# Feynman's propagator for a charged particle with time-dependent mass in a crossed time-varying electromagnetic field

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Through a sequence of time-dependent transformations and time substitution, we evaluate the propagator of a harmonically bound charged particle with time-dependent mass in a time-varying electromagnetic field by relating it to those of free particles. From this propagator we derive the wave functions. The propagators beyond and at caustics are then investigated, respectively, by including the Maslov phase factor and by using the modified semigroup property. Finally, we calculate explicitly the propagator for the constant-damping case.

## I. INTRODUCTION

In recent years there has been increasing interest in quantizing the harmonic oscillator with variable mass[1–9] and in quantizing the charged particle in a time-varying electromagnetic field.[4–6] In this regard, it would be worthwhile to investigate quantum mechanically the dynamical system that combines these two problems. In this paper we quantize a harmonically bound charged particle with time-variable mass in a time-dependent crossed electromagnetic field by using Feynman's path-integral approach. Although the entire expression obtained for the propagator has no obvious physical interpretation it is important to note its relevance in summarizing a great number of previous results obtained by various authors.[1–6,9–14] In addition, we observe also that some special cases of this Lagrangian connects to some propagators not previously calculated. For example, one can mention the case of constant mass, which has, for the first time, an expression valid beyond and at caustics. Another important, physical case is that of time-dependent frequency and mass, but without electric and magnetic fields. In particular, this case can be mapped in a system of electric and magnetic fields in the interior of a Fabry-Perot cavity.[1,2,9] In Sec. II, through a rotation, an extended Galilean transformation, a linear space transformation, and a time substitution, we are able to obtain the propagator of our system by connecting it to propagators of free particles and to derive the wave functions by expanding the propagator. In Sec. III we investigate the propagator beyond and at caustics by including the Maslov correction factor and by using the modified semigroup property of the propagator, respectively. In order to illustrate our method, we evaluate the propagator explicitly for the damping case in Sec. IV.

## II. PROPAGATOR AND THE WAVE FUNCTIONS

For a charged particle of variable mass $m(t)$ and charge $q$ moving classically in a time-dependent electromagnetic field, the Lagrangian has the form ($c=1$ throughout this paper)

$$L(\dot{\mathbf{r}},\mathbf{r},t)=\tfrac{1}{2}m(t)\dot{\mathbf{r}}^2+q\,\mathbf{A}(\mathbf{r},t)\cdot\dot{\mathbf{r}}-q\Phi(\mathbf{r},t),\quad m(t)>0 \tag{1}$$

with $\mathbf{A}(\mathbf{r},t)$ and $\Phi(\mathbf{r},t)$ being the vector and scalar potentials. Choosing the symmetric gauge,[15] $\mathbf{A}(\mathbf{r},t)=(-\frac{1}{2}B(t)x_2,\frac{1}{2}B(t)x_1,0)$, for the vector potential, we consider the potentials

$$\mathbf{A}(\mathbf{r},t)=\tfrac{1}{2}B(t)\hat{\mathbf{k}}\times\mathbf{r} \tag{2}$$

and

$$\Phi(\mathbf{r},t)=\frac{q}{2m(t)}\phi^2(t)\mathbf{r}^2+\boldsymbol{\epsilon}(t)\cdot\mathbf{r}\ , \tag{3}$$

which are more general than the case studied by Sökmen.[6] Here $m(t)$, $B(t)$, $\phi(t)$, and $\epsilon_i(t)$, the components of the vector $\boldsymbol{\epsilon}(t)$, are arbitrary piecewise continuous functions of time, and the position vector $\mathbf{r}=(x_1,x_2,x_3)$. With the preceding potentials, the corresponding electric and magnetic fields become

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)=B(t)\mathbf{k} \tag{4}$$

and

$$\mathbf{E}(\mathbf{r},t)=-\nabla\phi(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=-\frac{q}{m(t)}\phi^2(t)\mathbf{r}+\frac{B(t)}{2}\hat{\mathbf{k}}\times\mathbf{r}+\boldsymbol{\epsilon}(t)\ . \tag{5}$$

