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Colored noise: A perspective from a path-integral formalism

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A stationary distribution for Langevin equations driven by colored noise is obtained, in the weak-noise limit, from the configuration-space Lagrangian-like function. The derivation makes no explicit use of Markovian, Fokker-Planck, or small-correlation-time approximations. Markovian approximations based on the Lagrangian, which do not involve truncated expansions, are also discussed.

This paper aims to introduce and ellaborate the useful alternative point of view regarding path integrals in the problem of stochastic differential equations (SDE) driven by colored noise. We consider a SDE for a variable $q(t)$ of the form

$$\dot{q}(t)=f(q)+g(q)\xi(t)\,, \tag{1}$$

where $\xi(t)$ is an Ornstein-Uhlenbeck process that is Gaussian, of zero mean, and with correlation

$$C(t,t')=\langle\xi(t)\xi(t')\rangle=(D/\tau)\exp(-|t-t'|/\tau)\,.$$

The problem is the calculation of the statistical properties associated with the non-Markovian process $q(t)$. Results concerning the stationary distribution, steady-state dynamics, and transient dynamics for the process $q(t)$ are reviewed in Ref. 1. A large number of papers on this subject have been recently published and a sample of recent contributions is given in Refs. 2–5. Statements about confusion and controversy are often made.[2,3] A fair conclusion seems to be[3(a)] that useful insights are shared among many contributors, but a clear and useful theoretical understanding is still lacking. Many available developments [Refs. 1, 2(a), 2(b), 2(d), and 2(e)] aim to derive Fokker-Planck-type equations for the probability density $P(q,t)$ of the process $q(t)$ by systematic truncations or resummations of expansions in the parameters D and τ. These type of expansions have proved themselves to be useful in many specific applications,[1] but their validity has often been challenged.[1] An interesting different approximation, which does not invoke an expansion in powers of τ, is given in Ref. 6 where $q(t)$ is replaced by a Markovian, τ-dependent process. However, the extension and thorough justification of this result remains unclear. Early analogical[1,7] and numerical simulations[1,8] made clear the existence of qualitative facts predicted by the theory, but simulations aimed at discriminating different theoretical results have often added to the confusion, as discussed in Ref. 3(b), because of lack of accuracy or inadequate domain of parameters explored. More careful numerical studies have been recently reported.[3] In view of this general situation it is clear that there is a need for exact results and new methods which do not use τ as a small parameter. In this regard an important result[9] which has not been exploited is the form of the Lagrangian-like function[10] in q space featuring in the path-integral formulation of (1). For additive noise ($g=1$) the action integral is

$$S=\int_{t_0}^{t}\mathcal{L}\,dt+\frac{\tau}{2D}[\dot{q}-f(q)]^2_{t=t_0}\,, \tag{2a}$$

where

$$\mathcal{L}=\{\tau\ddot{q}+[1-\tau f'(q)]\dot{q}-f(q)\}^2/(4D)\,. \tag{2b}$$

The last term in (2a) involves non-Markovian initial-condition effects. Steady-state distributions and the approximations considered in this paper are independent of this term, which is neglected in our present discussion. Equation (2) is an exact result, which contains all required information on the process $q(t)$, and which only involves terms linear and quadratic in τ. Although functional methods have been widely used in the problem of colored noise[1] it appears that the methodology of a genuine path-integral approach in terms of a Lagrangian has not been considered at length. An exception is recent works aimed to calculate passage times[4] which, however, use the path integral in an extended phase space instead of the q-configuration space. Our purpose in this paper is to clarify the contents of the simple result (2) and to show the possibilities which the Lagrangian (2) opens for carrying out practical calculations. We are concerned here with a matter of principles and not of detailed comparisons with specific models and simulations. We show how a consistent Markovian Fokker-Planck approximation can be obtained from (2), which turns out to coincide with that of Ref. 6. The most relevant result in this paper is the calculation of the steady-state probability distribution in the limit of small noise intensity D. It is a direct calculation starting from the Lagrangian (2) so that no underlying Fokker-Planck approximation is invoked.

