}, \quad \beta = B(t)e^{-(\gamma - \frac{1}{2}q)t - iqw_2} \ . \qquad (3.3)$$

Substituting these expressions into (3.2) implies $A(t) = \text{const}B(t)$, where the constant of proportionality is $A(0)/B(0)$. The equations for $A(t)$ and $B(t)$ thus decouple, and therefore can be integrated separately to yield solution (3.1). The solution is thus reduced to (stochastic) quadratures, and by construction must be the unique solution to the initial value problem.[11] All statistical properties of solutions of the Fokker-Planck equation (2.3) may then be obtained from Eq. (3.1) by performing an ensemble average and averaging over the initial distribution of α_0 and β_0.

We observe immediately that there is no time-independent solution for $\gamma \le \frac{1}{2}$. For these values the oscillatory factors in the numerators of Eq. (3.1) remain undamped, effectively preventing the solution from approaching the equilibrium $\alpha = \beta = 0$ attained for $\gamma > \frac{1}{2}$. This behavior contrasts with that of the purely classical calculation ($q = 0$) which converges to this solution for all positive semidefinite γ. More critically, the solution for $\gamma < \frac{1}{2}$ appears to disagree with the vacuum steady state predicted—without ambiguity—by the Fock representation (see Ref. 6 for example). There are, however, acceptable positive P representations for the vacuum state which demand only that the ensemble averages of all normally ordered physical quantities are zero, as opposed to the stochastic variables themselves. In fact solution (3.1) is not consistent with zero ensemble averages. This can be checked by direct numerical averaging but may be anticipated by the scaling argument given in Sec. IV.

This unphysical quantum behavior can also be anticipated by examining the stability of the deterministic part of the Stratonovich equations (3.2). The eigenvalues of the linearized system are

$$\lambda_1 = -(\gamma - \tfrac{1}{2}q) - \alpha\beta, \quad \lambda_2 = -(\gamma - \tfrac{1}{2}q) - \alpha^*\beta^* \ ,$$
$$\lambda_3 = -(\gamma - \tfrac{1}{2}q) - 3\alpha\beta, \quad \lambda_4 = -(\gamma - \tfrac{1}{2}q) - 3\alpha^*\beta^* \ . \qquad (3.4)$$

This immediately shows that in the quantum case ($q = 1$) the phase-space point $\alpha = \beta = 0$ corresponds to an unstable equilibrium for $\gamma < \frac{1}{2}$. In the case $\gamma = \frac{1}{2}$ this point is marginally stable but any approach towards the equilibrium which does not maintain the phase relationship $\alpha = \beta^*$ is unstable, with the most unstable trajectory being the "out of phase" trajectory $\alpha = -\beta^*$. Quantum noise thus effectively prevents the equilibrium being attained.

We note that in Ref. 6 it is argued, on the basis of numerical simulations and a partial analytic treatment, that the steady-state solution is not attainable for $\gamma \le \frac{1}{2}$. Our complete analytic solution demonstrates this fact explicitly. The essential conclusion emerging from both studies is that the PPFP for the present system is fatally flawed, at least in the regime $\gamma \le \frac{1}{2}$. For $\gamma > \frac{1}{2}$, the present solution does indeed give the correct steady state, but the question of whether or not it gives the correct time evolution remains for future work.

IV. NUMERICAL SIMULATIONS—SPIKING BEHAVIOR

Previous investigators have observed that spiking in the phase-space variables can occur during numerical simulation of the positive-P stochastic equations. It is generally assumed—not unreasonably—that such behavior is of numerical origin rather than an inherent property of the stochastic equations. Our purpose here is to investigate this matter in the light of the analysis presented in Sec. III.