We now separate the Lagrangian (1) into the following form

$$L(\dot{\mathbf{r}},\mathbf{r},t)=L_\perp(\dot{\mathbf{r}}_\perp,\mathbf{r}_\perp,t)+L_\parallel(\dot{x}_3,x_3,t) \tag{6}$$

where

 

$$L_\perp(\dot{\mathbf{r}}_\perp,\mathbf{r}_\perp,t)=\tfrac{1}{2}m(t)\dot{\mathbf{r}}_\perp^2+\frac{qB(t)}{2}(\dot{x}_1x_2-\dot{x}_2x_1)$$
$$-\frac{q}{2m(t)}\phi^2(t)\mathbf{r}_\perp^2-q\epsilon_\perp(t)\cdot\mathbf{r}_\perp\ , \tag{7}$$

with $\mathbf{r}_\perp$ and $\epsilon_\perp(t)$ being the components of $\mathbf{r}$ and $\epsilon(t)$ perpendicular to the magnetic field $\mathbf{B}(\mathbf{r},t)$, and

$$L_\parallel(\dot{x}_3,x_3,t)=\tfrac{1}{2}m(t)\dot{x}_3^2-\frac{q}{2m(t)}\phi^2(t)x_3^2-q\epsilon_3(t)x_3\ . \tag{8}$$

In order to decouple the coordinates $x_1$ and $x_2$ in (7), we introduce a rotation about three-axis by a time-dependent angle $\omega(t)=qB(t)/2m(t)$, or

$$\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}=\begin{bmatrix}\cos[\alpha(t)] & \sin[\alpha(t)] & 0\\ -\sin[\alpha(t)] & \cos[\alpha(t)] & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\quad \mathbf{R}=\mathcal{R}\mathbf{r} \tag{9}$$

with $\alpha(t)=\int^t\omega(s)ds$ and $\mathcal{R}$ is the notation matrix. Under the above rotation, the Lagrangian (7) can be written as

$$L_\perp(\dot{\mathbf{R}}_\perp,\mathbf{R}_\perp,t)=\frac{m(t)}{2}[\dot{\mathbf{R}}_\perp^2-\Omega^2(t)\mathbf{R}_\perp^2]-q\mathcal{E}_\perp(t)\cdot\mathbf{R}_\perp\ , \tag{10}$$

where $\Omega^2(t)=\omega^2(t)+q\phi^2(t)/m^2(t)$ and $\mathbf{R}_\perp=(X_1,X_2)$ and $\mathcal{E}_\perp(t)=\mathcal{R}\epsilon_\perp$. With the help of (7)–(10), the Lagrangian (6) can be put in the form

$$L(\dot{\mathbf{r}},\mathbf{r},t)=\sum_{i=1}^{3}L_i(\dot{X}_i,X_i,t)\ , \tag{11}$$

where

$$L_i(\dot{X}_i,X_i,t)=\tfrac{1}{2}m(t)[\dot{X}_i^2-\Omega_i^2(t)X_i^2]-q\mathcal{E}_i(t)X_i\ , \tag{12}$$

with $\Omega_1^2(t)=\Omega_2^2(t)=\Omega^2(t)$ and $\Omega_3^2(t)=q\phi^2(t)/m^2(t)$, which is the Lagrangian of the time-dependent forced harmonic oscillator with time-varying mass and frequency.

Performing an extended Galilean transformation[7,8]

$$y_i=X_i+\eta_i(t) \tag{13}$$

and choosing $\eta_i(t)$ to satisfy

$$\ddot{\eta}_i+\frac{\dot{m}(t)}{m(t)}\dot{\eta}_i+\Omega_i^2(t)\eta_i=\frac{q}{m(t)}\mathcal{E}_i(t)\ , \tag{14}$$

the Lagrangian (12) becomes after lengthy manipulations

$$L_i(\dot{y}_i,y_i,t)=\tfrac{1}{2}m(t)[\dot{y}_i^2-\Omega_i^2y_i^2]+\dot{\Lambda}_i(y_i,\eta_i,t) \tag{15}$$

with

$$\dot{\Lambda}_i(y_i,\eta_i,t)=\tfrac{1}{2}m(t)\dot{\eta}_i(\eta_i-2y_i)$$
$$+\frac{q}{2}\int^t\mathcal{E}_i(\lambda)\eta_i(\lambda)d\lambda\ . \tag{16}$$