The Lagrangian (2) was obtained in Ref. 9 using the Jacobian of the transformation that connects the noise variable ξ with the coordinate q.[11] The explicit form of (2) assumes an initial preparation in which q and ξ are uncorrelated, and stationary initial conditions for ξ. Other alternative paths to (2) help in clarifying its meaning. We mention two of these paths which start by writing the two-variable Markovian Fokker-Planck equation associated with (1) in the enlarged (q,ξ) space. A Lagrangian in the (q,ξ) configuration space does not follow in the usual way[11] because the diffusion matrix is singular. However, the path-integral formulation for the Markovian process in the phase space spanned by (q,ξ) and their conjugate momenta is standard. A first path to obtain the Lagrangian (2) is to make a Legendre transformation[12] of the Hamiltonian in that phase space and to use Eq. (1) to eliminate the dependence on ξ.[13] A second path to (2) uses Phythian's[14] approach: Integrating in the extended phase space over ξ and over its conjugate moment we obtain a weighting function $A(q,\hat{q})$ for the paths in the restricted phase space spanned by q and its conjugate momentum $\hat{q}$

$$A(q,\hat{q}) = -i\int ds\,\hat{q}(s)\{\dot{q}(s)-f[q(s)]\} - \tfrac{1}{2}\int ds\int ds'\,\hat{q}(s)C(s,s')\hat{q}(s') . \quad (3)$$

The remaining integration over $\hat{q}$ is possible if one is able to invert the correlation function of the noise in the following sense:

$$\int_{t_0}^{t} C(s,s')R(s',s'')ds' = \delta(s-s'') . \quad (4)$$

This formal inversion, in the case of the Ornstein-Uhlenbeck noise, leads to (apart from surface terms)

$$R(s,s') = [\delta(s-s') - \tau^2\delta''(s-s')]/2 , \quad (5)$$

a remarkable result which directly leads to (2). While the formal expansion of C in the sense of distributions has all the powers of τ and all contributions $\delta^{(n)}(s-s')$ [(n) indicating the nth derivative with respect to the time s'], its inverse has only contributions up to the second order in τ, a result which is apparent in the form of the Lagrangian (2) which involves no power series in τ. The result (2) can be easily generalized for the multiplicative case [$g(q)\neq 1$]. In such a case, for example, by means of the inversion approach (5), we find

$$\mathcal{L} = (\tau\{\ddot{q}-f(q)\dot{q}-g'(q)\dot{q}[\dot{q}-f(q)]/g(q)\} + \dot{q}-f(q))^2/[4Dg(q)^2] . \quad (6)$$

From the structure of the Lagrangian-like functions (2) or (6) we see that the non-Markovian character of the process $q(t)$ reflects itself in the presence of terms involving $\ddot{q}$. Then, in order to have a consistent Markovian approximation to the process, it is clear that we need to remove such dependence from the Lagrangian, as well as the last terms in (2a), instead of making "expansions" in τ. The most drastic of all possible Markovian approximations is to take $\ddot{q}=0$. In the additive case it leads us immediately to a true Fokker-Planck Lagrangian corresponding to an effective multiplicative white noise SDE, which, by comparison with the general structure[11] for the Lagrangian of Fokker-Planck operators, coincides with the adiabatic approximation of Ref. 6. This justifies the dynamical contents of the Fokker-Planck equation proposed by Jung and Hänggi.[6] For the multiplicative case (6) and in the Markov approximation $\ddot{q}=0$, we are still left with a Lagrangian that has $\dot{q}^4$ contributions. These contributions are not allowed in a Lagrangian associated with a Markovian Fokker-Planck equation.[11] The easiest way to recover a desired Fokker-Planck approximation is to eliminate all powers of $\dot{q}$ larger than quadratic [that means to neglect the term $g'(q)\dot{q}^2$ in the Lagrangian (6)]. The Fokker-Planck equation (FPE) that results in such a case is[15]

