Dissipative quantum dynamics of a charged particle in a magnetic field

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We calculate the position autocorrelation and magnetic moment of a charged particle at finite temperatures moving in two dimensions in the presence of a magnetic field and a dissipative environment. We show that the time-dependent position-autocorrelation function does not change qualitatively from the free-particle behavior. This implies that orbital magnetic moment of a charged particle is zero at all temperatures.

I. INTRODUCTION

In this paper we consider the problem of a quantum-mechanical charged particle (henceforth referred to as an electron) moving in two dimensions in the presence of a magnetic field and coupled to a dissipative environment.[1–4] The main result of this work is that the time dependence of the position-autocorrelation function is qualitatively the same as that of a free particle, the only difference being that the overall coefficient is decreased from the free-particle value by a factor depending on the magnetic field. This is found to be the case even at zero temperature or equivalently at times $t \ll \hbar/k_B T$, where the dynamics is essentially quantum mechanical. Quantitatively,

$$\langle [R(t)-R(0)]^2\rangle = \begin{cases} \dfrac{4k_B Tt}{M\tau_R} \Big/ \left[\dfrac{1}{\tau_R^2}+\dfrac{e^2B^2}{M^2c^2}\right], & t \gg \hbar/k_B T \\ \dfrac{2\hbar}{M\tau_R}\dfrac{\ln\gamma t}{\left[\dfrac{1}{\tau_R^2}+\dfrac{e^2B^2}{m^2c^2}\right]}, & t \ll \hbar/k_B T \end{cases} \tag{1.1}$$

where $R(t)$ is the position of the electron at time t and B is the magnetic field in the z direction.

We then go on to calculate the orbital magnetic moment of the electron at finite as well as at zero temperature and show that the time-averaged magnetic moment goes to zero when there is a confining potential. These results are, of course, known[5] to hold at nonzero temperature $k_B T \gg \hbar/t$; we show that they even work at zero temperature (or $k_B T \ll \hbar/t$).

The plan of the paper is as follows. In Sec. II we introduce the general formalism and notation used in the rest of the paper. In Sec. III we discuss the general properties of the noise-auto-correlation function. In Sec. IV we calculate the position-auto-correlation function at both finite and zero temperatures. In Sec. V we calculate the magnetic moment of the electron with and without a confining potential, at zero and nonzero temperatures. We summarize our findings in Sec. VI.

II. MODEL AND BASIC NOTATION

The Hamiltonian on which our analysis is based is

$$H=\frac{1}{2M}\left[\mathbf{P}-\frac{e\,\mathbf{A}}{c}\right]^2+\sum_\alpha \frac{p_\alpha^2}{2m_\alpha}+\sum_\alpha \frac{1}{2}m_\alpha\omega_\alpha^2 x_\alpha^2 + \mathbf{R}\cdot\sum_\alpha C_\alpha \mathbf{x}_\alpha + R^2\sum_\alpha \frac{C_\alpha^2}{2m_\alpha\omega_\alpha^2}\ , \tag{2.1}$$

where $\mathbf{P}$, M, e, and $\mathbf{R}$ are the momentum, mass, charge, and position of the electron, respectively. $\mathbf{A}$ is the vector potential and c is the velocity of light. The electron is linearly coupled to a set of harmonic oscillators labeled by α with frequencies ω_α, mass m_α, coordinates $\mathbf{x}_\alpha$ and momenta $\mathbf{p}_\alpha$. C_α is the coupling between the electron and the oscillator path. The last term is added to ensure that the effective potential for the electron, obtained by eliminating x_α in favor of R, is zero apart from the magnetic field term. The equations of motion for the coordinates $\mathbf{R}(t)$ and momentum $\mathbf{P}(t)$ of the electron in the Heisenberg picture are

$$\begin{aligned}\dot{\mathbf{P}}(t) &= -\left[\sum_\alpha C_\alpha x_\alpha(t)+\frac{e}{c}(\mathbf{v}\times\mathbf{B})\right]+\mathbf{R}\sum_\alpha (c_\alpha^2/m_\alpha\omega_\alpha^2)\ ,\\ \ddot{\mathbf{R}}(t) &= -\left[\sum_\alpha \frac{c_\alpha}{M}x_\alpha(t)+\frac{e}{Mc}(\mathbf{v}\times\mathbf{B})\right] + \frac{\mathbf{R}}{M}\sum_\alpha C_\alpha^2/(m_\alpha\omega_\alpha^2)\ .\end{aligned} \tag{2.2}$$