Except for the term of the total derivative, $\dot{\Lambda}_i(y_i,\eta_i,t)$, we see that (15) is the Lagrangian of the harmonic oscillator with time-varying mass and frequency. Therefore we eliminated the time-dependent forced term $q\mathcal{E}_iX_i$ in (12) by using (13) and (14). Applying a linear space transformation and a time substitution introduced by Cheng,[9]

$$Y_i=y_i\left[\frac{\xi(t)}{\xi'}\right]\dot{\mu}_i^{1/2}\sec[\mu_i(t)],\quad u_i=\tan[\mu_i(t)]\ , \tag{17}$$

where $\xi(t)=\sqrt{m(t)}$ and $s_i(t)$ and $\mu_i(t)$ satisfy

$$\ddot{s}_i+\left[\Omega_i^2(t)-\frac{\ddot{\xi}(t)}{\xi(t)}\right]s_i=s_i^{-3},\quad s_i^2\dot{\mu}_i=1\ , \tag{18}$$

the Lagrangian (15) reads

$$L_i(\dot{Y}_i,Y_i,t)=\tfrac{1}{2}m'\tilde{Y}_i^2+\dot{\Lambda}_i(y_i,\eta_i,t)-\dot{F}_i(s_i,\xi_i,t)\ ,$$
$$\tilde{Y}_i=\frac{dY_i}{du_i} \tag{19}$$

with

$$F_i(s_i,\xi_i,t)=\tfrac{1}{4}m'y_i^2\{\sin[2\mu_i(t)]-2(\dot{s}_i\xi-s_i\dot{\xi})/s_iu_i\xi\}\ . \tag{20}$$

For later convenience we assume that $f'=f(t')$ and $f''=f(t'')$ for any function $f$ of time $t$. From the Lagrangian (19) we see that our system has been reduced to three one-dimensional free particles of mass $m'$.

Using the Van Vleck–Pauli formula,[10,16] we obtain the propagator as

$$K(y_i'',y_i';\tau)=\prod_{i=1}^{3}K_i(y_i'',y_i',\tau)$$
$$=\prod_{i=1}^{3}\left[\left[\frac{\partial Y_i'}{\partial y_i'}\right]\left[\frac{\partial Y_i''}{\partial y_i''}\right]\right]^{1/2}K_i^F(Y_i'',Y_i';U_i)\exp\left[\frac{i}{\hbar}[\Lambda_i(y_i,\eta_i,t)-F_i(s_i,\xi_i,t)]\Big|_{t'}^{t''}\right]\ , \tag{21}$$

where $\tau=t''-t$, $U_i=u_i''-u_i'$, and $K_i^F(Y_i'',Y_i';U_i)$ is the well-known propagator[16] of an one-dimensional free particle. With the help of (11)–(14) and (21), we obtain our main results

$$K_i(y_i'',y_i';\tau)=\left[\frac{\xi'\xi''(\dot\mu_i'\dot\mu_i'')^{1/2}}{2\pi i\hbar\sin(\mu_i''-\mu_i')}\right]^{1/2}\exp\left\{\frac{i}{2\hbar}\left[y_i^2\left[\frac{\dot s_i\xi-s_i\dot\xi}{\xi s_i}\right]m\dot\eta_i(\eta_i-2y_i)\right]\bigg|_{t'}^{a}\right\}$$
$$\times\exp\left[\frac{i}{2\hbar\sin(\mu''-\mu')}[(m''\dot\mu_i''y_i''^2+m'\dot\mu_i'y_i^2)\cos(\mu_i''-\mu_i')-2\xi'\xi''(\dot\mu_i'\dot\mu_i'')^{1/2}y_i'y_i'']\right]$$
$$\times\exp\left[\frac{iq}{2\hbar}\int_{t'}^{t''}\mathcal{E}_i(t)\eta_i(t)dt\right]. \tag{22}$$

