Photon distributions for nonclassical fields with coherent components

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(Received 24 October 1988; revised manuscript received 17 February 1989)

Nonclassical fields with coherent components generated in a large number of optical processes including losses can be characterized by a Gaussian Wigner function centered around the mean value of the field. The photon-number distributions for such fields are calculated and the numerical results are presented for a range of parameters. As a special case we give the number distributions for photons produced in down-conversion problems. We also give the analytical expression for the photoelectron distributions.

I. INTRODUCTION

In a recent paper[1] we have shown that the quantum fluctuations of the radiation field produced by a very large class of systems in nonlinear optics can be described in terms of a density matrix that possesses a Gaussian-Wigner function.[2] This is true even if losses and the coherent pumping effects are included. Even the fluctuations in a bistable system and in the context of more complex cavity problems can be described in a similar fashion. The parameters in the Gaussian Wigner function depend on the detailed microscopic properties of the quantum-mechanical system at hand. We thus consider a single-mode radiation field characterized by a Gaussian Wigner function

$$\Phi(z,z^*)=\frac{1}{\pi(\tau^2-4|\mu|^2)^{1/2}}\exp\left[-\frac{\mu(z-z_0)^2+\mu^*(z^*-z_0^*)^2+\tau|z-z_0|^2}{(\tau^2-4|\mu|^2)}\right]. \tag{1.1}$$

The parameters z_0, μ, and τ are related to the lower-order moments of the annihilation and creation operators a and $a^\dagger$

$$\langle a\rangle=z_0,\quad \langle a^2\rangle=-2\mu^*+z_0^2\,,$$
$$\langle (a^\dagger)^2\rangle=-2\mu+(z_0^*)^2,\quad \langle a^\dagger a\rangle=\tau-\tfrac{1}{2}+|z_0|^2\,. \tag{1.2}$$

Note that the coherent part z_0 of the field is essentially determined from the semiclassical analysis of the nonlinear problem. The parameters μ and τ satisfy a number of conditions that follow from the positive definiteness of ρ. We write

$$\mu=\frac{Q}{4}\sinh x e^{-i\theta},\quad \tau=\frac{Q}{2}\cosh x,\quad Q\geq 1,\quad z_0=r_0e^{i\varphi_0}\,. \tag{1.3}$$

The distribution function (1.1) leads to the squeezing in the component $(e^{-i\beta}a+a^\dagger e^{i\beta})$ of the field if

$$Q[\cosh x-\sinh x\cos(\theta-2\beta)]<1\,. \tag{1.4}$$

We remind the reader that the usual two-photon coherent state[3] is a special case of (1.1), i.e., (1.1) represents the field in the two-photon coherent state if $Q=1$. An operator representation of the density matrix can be constructed from (1.1). It can be shown[4] that the explicit form of the density matrix corresponding to (1.1) is

$$\rho=[\tfrac{1}{4}(e^{2\varphi}-1)]^{-1/2}\exp\{-2e^{-\varphi}\cosh^{-1}(\coth\varphi)[\mu(a-z_0)^2+\mu^*(a^\dagger-z_0^*)^2+\tau(a^\dagger-z_0^*)(a-z_0)+\tau/2]\}\,, \tag{1.5}$$

where

$$e^{2\varphi}=4(\tau^2-4|\mu|^2)\,. \tag{1.6}$$

The density matrix (1.5) in general represents a mixed state for $Q\neq 1$. This is expected since our quantum system possesses losses, and a system with losses in general cannot be characterized by a pure state. Depending on the parameters Q, x, θ, and φ, the field in the state (1.1) or (1.5) may possess sub-Poissonian statistics. Using the

Gaussian nature of the Wigner function the condition for the existence of sub-Poissonian statistics is found to be[2]

$$\langle (\Delta n^2) \rangle - \langle n \rangle < 0 ,$$

i.e. ,

$$\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle < 0$$
$$\Longrightarrow \tau^2 + 2|z_0|^2\tau - \tfrac{1}{4} - 2(z_0^*)^2\mu^* - 2z_0^2\mu + 4|\mu|^2 - |z_0|^2 - \tau + \tfrac{1}{2} < 0 . \quad (1.7)$$

The distribution function (1.1) carries information both on the amplitude and phase of the field. In this paper we examine the photon-number distributions associated with (1.1). In Sec. II we present the analytical formula for the number distribution for the field characterized by (1.1). In Sec. III we discuss several limiting cases of our general formula, and show how the known results are recovered as special cases. We also present the numerical results for the number distributions for a range of parameter values Q, x, and z_0. In Sec. IV we discuss the number distributions for the nonclassical fields produced in the down conversion process in a cavity. Finally, in Sec. V we show how the photoelectron counting distributions can be obtained from the number distributions (calculated in Sec. II) by changing the parameters Q, x, and z_0.