We first observe that solution (3.1) is generally well behaved. The only exception arises in the pathological case in which the denominator

$$D = 1 + 2\alpha_0\beta_0 \int_0^t dt \, e^{-2(\gamma - \frac{1}{2}q)t + iq(w_1 - 2_2)} \qquad (4.1)$$

vanishes. An upper bound for D can be derived by neglecting the stochastic phase factors

$$|D| \le 1 + \frac{\alpha_0\beta_0}{\frac{1}{2}q - \gamma}\left(e^{-2(\gamma - \frac{1}{2}q)t} - 1\right) \ . \qquad (4.2)$$

For the recalcitrant systems of interest ($\gamma \le \frac{1}{2}$) the exponential factor is dominant for all t sufficiently large; moreover, this factor exactly cancels the modulus of the numerator in the expression for the stochastic variables (3.1). This suggests that α and β should scale as $(\frac{1}{2} - \gamma)^{1/2}$ for large t. Thus, when stochastic phase factors are neglected, there appears no grounds for expecting rapid spiking behavior in the moduli of the stochastic variables.

Against this argument is the numerical evidence for spiking in α and β. Analytically, spiking can only be associated with rapid growth away from the unstable point $\alpha=\beta=0$ ($\gamma \le \frac{1}{2}$). Fast growth clearly requires that α and β are both large and sufficiently out of phase (3.4). However, since the phase relationships determining the solution are entirely stochastic in nature, local growth in the stochastic variables is invariably followed by decay. In this way excursions from the mean asymptotic trajectory are modulated nonlinearly by phase factors that depend on the details of the simulation. Thus, although each noise process possesses an "envelope" $w \propto (n\Delta t)^{1/2}=t^{1/2}$ after n iterations, we must evaluate the quadratures in (3.1) directly to realize a particular stochastic trajectory. The key question is whether these "excursions" about the equilibrium can explain the phenomena of spiking.

To investigate further we evaluate solution (3.1) by employing a simple quadrature formula to compute the integrals. Figure 1(a) shows the time development of $|\alpha|$ and $\text{Re}(\alpha)$ for a typical stochastic trajectory in the "stable" regime $\gamma > \frac{1}{2}$. In this figure $\gamma=0.6$ and the integrals were evaluated assuming the quadrature interval 0.003 (any value which ensures $|\lambda\Delta t| \ll 1$ locally will provide adequate resolution). Despite the stochastically changing phase of α and β the equilibrium point $\alpha=\beta=0$ is eventually attained. Figure 1(b) shows a time plot of the real part of the dominant eigenvalues (λ_3,λ_4) along with $|\alpha|$ for the same parameters as Fig. 1(a). As predicted by Eq. (3.4) the eigenvalue attains the stable limit value $\frac{1}{2}-\gamma=-0.1$. Although temporary excursions into the deterministically unstable (spiking) regime $\lambda>0$ are