Using Mehler's formula[17]

$$\frac{\exp[-(x^2+y^2-2xyz)/(1-z^2)]}{(1-z^2)^{1/2}}=\exp[-(x^2+y^2)]\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(x)H_n(y)\ , \tag{23}$$

with $x=(m'\mu_i'/\hbar)^{1/2}y_i', y=(m''\mu_i''/\hbar)^{1/2}y_i''$, $z=\exp[-i(\mu_i''-\mu_i')]$, and $n=n_i$, we can rewrite (22) as

$$K_i(y_i'',y_i';\tau)=\sum_{n_i=0}^{\infty}\psi_{n_i}^*(y_i',t')\psi_{n_i}(y_i'',t'')\ ,$$

where the wave functions are of the form

$$\psi_{n_i}(y_i,t)=\exp[-i(n_i+\tfrac{1}{2})\mu_i(t)]\varphi_{n_i}(y_i,t) \tag{24}$$

$$\varphi_{n_i}(y_i,t)=\left[\frac{1}{2^{n_i}n_i!}\left[\frac{m(t)\dot\mu_i}{\pi\hbar}\right]^{1/2}\right]^{1/2}\exp\left\{\frac{im(t)}{2\hbar}\left[y_i^2\left[\frac{\dot s_i\xi-s_i\dot\xi}{\xi s_i}-\dot\mu_i\right]\dot\eta_i(\eta_i-2y_i)\right]\right\}$$
$$\times\exp\left[\frac{iq}{2\hbar}\int^{t}\mathcal{E}_i(\lambda)\eta_i(\lambda)d\lambda\right]H_{n_i}\left[\left[\frac{m(t)\dot\mu_i}{\hbar}\right]^{1/2}y_i\right]. \tag{25}$$

Here $H_{n_i}(\cdot)$ is the $n_i$th Hermite polynomial. Making use of the transformation

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}\cos[\alpha(t)] & -\sin[\alpha(t)] & 0\\ \sin[\alpha(t)] & \cos[\alpha(t)] & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}\eta_1\\\eta_2\\\eta_3\end{bmatrix}, \tag{26}$$

we can now easily express the propagator (21) and the wave functions (24) in terms of the original coordinates $x_1$, $x_2$, $x_3$.

## III. PROPAGATOR BEYOND AND AT CAUSTICS

Applying the extended Feynman's formula[18,19] to (22), we obtain the propagator beyond caustics ($\phi_i=\mu_i''-\mu_i'$)

$$K_i(y_i'',y_i';\ \phi_i\neq k_i\pi)=M(\phi_i)\left[\frac{\xi'\xi''(\dot\mu_i'\dot\mu_i'')^{1/2}}{2\pi i\hbar|\sin\phi_i|}\right]^{1/2}\exp\left\{\frac{im(t)}{2\hbar}\left[y_i^2\left[\frac{\dot s_i\xi-s_i\dot\xi}{\xi s_i}\right]+\dot\eta_i(\eta_i-2y_i)\right]\bigg|_{t'}^{t''}\right\}$$
$$\times\exp\left[\frac{i}{2\hbar\sin\phi_i}[(m''\dot\mu_i''y_i''^2+m'\dot\mu_i'y_i'^2)\cos\phi_i-2\xi'\xi''(\dot\mu_i'\dot\mu_i'')^{1/2}y_i'y_i'']\right]$$
$$\times\exp\left[\frac{iq}{2\hbar}\int_{t'}^{t''}\mathcal{E}_i(t)\eta_i(t)dt\right],\quad k_i=0,1,2,\ldots \tag{27}$$

where the Malov correction factor $M(\phi_i)=\exp[-i\pi\,\mathrm{Int}(\phi_i/\pi)/2]$ and $\mathrm{Int}(\phi_i/\pi)$ stands for the greatest integer which is less than or equal to $\phi_i/\pi$. We can see that (27) is invalid at caustics or when $\phi_i=k\pi$. We now introduce the modified semigroup property of the propagator.[19]

$$K_i(y_i'',y_i';\ \phi_i=k_i\pi)=\exp(-ik_i\pi/2)|K_i(y_i'',y_i;t''-t)|\,|K_i(y_i,y_i';t-t')|$$
$$\times\int_{-\infty}^{\infty}\exp\left\{\frac{i}{\hbar}[S_{\mathrm{cl}}(y_i'',y_i;t''-t)+S_{\mathrm{cl}}(y_i,y_i';t-t')]\right\}dy_i \tag{28}$$

since the Maslov correction factor is already known at caustics. $S_{\rm cl}(\ )$ is the classical action functional. Evaluating the integral in (28), we finally obtain the propagator at caustics,