II. PHOTON-NUMBER DISTRIBUTION FOR THE FIELD CHARACTERIZED BY (1.1)

The number distribution can be obtained as a phase-space integral involving the Wigner function (1.1) and Laguerre polynomials. It is known that $p(n)$ is related to the Wigner function via

$$p(n) = \int d^2z\, \Phi(z) 2(-1)^n L_n(4|z|^2) \exp(-2|z|^2) , \quad (2.1)$$

which can be formally written as

$$p(n) = 2(-1)^n L_n \left[-\frac{\partial}{\partial \lambda} \right] I_1(\lambda_1) \Bigg|_{\lambda_1 = 1/2} . \quad (2.2)$$

Here I_1 is an integral over the quadratic forms

$$I_1(\lambda_1) = \int d^2z\, \Phi(z) \exp(-4\lambda_1 |z|^2) . \quad (2.3)$$

In a previous work[1] we calculated the number distributions[5-8] for the special case when the coherent part z_0 of the field was zero. We now generalize this to the case when $z_0 \neq 0$. This generalization is especially important if the field contains a small number of coherent photons. It is, for example, now known[9] that the interaction of atoms with coherent fields with a small number of photons results in important quantum-mechanical features. It may be noted that the field characterized by (1.1) includes as special cases most of the previously studied fields, except those for which the fluctuation behavior is dominated by nonlinear Langevin equations. It may be added that the field generally acquires a nonzero coherent part if the nonlinear optical processes above threshold are considered.

To evaluate $p(n)$, we first calculate the integral I_1 which can be evaluated by using

$$\int d(x_i) \exp\left[-\sum_{i,j} x_i A_{ij} x_j + \sum_i h_i x_i \right] = \frac{(\pi)^{m/2}}{\sqrt{\det A}} \exp\left[\frac{1}{4} \sum_{i,j} h_i (A^{-1})_{ij} h_j \right] , \quad (2.4)$$

where m is the total number of variables x_i. The integral I_1 can be reduced to the form in (2.4) by writing $z = x_1 + i x_2$. The calculations show that

$$I_1(\lambda_1) = \frac{1}{2Q} I_+(\lambda_1) I_-(\lambda_1) ,$$
$$I_\pm(\lambda_1) = \frac{1}{\sqrt{\nu_\pm(\lambda_1)}} \exp\left[-\frac{4r_\pm^2}{\omega_\pm(\lambda_1)} \right] , \quad (2.5)$$

$$r_+ = r_0 \cos\alpha, \quad r_- = r_0 \sin\alpha, \quad \nu_\pm = \lambda_1 + \frac{e^{\pm x}}{2Q} ,$$
$$\omega_\pm = \frac{1}{\lambda_1} + \frac{2Q}{e^{\pm x}}, \quad \alpha = \varphi_0 - \frac{\theta}{2} .$$

On using (2.2) and (2.5) and elementary algebra, we get

$$p(n) = (-1)^n \frac{1}{Q} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{m!} \left[-\frac{\partial}{\partial \lambda_1} \right]^m (I_+ I_-) \Bigg|_{\lambda_1 = 1/2} , \quad (2.6)$$

$$\left[-\frac{\partial}{\partial \lambda_1} \right]^m I_+ I_- = \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} \left[\left[-\frac{\partial}{\partial \lambda_1} \right]^l I_+ \right] \left[\left[-\frac{\partial}{\partial \lambda_1} \right]^{m-l} I_- \right] . \quad (2.7)$$

The derivatives in (2.7) have been calculated and we quote the result

$$\left[-\frac{\partial}{\partial\lambda_1}\right]^l I_\pm \Bigg|_{\lambda_1=1/2} = \frac{1}{\nu_\pm^{l+(1/2)}} \exp\left[-\frac{4r_\pm^2}{\omega_\pm}\right] l! L_l^{-1/2}\left[-\frac{c_\pm}{\nu_\pm}\right], \quad (2.8)$$

where L_l^ν is the associated Laguerre polynomial and

$$c_\pm = \left[\frac{e^{\pm x} r_\pm}{Q}\right]^2 . \quad (2.9)$$

The relation (2.8) can be proved by (i) using the generating function for the associated Laguerre polynomials,[10] (ii) writing $\lambda_1=\lambda+\frac{1}{2}$, and (iii) by expanding in powers of λ:

$$\frac{1}{\left[\lambda+\frac{1}{2}+e^{\pm x}/2Q\right]^{1/2}} \exp\left[\frac{-4r_\pm^2(\lambda+\frac{1}{2})(e^{\pm x}/2Q)}{\lambda+\frac{1}{2}+e^{\pm x}/2Q}\right] = \sum_n \frac{1}{\sqrt{\beta_\pm}} L_n^{-1/2}\left[\frac{-4r_\pm^2 e^{\pm 2x}}{4Q^2\beta_\pm}\right]\left[-\frac{\lambda}{\beta_\pm}\right]^n \times \exp\left[\frac{4r_\pm^2 e^{\pm 2x}}{4Q^2\beta_\pm} - \frac{4r_\pm^2 e^{\pm x}}{2Q}\right], \quad \beta_\pm = \frac{1}{2}+\frac{e^{\pm x}}{2Q} . \quad (2.10)$$

The coefficient of λ^l will give the lth derivative. On using (2.6)–(2.8), our final result for $p(n)$ is

$$p(n) = (-1)^n \Phi_0(z_0) \sum_{m=0}^{n}\sum_{l=0}^{m} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} \left[\frac{1}{\nu_+}\right]^l \left[\frac{1}{\nu_-}\right]^{m-l} L_l^{-1/2}\left[-\frac{c_+}{\nu_+}\right] L_{m-l}^{-1/2}\left[-\frac{c_-}{\nu_-}\right], \quad \nu_\pm = \frac{1}{2}+\frac{e^{\pm x}}{2Q} , \quad (2.11)$$

where Φ_0 is the distribution

$$\Phi_0(z_0) = \frac{1}{\pi[(\tau+\frac{1}{2})^2-4|\mu|^2]^{1/2}} \exp\left[-\frac{\mu z_0^2+\mu^*(z_0^*)^2+(\tau+\frac{1}{2})|z_0|^2}{(\tau+\frac{1}{2})^2-4|\mu|^2}\right] . \quad (2.12)$$

The result (2.11) is our key result. It gives the number distribution for fields produced in a nonlinear optical process when (a) the phase correlations can be nonzero ($x\neq 0$), (b) the coherent component of the field can be finite ($z_0\neq 0$), and (c) the losses could be important $Q\neq 1$. Note that in most cases the losses would consist of cavity losses and spontaneous-emission losses, and thus could not be ignored.

