Experimental study of interface instability in a Hele-Shaw cell

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Experiments on the stability of the circular interface formed in a Hele-Shaw cell when air displaces a viscous oil have shown that available theories underestimate the wavelength of the most unstable wave when the capillary number (N_{ca}) is large. Apparently this is caused by a modification of the interface boundary condition by three-dimensional effects. At small values of N_{ca} the results overlap several of the theories and we are unable to choose among them on the basis of the present experiments.

INTRODUCTION AND THEORY

When a less viscous fluid displaces a more viscous one in a Hele-Shaw cell, the moving interface between the two fluids becomes unstable by mechanisms that are, in general, well understood.[1-3] However, since these early works a number of questions have arisen concerning the exact boundary conditions to be applied at the interface, and application of these has slightly modified the theory given by the authors introduced above. In particular, in the earlier theories the pressure drop across the interface was assumed, *ad hoc*, to be of the form

$$\Delta p = -\sigma \left[\frac{2}{b} + \frac{1}{\Re} \right] ,$$

where σ is the surface tension between the two interpenetrating fluids, b is the gap between the plates, and $\Re$ the radius of curvature of the interface in the plane of the plates. Here it is assumed that the contact angle between the interface and the plate surfaces is zero [Fig. 1(a)].

Under these conditions the wave number of the most unstable wave, i.e., that with the maximum growth rate, was found to be

$$k_{max} = 2\frac{N_{ca}^{1/2}}{b} , \qquad (1)$$

where N_{ca} is the capillary number, equal to $V\mu/\sigma$, V is the velocity of the interface, and μ the viscosity of the displaced fluid, while the viscosity of the displacing fluid is assumed to be much smaller than μ.

A more precise derivation of this boundary condition by Park and Homsy[4] led to an expression for $\Delta p = -\sigma(2/b + \pi/4\Re)$ for N_{ca} much less than unity, which in turn modifies the relationship for k_{max} to

$$k_{max} = 2.26\frac{N_{ca}^{1/2}}{b} . \qquad (2)$$

Further progress was made by Schwartz[5] who used the full boundary condition derived by Park and Homsy,[4] taking into account first-order wetting effects, that is, the influence of the viscous layer left behind on the plate as the interface moves, Fig. 1(b). This condition can be written, following Schwartz, as

$$\Delta p = -\sigma \left[\frac{2}{b} + \frac{\pi}{4\Re} \right] - \frac{2\sigma J}{b} N_{ca}^{2/3} ,$$

where in the Park-Homsy formulation $J \approx 3.8$. Defining a modified wave number as $A = kbN_{ca}^{-1/2}$ led to an equation for the wave number with the maximum growth rate A_{max} as

$$\frac{\pi J}{216} N_{ca}^{1/6} A_{max}^3 + \frac{\pi}{16} A_{max}^2 - 1 = 0 , \qquad (3)$$

which is plotted together with (1) and (2) in Fig. 2 for $J = 3.8$ and leads to longer wavelengths than the simpler version of the Park-Homsy boundary condition.

With this as a basis it is a simple matter to modify Schwartz's result to take account of the wetting effect more precisely. We make use of the numerical results of Ref. 6, which appear, based on comparison with the

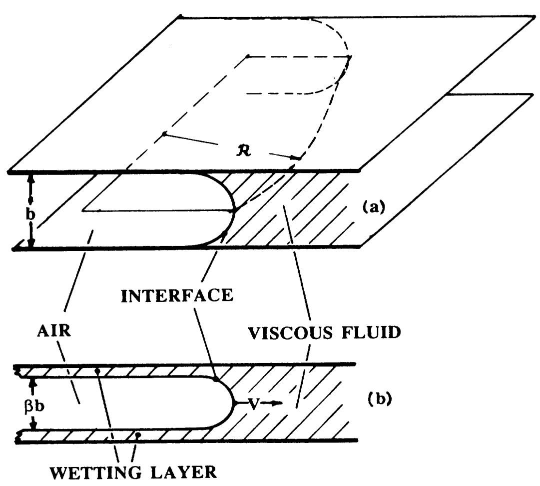

FIG. 1. Interface conditions: (a) The *ad hoc* Chouke *et al.* and Saffman-Taylor and the simpler of the Park-Homsy conditions. (b) The correct conditions with the dynamic, two-dimensional wetting-layer effect included.