$$K_i(y_i'',y_i';\varphi_i=k_i\pi)=[\xi'\xi''(\dot\mu_i'\dot\mu_i'')^{1/2}]^{1/2}\exp(-ik_i\pi/2)$$
$$\times\exp\left\{\frac{im(t)}{2\hbar}\left[y_i^2\left[\frac{\dot s_i\xi-s_i\dot\xi}{\xi s_i}\right]+\dot\eta_i(\eta_i-2y_i)\right]\Bigg|_{t'}^{t''}\right\}$$
$$\times\exp\left\{\frac{iq}{2\hbar}\int_{t'}^{t''}\mathcal{E}_i(t)\eta_i(t)dt\right\}\delta(\xi''\dot\mu_i''^{1/2}\sin(\mu_i-\mu_i')y_i''-\xi'\dot\mu_i'^{1/2}\sin(\mu_i''-\mu_i)y_i')\ . \tag{29}$$

We should mention that the classical paths of interest for the calculation of the propagator are those for which the initial and final positions are specified. When the arguments of the Dirac δ function in (29) vanish, there exists an infinite number of such classical paths, which should be expected for a system described by a quadratic Lagrangian. Therefore, the total propagator (21) can be obtained by combining (22) or (27) or (29) for each coordinate $y_1$, $y_2$, and $y_3$.

## IV. CONSTANT DAMPING CASE

In this section we consider the following Lagrangian

$$L(\dot{\mathbf r},\mathbf r,t)=\frac{m_0e^{\gamma t}}{2}\left[\dot{\mathbf r}^2+\frac{qB_0}{m_0}(\dot x_1x_2-x_1\dot x_2)-\frac{q^2\phi_0^2}{m_0^2}\mathbf r^2+\frac{2}{m_0}[\epsilon_1x_1+\epsilon_3x_3\cos(\omega t)]\right]\ , \tag{30}$$

where $\gamma$, $B_0$, $\phi_0$, $\epsilon_1$, and $\epsilon_3$ are constants. The corresponding equations of motion are given by

$$\begin{aligned}&\ddot x_1+\gamma\dot x_1+\Omega_0^2x_1=-2\omega_0\dot x_2-\gamma\omega_0x_2+q\epsilon_1/m_0\ ,\\&\ddot x_2+\gamma\dot x_2+\Omega_0^2x_2=2\omega_0\dot x_1+\gamma\omega_0x_1\ ,\\&\ddot x_3+\gamma\dot x_3+\Omega_0^2x_3=\epsilon_3\cos(\omega t)\end{aligned} \tag{31}$$

with $\Omega_0^2=q^2\phi_0^2/m_0^2$ and $\omega_0=qB_0/2m_0$. Therefore the Lagrangian (30) represents a damped charged particle of mass $m_0$ in a crossed time-dependent electromagnetic field.[20]

Equation (14) now becomes

$$\ddot\eta_i+\gamma\dot\eta_i+\Omega_i^2\eta_i=q\mathcal{E}_i/m_0\ , \tag{32}$$

where $\Omega_1^2=\Omega_2^2=\Omega_0^2+\omega_0^2$, $\Omega_3^2=\Omega_0^2$, $\mathcal{E}_1=\epsilon_1\cos(\omega_0t)$, $\mathcal{E}_2=-\epsilon_1\sin(\omega_0t)$, and $\mathcal{E}_3=\epsilon_3\cos(\omega t)$. Without loss of generality, hereafter we only investigate (32) for the underdamping case or $\Omega_i^2>\gamma^2$. The particular solutions of (32) are of the form

$$\begin{aligned}&\eta_1=\frac{q\epsilon_1\sin(\omega_0t+\beta_1)}{m_0(\Omega_0^4+\gamma^2\omega_0^2)^{1/2}},\quad\beta_1=\tan^{-1}\left[\frac{\Omega_0^2}{\gamma\omega_0}\right]\\&\eta_2=\frac{q\epsilon_1\sin(\omega_0t+\beta_2)}{m_0(\Omega_0^4+\gamma^2\omega_0^2)^{1/2}},\quad\beta_2=\beta_1+\pi/2\\&\eta_3=\frac{q\epsilon_3\sin(\omega t+\beta_3)}{m_0[(\Omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{1/2}},\quad\beta_3=\tan^{-1}\left[\frac{\Omega_0^2-\omega^2}{\gamma\omega}\right]\end{aligned} \tag{33}$$