The equation of motion for the oscillator coordinate $x_\alpha(t)$ is

$$\ddot{\mathbf{x}}_\alpha(t) = -\omega_\alpha^2\mathbf{x}_\alpha - \frac{\mathbf{R}(t)C_\alpha}{m_\alpha}\ . \tag{2.3}$$

Now eliminating $x_\alpha(t)$ and combining Eqs. (2.2) and (2.3) we find that

$$\ddot{\mathbf{R}}(t)+\frac{e}{Mc}(\mathbf{v}\times\mathbf{B})+\int K(t-t')\mathbf{v}(t') = -\mathbf{A}(t)\ , \tag{2.4}$$

where the memory kernel $K(t-t')$ is

$$K(t-t')=\sum_\alpha \frac{MC_\alpha^2}{m_\alpha}\cos[\omega_\alpha(t-t')] \tag{2.5}$$

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and

$$\mathbf{A}(t)=\sum_\alpha \frac{M}{m_\alpha}\cos[\omega_\alpha(t-t_0)]\mathbf{R}(t_0)$$

$$+\sum_\alpha C_\alpha\left[\frac{\hbar}{m_\alpha\omega_\alpha}\right]^{1/2}[e^{-i\omega_\alpha(t-t_0)}b_\alpha(t_0)+e^{i\omega_\alpha(t-t_0)}b_\alpha^\dagger(t_0)] , \quad (2.6)$$

where $b(t)$ and $b^\dagger(t)$ are two-dimensional annihilation and creation operators satisfying the commutation relation

$$[b,b^\dagger]=1 .$$

Now defining $Z=X+iY$, where X and Y are the two components of R, it is easy to see that

$$\ddot{Z}(t)+\frac{ieB}{Mc}\dot{Z}(t)+\int_0^t K(t-t')\dot{Z}(t')dt'=-A(t) , \quad (2.7)$$

where

$$A(t)=\sum_\alpha \frac{M}{m_\alpha}\frac{C_\alpha^2}{\omega_\alpha^2}\cos[\omega_\alpha(t-t_0)][X(t_0)+iY(t_0)]$$

$$+\sum_\alpha C_\alpha\left[\frac{\hbar}{2m_\alpha\omega_\alpha}\right]^{1/2}\times\left\{e^{-i\omega_\alpha(t-t_0)}[b_\alpha(t_0)+ia_\alpha(t_0)]+e^{i\omega_\alpha(t-t_0)}[b_\alpha^\dagger(t_0)+ia_\alpha^\dagger(t_0)]\right\} . \quad (2.8)$$

This is an exact equation for the dynamics of the charged particle and has been obtained by integrating out the fast degrees of freedom in the spirit of the Mori's[6] formalism. This we could do exactly because of the simple model system we have chosen. Note that no irreversible effects have been put in so far.

We introduced relaxation into the system by making the bath of oscillators take a continuous set of frequencies, which means the Poincare recurrence time is infinite. This approach is pioneered by Senitzky[7] and used by Caldeira and Leggett[2] to tackle the problem of macroscopic quantum tunneling and coherence. In order to allow detailed analysis we choose the memory kernel $K(t)$ to be of the form

$$K(t)=\frac{\gamma}{\tau_R}\Theta(t)e^{-\gamma t} , \quad (2.9)$$

where $\Theta(t)$ is the unit step function. γ is the bandwidth of the oscillators and τ_R will turn out to be the relaxation time of the velocity of the particle. Let us now discuss the initial conditions for this problem.

The kernel has a memory time which is of the order γ^{-1}. Thus, if we choose to look at times much larger than γ^{-1}, the initial position of the particle is irrelevant. We assume that the initial coordinates of the oscillators are in thermal equilibrium at temperature T. Therefore

$$\langle b_\alpha^\dagger(t_0)b_{\alpha'}(t_0)\rangle=\frac{\delta_{\alpha\alpha'}}{(e^{\beta\hbar\omega_\alpha}-1)}, \quad \beta=\frac{1}{k_BT} . \quad (2.10)$$

Since we are looking at time scales $t\gg\gamma^{-1}$, all single time averages are time independent and all two-time correlations will depend only on the time difference. We must emphasize the point that so far whatever we have done is entirely consistent with microscopic physics. It can easily be shown that the commutation relation, $[x(t),p(t)]=i\hbar$ holds for all times. This should be contrasted with earlier approaches,[8,9] where phenomenological Hamiltonians were written down to incorporate dissipation, which had serious problems of interpretation.