III. LIMITING CASES AND NUMERICAL RESULTS

Before we discuss the consequences of our basic formula (2.11), we demonstrate how the result (2.11) leads to the known results in special cases.

A. Field without phase correlations: $x=0$

Consider first the case $x=0$, i.e., $\mu=0$. Such a field consists of a mixture of a coherent and an incoherent field. In this case various parameters simplify to

$$\nu_\pm = \frac{Q+1}{2Q} = \nu, \quad Q=2\tau, \quad \omega_\pm = 2(Q+1) = \omega ,$$
$$\frac{4r_0^2}{\omega} = \frac{|z_0|^2}{\tau+\frac{1}{2}}, \quad c_+ + c_- = \left[\frac{r_0}{Q}\right]^2 = c , \quad (3.1)$$

and $p(n)$ reduces to

$$p(n) = \frac{(-1)^n}{\tau+\frac{1}{2}} \exp\left[-\frac{|z_0|^2}{\tau+\frac{1}{2}}\right] \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^m \frac{1}{\nu^m} \sum_{l=0}^{m} L_l^{-1/2}\left[-\frac{c_+}{\nu}\right] L_{m-l}^{-1/2}\left[-\frac{c_-}{\nu}\right] . \quad (3.2)$$

On using the identities[11]

$$\sum_{m=0}^{n} L_m^\alpha(x) L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y) , \quad (3.3)$$

$$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^m y^m L_m(-x) = (1-y)^n L_n\left[-\frac{xy}{y-1}\right] , \quad (3.4)$$

Eq. (3.2) simplifies to

$$p_n = \frac{(\tau-\frac{1}{2})^n}{(\tau+\frac{1}{2})^{n+1}} \exp\left[-\frac{|z_0|^2}{\tau+\frac{1}{2}}\right] L_n\left[-\frac{|z_0|^2}{\tau^2-\frac{1}{4}}\right] . \quad (3.5)$$

This is the standard result for a field which is a superposition of coherent and incoherent fields. It reduces, respectively, to Poisson and Bose-Einstein distributions in the limiting cases of a pure coherent field ($\tau=\frac{1}{2}$) and incoherent field ($|z_0|=0$).

B. Field without coherent component: $z_0=0$

The result of our previous publication can be obtained by letting $z_0\rightarrow 0$, whence

$$p(n)=\frac{(-1)^n}{Q\sqrt{\nu_+\nu_-}}\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^m\binom{n}{m}\times\frac{(\frac{1}{2})_l(\frac{1}{2})_{m-l}}{l!(m-l)!}\frac{1}{\nu_+^l\nu_-^{m-l}}, \tag{3.6}$$

where

$$(\alpha)_l=\alpha(\alpha+1)\cdots(\alpha+l-1). \tag{3.7}$$

On using the definitions of the hypergeometric function $F_1(\alpha,\beta,\beta',\gamma,x,y)$ of two variables[12]

$$F_1(\alpha,\beta,\beta',\gamma,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n \tag{3.8}$$

and the relation[13]

$$F_1(\alpha,\beta,\beta',\beta+\beta',x,y)=(1-y)^{-\alpha}F\left[\alpha,\beta,\beta+\beta',\frac{x-y}{1-y}\right], \tag{3.9}$$

the result (3.6) can be expressed in two different forms:

$$p(n)=\frac{1}{Q\sqrt{\nu_+\nu_-}}\left[\frac{1}{\nu_-}-1\right]^n F\left[-n,\frac{1}{2},1,\frac{\nu_--\nu_+}{\nu_+(\nu_--1)}\right] \tag{3.10}$$

$$=(-1)^n\frac{1}{Q\sqrt{\nu_+\nu_-}}F_1\left[-n,\frac{1}{2},\frac{1}{2},1,\frac{1}{\nu_+},\frac{1}{\nu_-}\right] \tag{3.11}$$

$$=\frac{2}{(Q^2+2Q\cosh x+1)^{1/2}}\left[\frac{Qe^x-1}{Qe^x+1}\right]^n\times\sum_{m=0}^{n}\binom{-\frac{1}{2}}{m}\binom{n}{n-m}\left[\frac{4Q\sinh x}{Q^2+2Q\sinh x-1}\right]^m. \tag{3.12}$$

For a squeezed vacuum $Q=1$, (3.12) reduces to

$$p(n)=\frac{1}{\cosh\frac{x}{2}}\tanh^n\frac{x}{2}\sum_{m=0}^{n}\binom{-\frac{1}{2}}{m}\binom{n}{m}2^m. \tag{3.13}$$