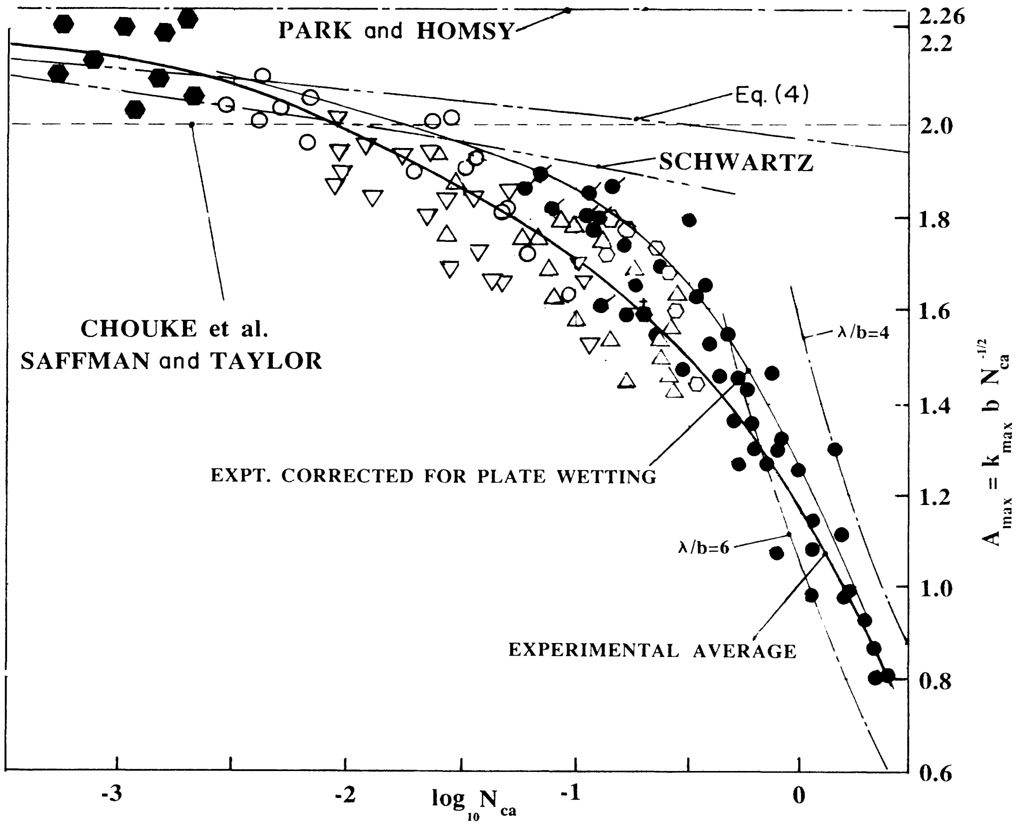

FIG. 2. Modified wave number for the most unstable wave vs capillary number. The various theoretical results are shown. The experimental points are for values of b and μ equal to ○, 0.034 cm, 1.17 P; ▽, 0.065 cm, 1.17 P; △, 0.128 cm, 1.17 P; ⬡, 0.191 cm, 1.17 P; ⬡, 0.0153 cm, 0.061 P; ⬢, 0.065 cm, 9.95 P; and ●, 0.128 cm, 9.95 P, while the heavy line is the average drawn through these points. The lighter solid curve is the experimental average corrected for the effects of plate wetting. The solid point ● corresponds to the case shown in Fig. 3.

dynamically similar flow in a tube, to accurately describe the interfacial pressure drop for all values of N_{ca}. An empirical fit to their curve of N_{cp} versus N_{ca} (their Fig. 5) gives

$$N_{cp} = -1.92 N_{ca}^{0.61} ,$$

where cp is the pressure drop across the interface. Whence Eq. (3) becomes

$$\frac{1.92\pi}{216} N_{ca}^{0.11} A_{max}^3 + \frac{\pi}{16} A_{max}^2 - 1 = 0 . \qquad (4)$$

This result is plotted in Fig. 2 as well. As expected, the smaller pressure drop associated with the correctly calculated flow results in shorter wavelengths than those found in the Schwartz formulation and is actually closer to the Park-Homsy result.