$$\partial_t P(q,t) = -\partial_q\left[\left(\frac{(1-\tau g'f/g)f}{(1-\tau f'+\tau g'f/g)}+D'(q)/2\right)P\right] + \partial_q^2 D(q)P ,$$

$$D(q) = Dg(q)^2/[1-\tau f'(q)+\tau g'(q)f(q)/g(q)]^2 . \quad (7)$$

At this point it is worth noting that it is not the smallness of τ that leads to a Markov approximation. Even in the case of keeping only terms up to first order in τ in the Lagrangian the problem will still be non-Markovian due to the presence of the $\ddot{q}$ contribution. That notwithstanding, keeping all powers in τ but neglecting the terms with $\ddot{q}$, we get independent of the magnitude of τ, a Markovian approximation without increasing the complexity of the calculation. This fact comes directly from the inverse of the correlation function (5), which has, at most, contributions up to τ^2.

Let us next consider the stationary distribution of the process $q(t)$ defined by (1). A form of such distribution for $g=1$ which seems to be favored by empirical and numerical considerations[2(e),3(a)] for weak noise is

$$P_{st}(q) = N\,|\,1-\tau f'(q)\,|\exp[-\Phi(q)/D] , \quad (8)$$

$$\Phi(q) = -\int f(q)dq + \tau f(q)^2/2 . \quad (9)$$

This form can be justified in several ways. First, in the limit $D\rightarrow 0$ the dominant contribution is given by the potential $\Phi(q)$ in (9) and it coincides with the result obtained by an *ad hoc* exponentiation proposed in Ref. 8 (see also Ref. 1) to solve a Fokker-Planck equation obtained to first order in τ. A more refined justification of this procedure is to look for solutions of the first order in τ Fokker-Planck equation of the form $P_{st}\simeq e^{-\Phi/D}$. The equation obtained[16] for Φ in this way is solved by (9). A second justification is that Eqs. (8) and (9) are the formal solution of a Fokker-Planck approximation[17] which is intermediate[1] between the small τ and small D equations of Ref. 8. These two justifications explicitly rely on small τ assumptions. However, it turns out that Eqs. (8) and (9) are also the stationary solution of the Fokker-Planck equation proposed in Ref. 6. In addition, the distribution of Eqs. (8) and (9) is known[2(e),6,8,17] to become exact in the limit $\tau\rightarrow\infty$. We recall that the approximation of Ref. 6 is not based on the smallness of τ, but on a Fokker-Planck equation describing a rather different dynamics than those of Refs. 8 and 17. Our results above favor the point of view of Ref. 6 in the sense that Eqs. (8) and (9) are justified as the long-time limit of the Fokker-Planck dynamics obtained as a consistent Markovian approximation valid for all values of τ. In the multiplicative case the stationary solution of (7) has a potential

$$\Phi = -\int [f(q')/g(q')^2][1-\tau f'(q') + \tau f(q')g'(q')/g(q')]dq' , \quad (10)$$

and it also coincides with that indicated in Ref. 6. However, we will now show that the wisdom of Eqs. (8) and (9) goes beyond the Markoviau Fokker-Planck approximation discussed above. We do that by a direct calculation of (9) starting from the Lagrangian (2). The function Φ is defined as the action of the minimizing path in q space and it is calculated solving a generalized Hamilton-Jacobi-like equation associated with (2). This approach to the problem gives a rather safe first-principle ground to (9) without relying on underlying Fokker-Planck approximations.

It is known[12(b)] that the nonequilibrium potential Φ can be obtained in the weak-noise limit as

$$D^{-1}\Phi(q)=\min\int \mathcal{L}(q,\dot{q},\ddot{q})dt\,. \tag{11}$$

To minimize the action integral we need to choose an appropriate trajectory solution of the associated Euler-Lagrange equations. In order to clarify ideas we consider the linear case, that is $f(q)=-aq$, $g=1$. The Euler-Lagrange equations have two solutions, one is the deterministic trajectory going to the attractor $q=0$, a second solution, which we call antideterministic, starts in the attractor and goes outwards. It is this second solution starting at times $t=-\infty$, and specified by values q and $\dot{q}$ at $t=0$, that is of relevance here. Integrating the Lagrangian along this trajectory one obtains