Let us define the Fourier transform of $f(t)$:

$$f(\omega)=\int_{-\infty}^{\infty}e^{i\omega t}f(t)dt .$$

The autocorrelation function of the fluctuating force per unit mass

$$\phi_T(t)=\tfrac{1}{2}\langle A^*(t+t')A(t)+A^*(t')A(t+t')\rangle \quad (2.11)$$

takes the form

$$\phi_I(t)=\frac{2\hbar\gamma^2}{M\tau_R}\int_{-\infty}^{\infty}1\frac{dw}{2\pi}e^{i\omega t}\frac{\omega}{\omega^2+\gamma^2}\coth\left[\frac{\beta\hbar\omega}{2}\right] . \quad (2.12)$$

The factor of 2 is important and it comes from the fact that A is a combination of A_x and A_y. The autocorrelation function of the position of the particle can be found very easily by Fourier transforming the equations of motion. We find

$$C_{ZZ}(t)=\tfrac{1}{2}\langle Z(t)Z^*(t+t')+Z^*(t)Z(t+t')\rangle , \quad (2.13)$$

where

$$C_{ZZ}(t)=\frac{2\hbar\gamma^2}{M\tau_R}\int_{-\infty}^{\infty}d\omega\frac{1}{2\pi}e^{i\omega t}\frac{\omega}{\omega^2+\gamma^2}M^2|\chi(\omega)|^2\times\coth(\beta\hbar\omega/2) \quad (2.14)$$

and

$$\chi(\omega)=\frac{1}{\omega^2+i\omega K(\omega)-(eB/Mc)\omega} . \quad (2.15)$$

We shall use this equation exclusively for the calculations which follow.

III. PROPERTIES OF THE NOISE-NOISE CORRELATION FUNCTION

Properties of the noise-autocorrelation function have been dealt with exclusively in the paper by Aslangul *et al.* We merely sketch them here for completeness and because we shall use them in the calculations that follow. From Eq. (2.12) it is clear that

$$\langle A|(\omega)|^2\rangle=\frac{2\hbar\gamma^2}{M\tau_R}\frac{\omega}{\omega^2+\gamma^2}\coth(\beta\hbar\omega/2) . \quad (3.1)$$

For $\beta\hbar\omega\ll1$ (i.e., high temperatures) this is approximately

$$\langle |A(\omega)|^2\rangle = \frac{4k_BT}{M\tau_R}\frac{\gamma^2}{\gamma^2+\omega^2} , \tag{3.2}$$

which for frequencies $\omega \ll \gamma$ becomes

$$\langle |A(\omega)|^2\rangle = \frac{4k_BT}{M\tau_R} . \tag{3.3}$$

This is ordinary white noise. But if $\beta\hbar\omega \gg 1$ then

$$\langle |A(\omega)|^2\rangle = \frac{2\hbar\gamma^2}{M\tau_R}\frac{\omega}{\omega^2+\gamma^2} . \tag{3.4}$$

Note that this form of $A(\omega)$ implies that

$$\langle A^2(t)\rangle = \int_{-\infty}^{\infty} d\omega |A(\omega)|^2 e^{i\omega t} = \int_{-\infty}^{\infty} d\omega \frac{\omega}{\omega^2+\gamma^2} ,$$

which diverges at large frequencies. On physical grounds we would like $\langle A^2(t)\rangle$ to be finite. We shall thus impose a high-frequency cutoff $\omega_{\rm max}$. It is reasonable to take $\omega_{\rm max} \approx \gamma^{-1}$, since we are in any case interested in the physics of time scales much larger than γ^{-1}. Thus, if we make measurements at a fixed temperature T, we should see quantum behavior for frequencies $\omega \gg k_BT/\hbar = \omega_*$, i.e., for time scales $t \ll \hbar/k_BT$. For $\omega \ll \omega_*$ ($t \gg \hbar\omega_*^{-1}$) classical behavior will be seen, which, for $\omega \ll \gamma$, will be ordinary Brownian motion. Note that the crossover time scale ω_*^{-1} diverges as $T \to 0$.