which do not depend on the initial conditions of $\eta_i$. We should mention that for evaluating $\Lambda_i(y_i,\eta_i,t)$ in (21) we only need the steady state[21] of (32). Substituting (26) into (22), we obtain explicitly the propagators for the constant-damping charged particle

$$K_\perp(\mathbf r_\perp'',\mathbf r_\perp';\tau)=\frac{m_0\omega_1e^{\gamma(t'+t'')/2}}{2\pi i\hbar\sin(\omega_1\tau)}\exp\left[\frac{iq\epsilon_1}{2\hbar}\int_{t'}^{t''}[\eta_1\cos(\omega_0t)-\eta_2\cos(\omega_0t)]dt\right]$$
$$\times\exp\left[\frac{im\gamma}{4\hbar}\{[x_1\cos(\omega_0t)-x_2\sin(\omega_0t)+\eta_1]^2+[x_1\sin(\omega_0t)+x_2\cos(\omega_0t)+\eta_2]^2\}\Big|_{t'}^{t''}\right]$$
$$\times\exp\left[\frac{im}{2\hbar}(\dot\eta_1\{2[x_1\cos(\omega_0t)-x_2\sin(\omega_0t)]+\eta_1\}+\dot\eta_2\{2[x_1\sin(\omega_0t)+x_2\cos(\omega_0t)+\eta_2]\})\Big|_{t''}^{t'}\right]$$
$$\times\exp\left[\frac{im''\omega_1}{2\hbar\tan(\omega_1\tau)}\{[x_1''\cos(\omega_0t'')-x_2''\sin(\omega_0t'')+\eta_1]^2+[x_1''\sin(\omega_0t'')+x_2''\cos(\omega_0t'')+\eta_2]^2\}\right]$$
$$\times\exp\left[\frac{im'\omega_1}{2\hbar\tan(\omega_1\tau)}\{[x_1'\cos(\omega_0t')-x_2'\sin(\omega_0t')+\eta_1]^2+[x_1'\sin(\omega_0t')+x_2'\cos(\omega_0t')+\eta_2]^2\}\right]$$
$$\times\exp\left[-\frac{im_0\omega_1e^{\gamma(t'+t'')/2}}{\hbar\sin(\omega_1\tau)}\{[x_1'\cos(\omega_0t')-x_2'\sin(\omega_0t')+\eta_1][x_1''\cos(\omega_0t'')-x_2''\sin(\omega_0t'')+\eta_2]\right.$$
$$+[x_1'\sin(\omega_0t')+x_2'\cos(\omega_0t')+\eta_1]$$
$$\left.\times[x_1''\sin(\omega_0t'')+x_2''\cos(\omega_0t'')+\eta_2]\}\right],\quad\omega_1^2=\Omega_1^2-\gamma^2 \tag{34}$$

and

$$K_{\parallel}(x_3'',x_3';\tau)=\left[\frac{m_0\omega_3 e^{\gamma(t'+t'')/2}}{2\pi i\hbar\sin(\omega_3\tau)}\right]^{1/2}\exp\left[\frac{iq\epsilon_3}{2\hbar}\int_{t'}^{t''}\eta_3\cos(\omega t)dt\right]$$
$$\times\exp\left[\frac{im}{4\hbar}[\gamma(x_3+\eta_3)^2-\dot{\eta}_3(4x_3+2\eta_3)]\Big|_{t'}^{t''}\right]$$
$$\times\exp\left[\frac{i\omega_3}{2\hbar\sin(\omega_3\tau)}\{[m''(x_3''+\eta_3)^2+m'(x_3'+\eta_3)^2]\cos(\omega_3\tau)$$
$$-2m_0e^{\gamma(t'+t'')/2}(x_3'+\eta_3)(x_3''+\eta_3)\}\right],\quad \omega_3^2=\Omega_0^2-\gamma^2 \tag{35}$$

since $\dot{s}_i=0$ and $\mu_i=\omega_i t$.