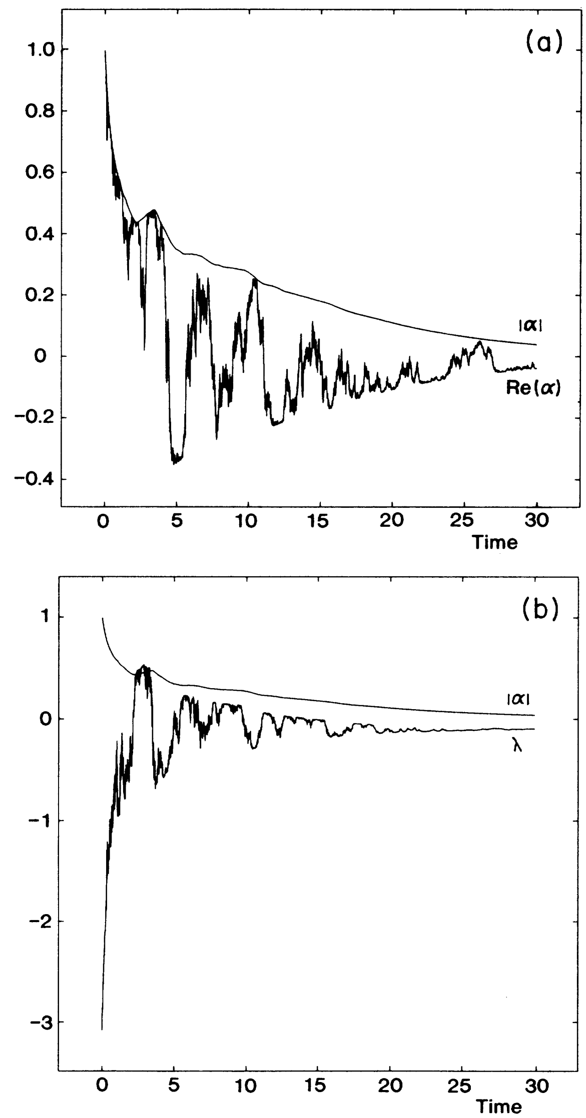

FIG. 1. (a) Time evolution of $|\alpha|$ and $\text{Re}(\alpha)$ for $\gamma=0.6$. The initial conditions are $\alpha_0=\beta_0=1$ and the quadrature interval is 0.003. (b) Time evolution of $|\alpha|$ and $\text{Re}(\lambda_3)$ for the parameters of (a). Excursions into the deterministically unstable regime ($\lambda>0$) become less apparent for large t.

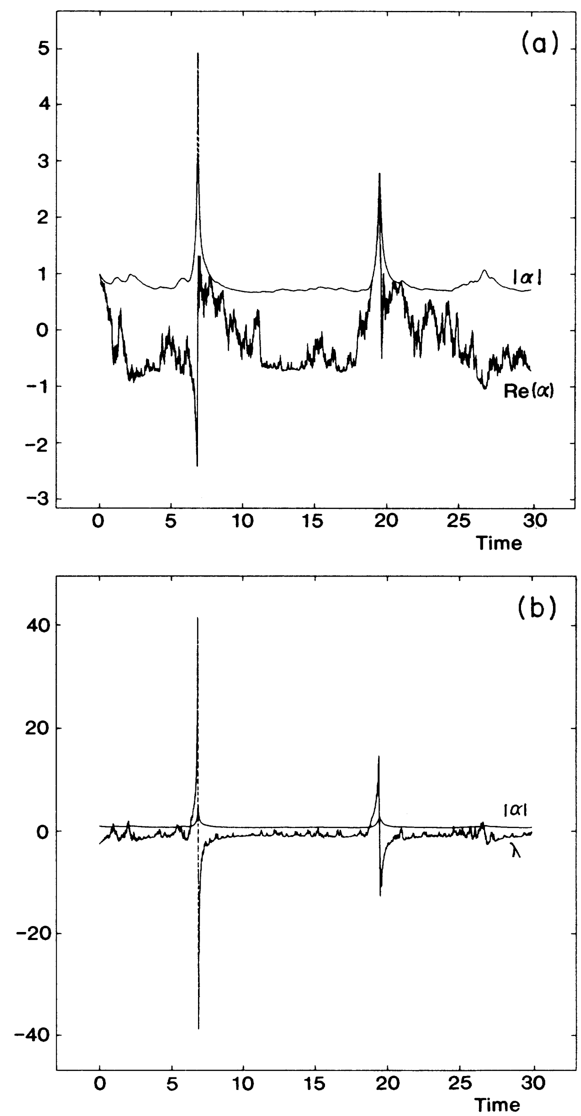

FIG. 2. (a) Time evolution of $|\alpha|$ and $\text{Re}(\alpha)$ for $\gamma=0.1$. The initial conditions are $\alpha_0=\beta_0=1$ and the quadrature interval is 0.003. (b) Time evolution of $|\alpha|$ and $\text{Re}(\lambda_3)$ for the parameters of (a).

clearly apparent, this tendency clearly diminishes as $t \to \infty$.