Algebraic manipulations enable one to reduce (3.13) to the known result[14]

$$p(2n)=\frac{(-1)^n}{\cosh\frac{x}{2}}\binom{-\frac{1}{2}}{n}\tanh^{2n}\frac{x}{2},\quad p(2n+1)=0. \tag{3.14}$$

We next examine the photon-number distribution for the general radiation field, i.e., when the phase correlations ($x\neq 0$), losses ($Q\neq 1$), and coherent component ($z_0\neq 0$) are important. We have carried out numerical calculations for a range of the parameter values $Q,x,\alpha,|z_0|$, and some representative results are shown in Figs. 1–5. In Fig. 1 we reproduce the known result[3,8,14] for the two-photon coherent state $Q=1$. With an increase in x, the field first acquires increasingly sub-Poissonian character, and then the sub-Poissonian character decreases and the field distribution becomes remarkably oscillatory. This oscillatory character has been associated[5,8] with the nonclassical nature of the light. In fact, when the field distribution becomes oscillatory, then the first two moments of the distribution are not enough to characterize the nature of the distribution. Figures 2 and 3 give the effects of losses in the nonlinear medium on the photon-number distribution. The distributions broaden with an increase in Q. The distribution becomes oscillatory for larger values of x. Note that the condition (1.4) for $\theta=\beta=0$ shows that larger value of Q require larger values of x for squeezing to exist. Thus in the pres-

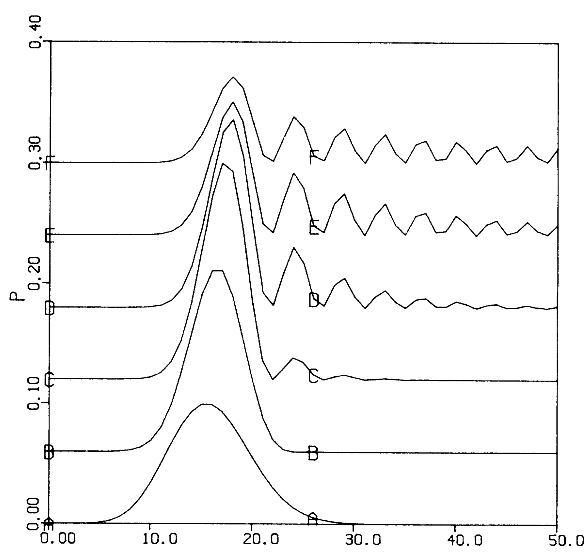

FIG. 1. Photon-number distribution $p(n)$ [$\equiv P-$(displacement)] as a function of n for a radiation field with a coherent component $z_0=4$ (average number of coherent photons 16). The field is characterized by a Gaussian Wigner function centered around $z_0=4$. The phase factor $A=(1/\pi)(\varphi_0-\theta/2)$ is chosen as zero and Q is set as 1. The curves A, B, C, D, E, and F are for the squeezing parameter $x=0$, 1, 2, 3, 4, and 5, respectively. For clarity, different curves are displaced, i.e., $P=p(n)+0.06x$.

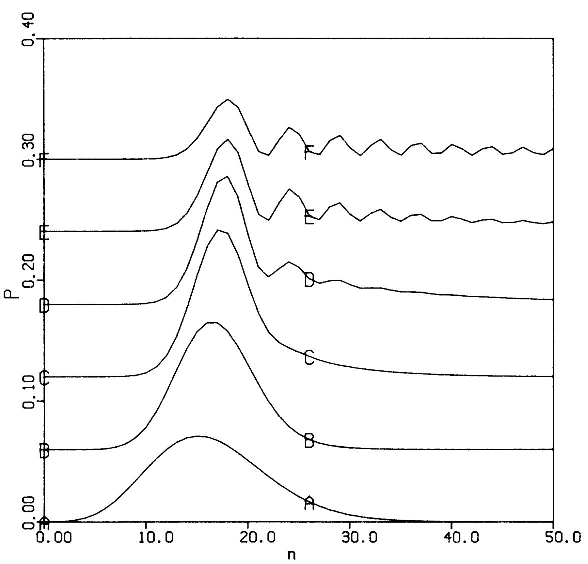

FIG. 2. Same as in Fig. 1 with $Q=2$.

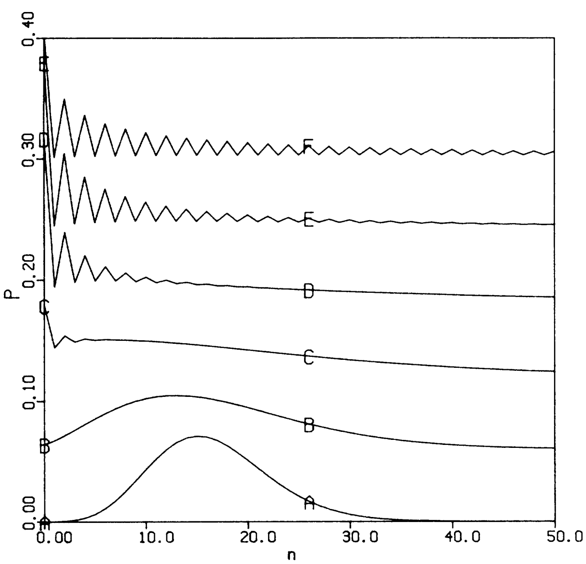

FIG. 4. Same as in Fig. 2, but with a different phase $A=0.5$.