IV. CALCULATION OF POSITION AUTOCORRELATION FUNCTION

Here we calculate the position-autocorrelation function and show that its qualitative behavior in both quantum ($t \ll \hbar/k_BT$) and classical ($t \gg \hbar/k_BT$) regimes is the same as that of a free particle. We proceed as follows. Upon simplification Eq. (2.14) this yields

$$C_{ZZ}(t) = \frac{2\hbar\gamma^2}{M\tau_R}\int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} \times \frac{e^{i\omega t}\coth(\beta\hbar\omega/2)}{\left[\omega^2 - \frac{\gamma}{\tau_R} - \omega_c\right]^2 + \gamma^2(\omega-\omega_c)^2} , \tag{4.1}$$

where $\omega_c = eB/Mc$ is the cyclotron frequency. For time scales that are sufficiently large (i.e., $t \gg \hbar/k_BT$) this integral is easily evaluated,

$$C_{ZZ}(t) = \left\{ 4k_BT \Big/ \left[M\tau_R \left[\frac{1}{\tau_R^2} + \omega_c^2 \right] \right] \right\} t . \tag{4.2}$$

This tells us, not surprisingly, that at any nonzero temperature the long-time behavior of the particle is going to be classical and diffusive but with a reduced diffusion coefficient. As is already known, the magnetic field does not confine a classical Brownian particle. We see quantum behavior by looking at time scales $t \ll \hbar/k_BT$. This behavior is easily extracted by setting $T=0$ in the integral, so that we are always in the quantum regime. One might wonder whether the combination of a magnetic field and "quantum dissipation" can confine a particle at zero temperature. It turns out that it cannot. The calculations go as follows:

$$C_{ZZ}(t) = \frac{2\hbar\gamma^2}{M\tau_R}\int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} \frac{e^{i\omega t}}{\omega\left[\left[\omega^2 - \frac{\gamma}{\tau_R} - \omega\omega_c\right]^2 + \gamma^2(\omega-\omega_c)^2\right]} . \tag{4.3}$$

Evaluating this integral and noting that there is an upper cutoff γ^{-1} to the frequency integral we get, for $t \gg \gamma^{-1}$, ω_c^{-1},

$$C_{ZZ}(t) = \frac{2\hbar}{M\tau_R}\frac{\ln(\gamma t)}{\left[\frac{1}{\tau_R^2} + \omega_c^2\right]} . \tag{4.4}$$

To summarize, the effect of the magnetic field on the position autocorrelation is a quantitative one. The functional dependence on time is unchanged. The classical diffusion and quantum subdiffusion still take place. Only the overall coefficient is smaller by a field-dependent factor. This is the first of our two principal results.

V. ORBITAL MAGNETIC MOMENT OF THE ELECTRON

We calculate the orbital magnetic moment of the electron in the presence of the dissipative environment. The idea is to test whether the time average orbital magnetic moment of the electron goes to zero even at $T=0$ due to the presence of dissipative environment. Indeed, we find that it does, if we are careful. Let us first show that our results agree with the known results in the classical limit, for example, at nonzero temperature, where all the quantum effects are washed out on a sufficiently long time scale. Then we do the calculation for zero temperature and show that the results hold there as well.

The orbital magnetic moment μ of the electron is given by

$$\mu = \frac{-|e|}{2c}\langle \mathbf{v}\times\mathbf{R}\rangle , \tag{5.1}$$

which implies that

$$\mu = \frac{-|e|}{2c}{\rm Im}\langle (Z^*\dot{Z})\rangle . \tag{5.2}$$

Now let us take $\beta\hbar\omega \ll 1$,

$$\mu = -\frac{|e|}{2c}\frac{2k_BT}{M\tau_R}\int_{-\infty}^{\infty} d\omega \frac{1}{2\pi}\frac{1}{\omega\left[(\omega-\omega_c)^2+\frac{1}{\tau_R^2}\right]}$$

$$= -\frac{|e|}{2cM}\frac{k_BT\omega_c}{\left[\frac{1}{\tau_R^2}+\omega_c^2\right]} . \quad (5.3)$$

This seeming contradiction to the Bohr–van Leeuwen theorem is a result of the fact that we have not bounded the motion of the particle. The "skipping orbits" which are the usual dynamical explanation of the Bohr–van Leeuwen result can only arise if the particle is confined externally by walls or by a potential. To see this we impose a confining potential which we take to be the harmonic-oscillator potential in two dimensions. Then the equation of motion becomes

$$\ddot{Z}(t)+\int_0^t K(t-t')\dot{Z}(t')+\frac{ieB}{Mc}\dot{Z}(t)+\omega_0^2 Z(t) = -A(t) . \quad (5.4)$$