In contrast Figs. 2(a) and 2(b) display the same variables for an "unstable" case $\gamma=0.1$. Although $|\alpha|$ generally sits close to the "baseline" estimate $(\frac{1}{2}-\gamma)^{1/2}$, it is clear that phase fluctuations induced by quantum noise can drive large excursions away from this value. These excursions manifest themselves as spiking phenomena similar to those encountered in the numerical simulation experiments reported in the literature. The essential conclusion is that while poor time resolution and lack of stochastic accuracy can magnify the tendency to spike, the spiking itself is real and arises (analytically) via the nonlinear coupling of unstable deterministic behavior with the quantum-noise process. In the present problem the tendency to spike is present for all γ, but for $\gamma > \frac{1}{2}$ it is effectively suppressed by the exponential decay factors.

It appears that the same broad conclusion will hold for any nonlinear quantum system since the phase independence of α and β generally admits the existence of locally, deterministically unstable trajectories (especially in the region $\alpha=-\beta^*$). The exact nature of the instability depends on the eigenstructure of the particular problem and in some cases can lead to the severe problem of dynamical "stiffness."[5] In other special cases there are analytically forbidden regions of phase space which should never be accessed numerically.[12] These considerations do not, however, play an important role in the present problem.

Finally we mention that the Euler integration of the Ito SDE system gives results which are entirely consistent with our analytic solution provided the Euler time step is sufficiently small (typically the time step has to be of the order 10^{-3} or smaller).

V. CONCLUSIONS

We have constructed an analytic solution for the stochastic system which represents the positive-P Fokker-Planck (PPFP) equation for the problem of nonlinear optical damping with one- and two-photon losses. Although the overcompleteness of the coherent states means that any solution of the Fokker-Planck equation is not necessarily unique, the solution to the SDE system presented here is unique, and contains the totality of the statistical properties of the process described by the Fokker-Planck equation. This solution is expressed in the form of a stochastic quadrature and immediately makes explicit the fact that in the range $0<\gamma<\frac{1}{2}$ the steady-state solution $\alpha=0$, $\beta=0$ [corresponding to the positive-P solution $\delta(\alpha)\delta(\beta)$] cannot be attained except in the trivial case of zero initial conditions. This solution is also inconsistent with the more general requirements of zero near ensemble averages for all normally ordered physical quantities. We conclude therefore, in agreement with Ref. 6, that the PPFP description cannot be valid in the range $0 \le \gamma \le \frac{1}{2}$. The PPFP description may or may not be valid outside this range but at least it appears to give the correct steady-state behavior.

In Sec. IV we considered the problem of spiking in quantum-optics simulations. It was pointed out that the nonlinear interplay between the deterministically unstable $(\gamma < \frac{1}{2})$ and stochastic aspects invariably leads to spiking behavior in the dependent variables. In other words spiking is a natural aspect of PPFP simulations. Specifically, quantum noise, in modulating the phase relationships of the dependent variables, allows access into classically forbidden, potentially unstable regions of phase space [e.g., trajectories $\alpha=-\beta^*$ in Eq. (3.4)]. Although this property can lead to severe numerical difficulties in certain quantum-optical systems,[5] the form of the eigenspectrum (3.4) implies that the symptoms are relatively mild in the present application.

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[1] P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980).
[2] J. S. Satchell and Sarben Sarker, J. Phys. A **19**, 2737 (1986).
[3] H. J. Carmichael, J. S. Satchell, and Sarben Sarker, Phys. Rev. A **34**, 3166 (1986).
[4] M. Dörfle and A. Schenzle, Z. Phys. B **65**, 113 (1986).
[5] K. J. McNeil and I. J. D. Craig (unpublished).
[6] A. M. Smith and C. W. Gardiner, Phys. Rev. A **39**, 3511 (1989).
[7] G. S. Argawarl, Phys. Rev. A **1**, 1445 (1970).
[8] K. J. McNeil and D. W. Walls, J. Phys. A **7**, 617 (1974).
[9] R. Loudon and H. D. Simaan, J. Phys. A **8**, 539 (1975).
[10] See, for example, C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).
[11] See, for example, E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[12] M. Wolinsky and H. J. Carmichael, Phys. Rev. Lett. **60**, 1836 (1988).