ence of losses one needs a larger value of x to see the oscillatory character of $p(n)$. We next explore the changes in the nature of the distributions as the relative phase between z_0 and μ is changed. Figure 4 is to be compared with Fig. 2. Notice the considerable change in the nature of the distribution. Note also considerable broadening of the distribution when x is increased from zero to values of the order of 1. The peak of the distribution shifts progressively to lower photon number with the increase in x, though the average number of photons in the field increases with the increase in x as

$$\langle a^\dagger a \rangle = |z_0|^2 + \tfrac{1}{2}(Q \cosh x - 1) . \quad (3.15)$$

This is demonstrated more clearly in Fig. 5, where $p(n)$ is plotted for a number of different phase settings. Thus the number distribution is quite sensitive to the phase of z_0. This is also evident from the condition (1.7) for sub-Poissonian statistics, which depends on the phase factor $(2\varphi_0-\theta)$, i.e., on A. This is in contrast to the condition for squeezing (1.4) which is independent of the phase of z_0.

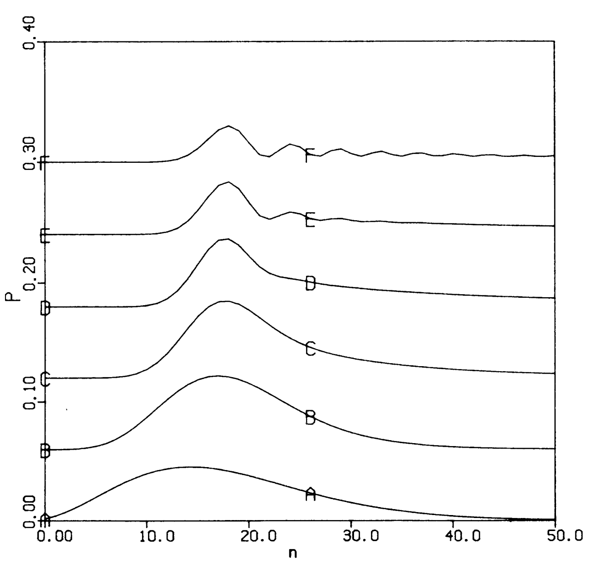

FIG. 3. Same as in Fig. 1 with $Q=5$.

IV. PHOTON-NUMBER DISTRIBUTIONS FOR FIELDS PRODUCED IN DOWN CONVERSION

The generation of nonclassical fields in down conversion has been studied by several authors[15,16] using master-equation techniques. While the previous work concerns the squeezing properties of the generated fields, the present work deals with the photon-number distribu-

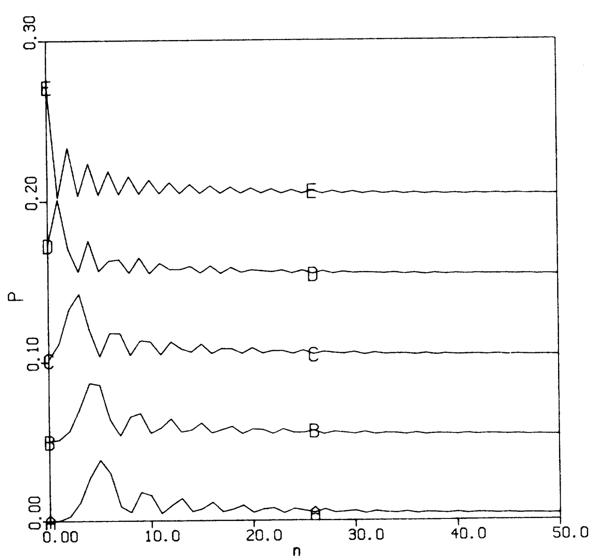

FIG. 5. Number distributions for different phase settings $A=0$, 0.125, 0.25, 0.375, and 0.5 and for $Q=5$, $x=5$, and $z_0=2$. Curves are displaced so that the quantity plotted is $P=p(n)+0.4A$.

tions for the generated fields. We recall the basic results on the generation of the nonclassical fields in the down-conversion process.

Let c and $c^\dagger$ (a and $a^\dagger$) be the annihilation and creation operators associated with the pump mode of frequency $2\omega_a$ (generated mode of frequency ω_a). The Hamiltonian describing the down conversion is

$$H=\hbar\omega_a a^\dagger a+2\hbar\omega_a c^\dagger c+\frac{i\hbar}{2}[\tilde{G}(a^\dagger)^2 c-\text{H.c.}]+i\hbar(\epsilon_c c^\dagger e^{-2i\omega_a t}-\text{H.c.}) . \quad (4.1)$$

On including the losses γ_a and γ_c associated with two modes, the density matrix for the system can be written as

$$\frac{\partial\rho}{\partial t}=-\frac{i}{\hbar}[H,\rho]-\gamma_a(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)-\gamma_c(c^\dagger c\rho-2c\rho c^\dagger+\rho c^\dagger c) . \quad (4.2)$$