The orbital magnetic moment of the electron, averaged up to a time τ, is

$$\mu(\tau) = \frac{|e|}{Mc\tau_R}\int d\omega \frac{1}{2\pi}\frac{\omega e^{i\omega\tau}}{(\omega^2-\omega_0^2-\omega\omega_c)^2+\frac{\omega^2}{\tau_R^2}} . \quad (5.5)$$

This integral, by inspection, goes as τ^{-2}, so that if averaging time τ is taken to infinity, we see that $\lim_{\tau\to\infty}\mu(\tau)=0$, for any $\omega_0 \neq 0$. Note that what we have done here is to take the limit of infinite averaging time τ first and then take $\omega_0 \to 0$. Had we worked with a finite τ then we would have seen that for $\omega_0\tau \gg 1$, the equilibrium statistical mechanics result of Bohr–van Leeuwen holds, while for $\omega_0\tau \ll 1$, the particle would not have explored most of the phase space and would have a nonzero orbital magnetic moment. Note also that in the presence of a confining potential where the "skipping cycles" do not occur as such, the cancellation is due to the restoring force of the confining potential which causes the guiding center of the Larmour orbit to move in a sense counter to the Larmour orbit itself. For a lucid discussion of this classical regime see Ref. 10.

Now we follow the same procedure at zero temperature. First we perform the calculation without a confining potential, and find that $\mu=0$. Then we go on to calculate the orbital moment when there is a confining potential and find that in this case it is zero again. The orbital moment of the electron at $T=0$ without the confining potential is

$$\mu = -\frac{|e|}{2c}\frac{2\hbar\gamma^2}{m\tau_R} \times \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi}\frac{1}{\gamma^2(\omega-\omega_c)^2+\left[\omega^2-\omega\omega_c-\frac{1}{\tau_R^2}\right]^2} . \quad (5.6)$$

This integral is straightforward, if a little tedious. We simply write down the final form for the magnetic moment

$$\mu = \frac{-|e|}{2Mc}\frac{\hbar}{M\tau_R}\omega_c\left[\frac{\sin(\theta/2)}{2d\left[\frac{\omega_c^2}{r^2}+d^2\sin^2\theta+\gamma^2\right]}\right] , \quad (5.7)$$

which is nonzero. In the above equation

$$\theta = \tan^{-1}\left[\frac{\omega_c\gamma}{\gamma^2-\omega_c^2-4\gamma\tau_R^{-1})}\right]$$

$$d = \left[\left[\gamma^2-\omega_c^2-\frac{l}{\tau_R^2}\right]^2+\gamma^2\omega_c^2\right]^{1/2} .$$

Now let us put in a confining potential as we did for the nonzero-temperature case, evaluating the integral and then letting the frequency of the confining potential go to zero. We find that the orbital magnetic moment, averaged up to a finite time τ, is given by

$$\mu(\tau) = -\frac{|e|}{2c}\frac{2\hbar\gamma^2}{M\tau_R} \times \int d\omega \frac{1}{2\pi}\frac{\omega^2}{\gamma^2+\omega^2}\frac{e^{i\omega\tau}}{|\omega^2-\omega_0^2-\omega\omega_c+ik(\omega)\omega|^2} . \quad (5.8)$$

As in the classical case (5.5) this gives $\lim_{\tau=\infty}\mu(\tau)=0$ for any $\omega_0 \neq 0$. Thus quantum dissipation with confining potential suffices to make the time-averaged magnetic moment go to zero even at zero temperature. This is the second principal result of this work.

VI. CONCLUSION

In this paper we have used the formalism developed in Refs. 1, 2, and 4 to calculate the dissipative dynamics of a charged spinless particle in the presence of a magnetic field. Firstly, we have obtained the position-autocorrelation function in the presence of a magnetic field and we have shown that it is qualitatively the same as that of a free particle provided there is dissipation. Secondly, we have calculated the time-averaged induced orbital magnetic moment of the electron and found it to be zero both for zero and finite temperatures in the presence of dissipation, showing therefore that the motion of the charged particle (as far as the magnetic moment is concerned) is rendered "classical."

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