Considering the case of $\epsilon_1=0$ (or $\eta_1=\eta_2=0$) and $\omega=0$ (or $\eta_3=q\epsilon_3/m_0\Omega_3^2$) and combining (34) and (35), we get the total propagator (after simplifications),

$$K(\mathbf{r}'',\mathbf{r}';\tau)=K_{\perp}(\mathbf{r}_{\perp}'',\mathbf{r}_{\perp}';\tau)K_{\parallel}(x_3'',x_3';\tau)$$
$$=\frac{m_0\omega_1 e^{\gamma(t'+t'')/2}}{2\pi i\hbar\sin(\omega_1\tau)}\left[\frac{m_0\omega_3 e^{\gamma(t'+t'')/2}}{2\pi i\hbar\sin(\omega_3\tau)}\right]^{1/2}\exp\left[\frac{iq^2\epsilon_3^2\tau}{2\hbar m_0\Omega_3^2}\right]$$
$$\times\exp\left[\frac{im_0\gamma}{4\hbar}[(x_1''^2+x_2''^2)e^{\gamma t''}-(x_1'^2+x_2'^2)e^{\gamma t'}]\right]$$
$$\times\exp\left[\frac{i\omega_1}{2\hbar\tan(\omega_1\tau)}[m''(x_1''^2+x_2''^2)+m'(x_1'^2+x_2'^2)]\right]$$
$$\times\exp\left[-\frac{im_0e^{\gamma(t'+t'')/2}}{\hbar\sin(\omega_1\tau)}[(x_1'x_1''+x_2'x_2'')\cos(\omega_0\tau)+(x_1'x_2''-x_1''x_2')\sin(\omega_0\tau)]\right]$$
$$\times\exp\left[\frac{i\omega_3}{2\hbar\sin(\omega_3\tau)}[(m''z''^2+m'z'^2)\cos(\omega_3\tau)-2m_0e^{\gamma(t'+t'')/2}z'z'']\right]$$
$$\times\exp\left[\frac{i\gamma}{4\hbar}(m''z''^2-m'z'^2)\right],\quad z=x_3+q\epsilon_3/m_0\Omega_3^2 \tag{36}$$

in which $K_{\perp}(\mathbf{r}_{\perp}'',\mathbf{r}_{\perp}';\tau)$ is in agreement with Eq. (35) in Ref. 17. For $\gamma=0$, the preceding equation reduces to

$$K(\mathbf{r}'',\mathbf{r}';\tau)=\frac{m_0\Omega_1}{2\pi i\hbar\sin(\Omega_1\tau)}\left[\frac{m_0\Omega_3}{2\pi i\hbar\sin(\Omega_3\tau)}\right]^{1/2}\exp\left[\frac{iq^2\epsilon_3^2\tau}{2\hbar m_0\Omega_3^2}\right]$$
$$\times\exp\left[\frac{im_0\Omega_1}{2\hbar\tan(\Omega_1\tau)}(x_1'^2+x_2'^2+x_1''^2+x_2''^2)\right]$$
$$\times\exp\left[-\frac{im_0}{\hbar\sin(\Omega_1\tau)}[(x_1'x_1''+x_2'x_2'')cos(\omega_0\tau)+(x_1'x_2''-x_2'x_1'')\sin(\omega_0\tau)]\right]$$
$$\times\exp\left[\frac{im_0\Omega_3}{2\hbar\sin(\Omega_3\tau)}[(z'^2+z''^2)\cos(\Omega_3\tau)-2z'z'']\right], \tag{37}$$

from which we see that the Bloch density matrix $K(\mathbf{r}'',\mathbf{r}';-i\hbar\beta)$ is exactly equivalent to that of Glasser.[12]

In this paper we obtained the propagator for a harmonically bound charged particle with time-dependent mass in a crossed time-varying electromagnetic field by relating the propagator to those of free particles through a sequence of space transformations and time substitution. Our results are more general than the previous known results,[6,13,14] in the sense that (1) the time-dependent mass has been taken into account, (2) the most general time-varying crossed electromagnetic field has been considered, (3) the Maslov correction factor (or Morse index) has been included in the propagator beyond caustics, and (4) the Dirac δ function has appeared in the propagator at caustics as it should be for a quadratic Lagrangian. In other words, for a quadratic Lagrangian there exists an infinite number of classical paths with fixed end points in which the arguments of the Dirac δ function vanish. As for the constant-damping case, we evaluated explicitly the propagator which includes the well known results[11,12] as special cases as we expect. As a final remark we should mention that the present method is inadequate to evaluate the propagator for an anisotropic bound charged particle in a time-varying crossed electromagnetic field.

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