On assuming $\gamma_c \gg \gamma_a$, the pump mode can be adiabatically eliminated. We moreover linearize the fluctuations around steady state, i.e., we set $a=\langle a\rangle+A$ and assume that the fluctuations in A are much smaller. Then the calculations show that the reduced density matrix for the mode A satisfies

$$\frac{\partial\rho_A}{\partial t}=-\kappa(A^\dagger A\rho_A-2A\rho_A A^\dagger+\rho_A A^\dagger A)-\frac{i}{\hbar}\left[\left[\frac{i\hbar}{2}\tilde{G}\langle c\rangle(A^\dagger)^2+\text{H.c.}\right],\rho_A\right] , \quad (4.3)$$

where

$$\gamma_c\langle c\rangle=\epsilon_c-\frac{\tilde{G}^*}{2}\langle a\rangle^2,\quad \gamma_a\langle a\rangle=\tilde{G}\langle a\rangle^*\langle c\rangle ,\quad \kappa=\gamma_a+\frac{|\tilde{G}|^2|\langle a\rangle|^2}{\gamma_c} . \quad (4.4)$$

It can be shown that the steady-state solution of (4.3) is characterized by a Gaussian Wigner function. It can be further shown that

$$\langle A^\dagger A\rangle+\tfrac{1}{2}=\langle a^\dagger a\rangle-|\langle a\rangle|^2+\tfrac{1}{2}=\frac{1}{2}\left[1-\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|^2\right]^{-1} . \quad (4.5)$$

$$\langle A^2\rangle=\langle a^2\rangle-\langle a\rangle^2=\frac{\tilde{G}\langle c\rangle}{2\kappa}\left[1-\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|^2\right]^{-1} . \quad (4.6)$$

On comparison with (1.2) we get the values of the parameters μ, τ, and z_0:

$$\tau=\frac{1}{2}\left[1-\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|^2\right]^{-1} ,\quad \mu=-\frac{(\tilde{G}\langle c\rangle)^*}{4\kappa}\left[1-\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|^2\right]^{-1} , \quad (4.7)$$

$$z_0=\langle a\rangle,\quad \gamma_a\gamma_c\langle a\rangle=\tilde{G}\langle a\rangle^*\left[\epsilon_c-\frac{\tilde{G}^*}{2}\langle a\rangle^2\right] .$$

Note that z_0 is zero below threshold ($\epsilon<1$), whereas above threshold ($\epsilon>1$) (4.7) shows that

$$|z_0|^2=2\left|\frac{\gamma_a\gamma_c}{\tilde{G}^2}\right|(\epsilon-1),\quad \left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|=\frac{1}{(2\epsilon-1)},\quad \epsilon=\frac{\tilde{G}\epsilon_c}{\gamma_a\gamma_c} . \quad (4.8)$$

The phase of z_0 is determined by

$$\arg[\tilde{G}\epsilon_c(z_0^*)^2]=0 . \quad (4.9)$$

The results of Sec. II can now be applied to obtain the number distribution associated with the field a. We show the numerical results in Figs. 6–8. The nature of the distribution depends on whether the system is below threshold (Fig. 6) or above threshold (Figs. 7 and 8). For small $|\tilde{G}\langle c\rangle/\kappa|\equiv\beta$ the average excitation of the mode a is close to zero. The excitation increases as β approaches threshold value. Near threshold average excitation is large. However, the distribution is such that the probability of finding a moderate number of photons in the field is rather small. In such situations the mean and variance do not adequately characterize the distribution. For example, for curve F in Fig. 6, the mean number of photons is about 8 but the number distribution is very broad. As one moves further away (Fig. 7) above threshold, the distribution starts acquiring a smooth form with a well-defined maximum and dispersion. Far away from threshold, the fluctuations become less and less important, and the field becomes more like a coherent state

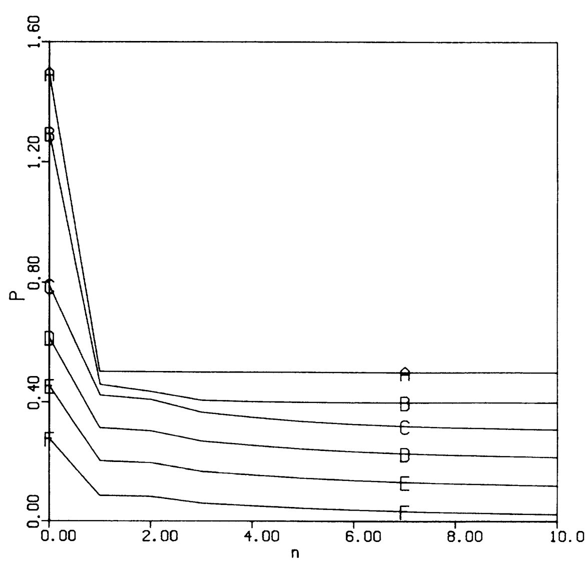

FIG. 6. Number distribution for the field produced in the down-conversion process. The parameters correspond to the system below threshold, and different curves A to F are for $|\tilde{G}\langle c\rangle/\kappa|=\beta=0.1$, 0.5, 0.9, 0.93, 0.95 and 0.97, respectively. For clarity each curve has been displaced by 0.1 units on the y axis.