$$\Phi=\min[(1+a\tau)(aq^2/2+\tau\dot{q}^2/2)]\,. \tag{12}$$

The minimizing path is the one for which $\dot{q}=0$. In this case (9) reproduces the exact stationary distribution of the linear case.[1] The same procedure can be, in principle, applied to a general nonlinear case, but the explicit form of the minimizing path is not trivially obtained.[18] Instead, we look for an equation for Φ in this general case. In the case of having a Lagrangian which is a function of q and $\dot{q}$ only, Φ fulfills[12] a Hamilton-Jacobi equation related to the Hamiltonian associated with the underlying Fokker-Planck equation. In our case, properly speaking, $\mathcal{L}$ is not a Lagrangian because it depends on $\ddot{q}$. However, using a generalized approach due to Buchdahl,[19] we obtain the desired Hamilton-Jacobi-like equation. For $g=1$ we find

$$\tau^{-2}\left(\frac{\partial\bar{\Phi}}{\partial\dot{q}}\right)^2-\tau^{-1}\{[1-\tau f'(q)]\dot{q}-f(q)\}\times\left(\frac{\partial\bar{\Phi}}{\partial\dot{q}}\right)+\dot{q}\left(\frac{\partial\bar{\Phi}}{\partial q}\right)=0\,. \tag{13}$$

The solution of (13) is a function $\bar{\Phi}$ of q and $\dot{q}$, and we are interested, as explicitly seen in the linear case, in this function for the value of $\dot{q}$ at which it has a minimum. The structure of (13) indicates that $\dot{q}=0$ is a minimum of $\bar{\Phi}$, so that we can look for a solution of (13) in the form

$$\bar{\Phi}(q,\dot{q})=\Phi(q)+\dot{q}^2\Phi_1(q)+\cdots\,. \tag{14}$$

From (13) and (14) and keeping terms up to second order in $\dot{q}$, we obtain the following equations for Φ and Φ_1:

$$\frac{d\Phi(q)}{dq}=-2f(q)\Phi_1(q)/\tau,\quad \Phi_1(q)=\tau[1-\tau f'(q)]/2\,. \tag{15}$$

This immediately reproduces (9). Similar steps for the multiplicative-noise case reproduce the potential Φ given in (10). It is clear that the only explicit argument needed to obtain the potential Φ in this scheme is one of weak-noise intensity (11), and not of small τ or any adiabatic or Markovian approximation.[20] We note, however, that our discussion based on (11) requires further elaboration when more than a single deterministic dynamical attractor exists and that the extent of the validity of the quadratic approximation in (14) might depend on the value of τ.

As a final point we consider the extension of the above results to the case in which a Gaussian white noise $\eta(t)$ is added to Eq. (1). For the SDE

$$\dot{q}(t)=f(q)+\xi(t)+\eta(t)\,, \tag{16}$$

with $\langle n(t)\eta(t')\rangle=2\epsilon\delta(t-t')$ we can repeat the procedure based on (4) to obtain the appropriate Lagrangian. It happens that the inverse of the correlation function of the Gaussian noise $\xi(t)+\eta(t)$ contains all the powers in τ and $\delta^{(n)}(s-s')$. The Lagrangian contains, then, arbitrary time derivatives of q. This identifies the difficulty of the nontrivial extension of the adiabatic approximation of Ref. 6 to this case. A Markovian Fokker-Planck approximation can be done in the same sense, that is, neglecting the Lagrangian terms containing time derivatives $q^{(m)}$ with $m>2$ and powers $\dot{q}^n$ with $n>2$. The Lagrangian obtained in this way corresponds to the FPE (7) and $g=1$ and τ and D replaced, respectively, by $\tilde{\tau}=\tau[D/(\epsilon+D)]^{1/2}$ and $\tilde{D}=D+\epsilon$. The fact that the stationary distribution of such an equation is not exact for the linear case $f(q)=-aq$ gives an idea of the limitations of this approximation.[21]

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