Available data on the stability of such an interface, as outlined, for example, in Ref. 7 show large error bars, and, as described in that paper, are subject to compromise due to side-wall interference effects. Also the range of N_{ca} studied has been somewhat restricted for practical reasons and so the full stability problem has not been investigated.

In what follows we have devised an experimental method that removes some of these difficulties. It is based, formally, on the work of Paterson[8] who calculated the stability of a radially expanding interface and found that the number of waves formed around a circle of radius R was given by

$$n_{max} = \frac{1}{\sqrt{3}} \left[12 N_{ca} \frac{R^2}{b^2} + 1 \right]^{1/2}$$

for the *ad hoc* boundary condition leading to Eq. (1). For most cases of interest the first term in the parentheses is much larger than unity so that the equation reduces to Eq. (1) when one realizes that $n_{max} = k_{max} R$. Thus the effects of the other assumptions concerning the pressure-jump boundary conditions can easily be incorporated into the circular geometry.

APPARATUS AND PROCEDURE

Two Hele-Shaw cells were used: the first consisted of two glass plates 1.9 cm thick and 60 cm in diameter, while the second was made of the same material but was only 25 cm in diameter. They were mounted in an aluminum ring that supported the bottom plate around its periphery. The top plate rested on six supports that were 2 cm square and of the appropriate thickness. Five

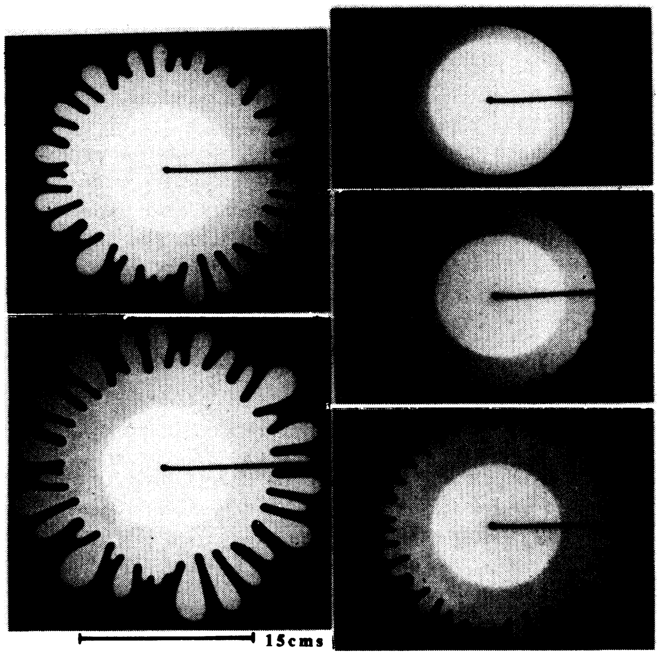

FIG. 3. Photographs of the evolving instability on an interface. The scale on the bottom is 15 cm long. Time interval between photographs is 0.3 s. Thirty-five waves can be counted. The change in grey level in the central region is due to the thicker layers of oil left behind by the rapidly moving interface after the initial circle of air was formed by slow air injection.

different thicknesses were used, resulting in gap widths (b) of 0.0153, 0.034, 0.065, 0.128, and 0.191 cm, as found by injecting known volumes of fluid and measuring the area of the circle of fluid thus formed. The gaps were uniform to within 0.001 cm, as measured in this way. In the larger cell the top plate was clamped into place by a circular ring and O-ring that was attached to the bottom ring by 12 machine screws; in the second it was held in place by a heavy retaining ring. A small hole was drilled into the center of each of the top plates and a thin-walled stainless-steel tube glued within to provide the source for the injected fluid, in this case air. The displaced fluids were a DC200 silicone oil of the following properties: (i) viscosity (μ) 1.17 P and surface tension (σ) 21.2 dyn/cm, (ii) $\mu=0.06$ P, $\sigma=19.9$ dyn/cm, and (iii) $\mu=9.95$ P, $\sigma=21.2$ dyn/cm, at the operating temperature of the cell.