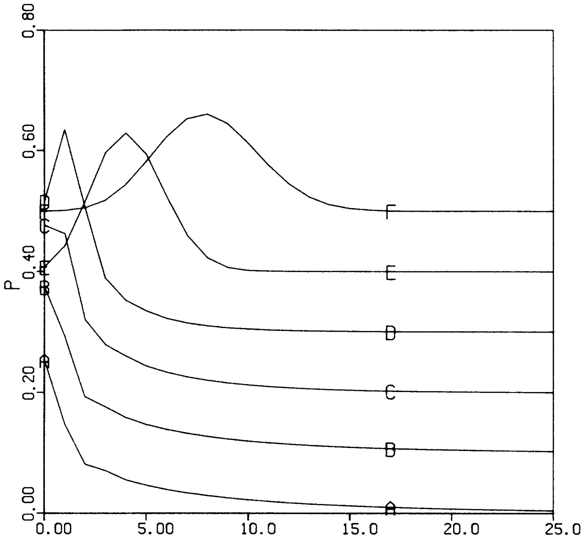

FIG. 7. Same as in Fig. 6 but now the system is above threshold with the parameter $\epsilon=|\tilde{G}\epsilon_c/\gamma_a\gamma_c|$ equal to 1.02, 1.03, 1.05, 1.1, 1.5, and 2. The phase parameter A is zero. The other parameter $|\gamma_a\gamma_c/\tilde{G}^2|$ has been set as 4.

with the mean number of photons equal to $|z_0|^2$. This is evident, for example, from the curve marked F in Fig. 7. The width of the distribution depends on the parameter A, as is seen, for example, by comparing curves F in Figs. 7 and 8. A comparison of Figs. 7 and 8 also shows how the number distributions are sensitive to the phase noise in the system. It may be noticed that, unlike our general results in Sec. II, the distribution in the down-conversion problem is determined by a single parameter $|\tilde{G}\langle c\rangle/\kappa|$ and does not exhibit any oscillatory character. This can be understood from the values of Q and the interpretation of $(Q-1)$ as a loss parameter. Note that $Q^2=1/(1-|\tilde{G}\langle c\rangle/\kappa|^2)$, and thus near threshold Q is rather large. Large values of Q destroy the oscillatory character of $p(n)$.

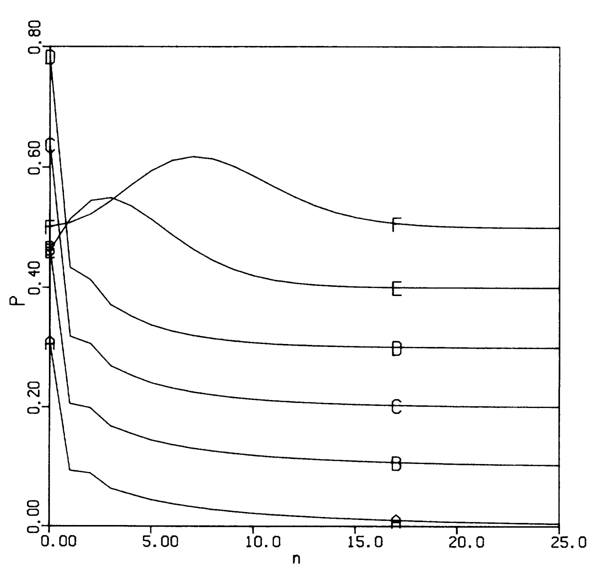

FIG. 8. Same as in Fig. 7, but now A is equal to 0.5.

V. PHOTOELECTRON COUNTING DISTRIBUTIONS

We conclude this paper by discussing the connection between the photon-number distributions and photon-counting distributions. It is well known that the photon-counting distribution $p_e(n,T)$ is related to the photon statistics by the Mandel formula

$$p_e(n,T)=\left\langle T_N\frac{W^n e^{-W}}{n!}\right\rangle . \tag{5.1}$$

Here $p(n,T)$ is the probability of counting n photoelectrons in time interval T; T_N stands for time-ordered normal-ordered products, and W is the integrated intensity given by

$$W=\alpha\int_0^T a^\dagger(\tau)a(\tau)d\tau . \tag{5.2}$$

The parameter α is related to the quantum efficiency of the detector. If the counting time T is much smaller than the coherence time of the field, then it is a good approximation to use

$$W\cong\alpha T a^\dagger a . \tag{5.3}$$

On combining (5.1) and (5.3) we get the counting distribution as

$$p_e(n,T)=\left\langle:\frac{(\alpha T)^n(a^\dagger a)^n e^{-\alpha T a^\dagger a}}{n!}:\right\rangle , \tag{5.4}$$

where : : stands for the normal ordering. Equation (5.4) can be simplified by using the Glauber-Sudarshan P function, which can be used, even though it does not exist for squeezed light. A simple calculation leads to

$$p_e(n,T)=\sum_{m=n}^{\infty}p(m)\begin{bmatrix}m\\ n\end{bmatrix}(\alpha T)^n(1-\alpha T)^{m-n} . \tag{5.5}$$

Note that if the quantum efficiency parameter is 1, then $p_e(n,T)=p(n)$. The photoelectron distributions can be obtained from the results of the preceding sections by doing the binomial averaging; an example of this averaging is given in Ref. 7. We next show that $p_e(n,T)$ for a fixed $\alpha T=\gamma$ can be obtained from $p(n)$ by changing the parameters x, Q, and z_0 to x_e, Q_e, and z_e defined by Eq. (5.9).

The counting distribution (5.4) can be written in terms of the P function as

$$\begin{aligned}p_e(n,T)&=\int d^2\alpha\, P(\alpha)\frac{(\gamma|\alpha|^2)^n e^{-\gamma|\alpha|^2}}{n!}\\ &=\int d^2\alpha\, P_e(\alpha)\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} ,\end{aligned} \tag{5.6}$$

where

$$P_e(\alpha)=\frac{1}{\gamma}P\left[\frac{\alpha}{\sqrt{\gamma}}\right] . \tag{5.7}$$

Thus the argument of P function in (5.6) is scaled by the factor $\sqrt{\gamma}$. It is clear now that, if the original P function corresponded to the Gaussian Wigner function (1.1), then the new P function would also correspond to (1.1) with

$$\begin{aligned} z_0 &\to z_e = z_0\sqrt{\gamma} \ , \\ \mu &\to \mu_e = \mu\gamma \ , \\ \tau &\to \tau_e = \gamma\tau + \tfrac{1}{2}(1-\gamma) \ . \end{aligned} \tag{5.8}$$

On using (1.3) and (5.8) we get relations between Q,x and Q_e, x_e,

$$\begin{aligned} Q_e \sinh x_e &= \gamma Q \sinh x \ , \\ Q_e \cosh x_e &= \gamma Q \cosh x + (1-\gamma) \ . \end{aligned} \tag{5.9}$$

Therefore, the photoelectron distribution $p_e(n,T) \equiv p_e(n,\gamma)$ can be obtained from photon-number distribution $p(n)$ by the relation

$$p_e(n,\gamma)\Big|_{z_0,Q,x} = p(n)\Big|_{\substack{z_0 \to z_e \\ Q \to Q_e \\ x \to x_e}} \ . \tag{5.10}$$

Thus our fundamental relation (2.11) can be used to give directly the counting distributions. All that one has to do is to change the parameters Q and x according to (5.9). As mentioned in the Introduction, the case $Q=1$ corresponds to the squeezed state,[3] and the effect of the binominal averaging on number distributions for squeezed states has been discussed previously.[1,6,7] In Fig. 9 we consider a typical case corresponding to $Q=2$ and $x=4$, and show the effect of finite efficiency of detection. The effect of the parameter $\gamma = \alpha T$ is to reduce the oscillatory character of the photon-number distribution.

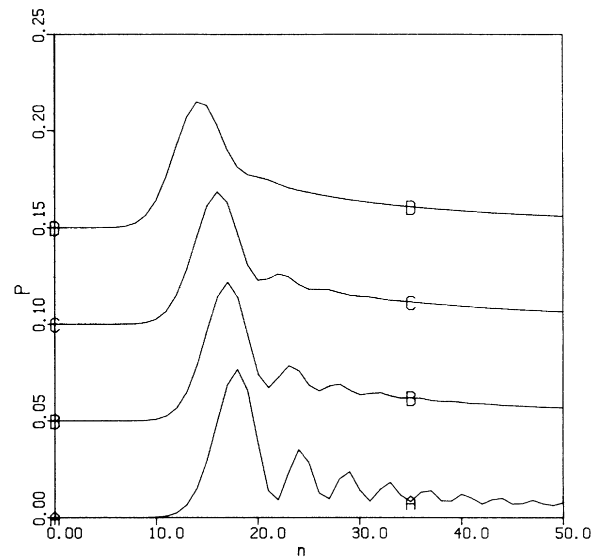

FIG. 9. Effect of finite quantum efficiency of the detector on counting distributions. Here $p_e(n,T)$ is plotted for $z_0=4$, $A=0$, $Q=2$, $x=4$, and for different values of $\alpha T=1$ (A), 0.95 (B), 0.9 (C), and 0.8 (D). Each curve is displaced by 0.05 units on the y axis. Note that curve A is identical to curve E of Fig. 2.

ACKNOWLEDGMENTS

G.S.A. is grateful to the Department of Science and Technology, Government of India for supporting this work. G. A. acknowledges the support by the Fonds zur Förderung der Wissenschaftlichen Forschung in Österreich under Contract No. P6690.

[1]G. S. Agarwal and G. Adam, Phys. Rev. A **38**, 750 (1988).
[2]G. S. Agarwal, J. Mod. Opt. **34**, 909 (1987).
[3]H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[4]Cf. G. S. Agarwal, Phys. Rev. A **3**, 828 (1971).
[5]W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987).
[6]A. Vourdas and R. M. Weiner, Phys. Rev. A **36**, 5866 (1987).
[7]G. J. Milburn and D. F. Walls, Phys. Rev. A **38**, 1087 (1988); these authors calculate the effects of losses by averaging $p(n)$ over a binomial distribution.
[8]W. Schleich, D. F. Walls, and J. A. Wheeler, Phys. Rev. A **38**, 1177 (1988).
[9]See, for example, H. I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).
[10]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965), p. 1038, formula 8.975-1.
[11]I. S. Gradshteyn and I. M. Ryzhik, Ref. 10, p. 1038, formula 8.974-4.
[12]I. S. Gradshteyn and I. M. Ryzhik, Ref. 10, p. 1053, formula 9.180-1.
[13]I. S. Gradshteyn and I. M. Ryzhik, Ref. 10, p. 1054, formula 9.182-1.
[14]W. Schleich and J. A. Wheeler, J. Opt. Soc. Am. B **4**, 1715 (1987).
[15]G. Milburn and D. F. Walls, Phys. Rev. A **27**, 392 (1983); M. J. Collett and C. W. Gardiner, *ibid.* **30**, 1386 (1984); M. J. Collett and D. F. Walls, *ibid.* **32**, 2887 (1985).
[16]A Wigner-function treatment of this problem has been given recently by G. S. Agarwal and S. Dutta Gupta, Phys. Rev. A **39**, 2961 (1989).