An initially circular interface of large diameter, approximately 15 cm, was formed by injecting air very slowly into the central supply hole. The inflow velocity was sufficiently slow that no unstable waves were formed. At the initiation of an experiment the flow rate was suddenly increased to a larger constant value, the interface velocity V increased, a thicker layer of oil was left on the plates, and after a short time instability waves appeared and grew on the interface (Fig. 3). The values of b, μ, and V used were such that at least 30 waves were always formed, so that enough waves were generated to give a good average value for the wave number of maximum growth rate. A video camera and recorder or a motor-driven 35-mm camera were used to record the interface position as a function of time, the interface radius at which instability first appeared R_i, and the number of waves formed $n_{\max}$.

By measuring the increase in area of the central circular region over a known time interval the "apparent" inflow rate can be calculated and set equal to $2\pi R_i V_i$, where R_i is the radius at which the instability first appeared and V_i is the actual interface velocity at that time. As discussed in the next section, it is possible to recast this result in terms of a true independent variable, i.e., the velocity at which the interface would move if there were no wetting of the glass plates. This has been done by using the results of Reinelt and Saffman[6] to calculate the total thickness $(1-\beta)b$ of the wetting films at the value of N_{ca} calculated using the measured velocity V_i. The corrected velocity is then βV_i and the actual inflow rate $2\pi R_i \beta V_i b$.

RESULTS

Some 110 experiments were run at the five values of b and three fluids mentioned previously and at several values of flow rate. From the measured values of V_i and R_i, and the number of waves formed $n_{\max}$, both N_{ca_i} and $k_{\max}$ could be found. Here it has been assumed that the observed wave number corresponds to that of the most unstable waves.

In order to emphasize the discrepancies between theory and experiment, it was found most convenient to plot the results in terms of the modified wave number

$$A_{\max}=k_{\max} b N_{\mathrm{ca}_i}^{-1/2}$$

and N_{ca_i} itself, in which case the simpler theories reduce to $A_{\max}=$const. The results are shown in Fig. 2 for a range of N_{ca_i} of almost four orders of magnitude. Data scatter is fairly large, representing variations of $\pm 10\%$ typically; however, the trends are quite clear and allow a number of interesting observations.

DISCUSSION

The results presented in the preceding section show clear deviations from all of the theoretical predictions at the larger values of N_{ca_i} whether based on the actual measured interface velocity (lower experimental curve) or velocity corrected for the presence of the wetting layers (upper curve). These smaller wave numbers are also consistent with the result discussed in Ref. 9 for the case of a plane interface driven by gravity. Furthermore, there is a clear tendency for the experimental results to approach a wavelength that is a constant multiple of the gap width b, for values of N_{ca} greater than unity, approximately. There are two possible reasons for this discrepancy, finite Reynolds-number $(N_{\mathrm{Re}}=V_i b/v)$ or three-dimensional effects.

If one looks at the Reynolds-number distribution of the plotted points there is a slight tendency for the very-low-Reynolds-number cases, i.e., the solid points of Fig. 2

($N_{\mathrm{Re}} < 0.07$), to be higher than the other points ($0.2 < N_{\mathrm{Re}} < 0.7$); however, the variation is well within the experimented inaccuracies and data scatter. That three-dimensional effects are a more likely source, for the observed differences can be inferred from the observation that the measured wavelengths at large N_{ca} approach a value of about five times the gap width. This is very reminiscent of the result of Paterson[10] for the interfacial stability of miscible fluids, i.e., those for which no surface tension exists. In this case the most favored wavelength was found to depend solely on viscous dissipation between the three-dimensional finger structures. This suggests that in the present case, even when N_{ca} is of order 1, that surface-tension effects are negligible and that here too the wavelength of maximum growth rate is determined solely by dissipation in a three-dimensional viscous flow.

At the smaller values of N_{ca_i} the experiments fall between the Schwartz[5] and Park-Homsy[4] theories and perhaps agree reasonably well with Eq. (4); however, here also the experimental scatter makes it difficult to choose between the theories. What is surprising is the observation that N_{ca} must be less than $O(10^{-2})$ before any of the theories can be considered valid.

It has been suggested[11] that the use of the extended boundary conditions presented by Reinelt[12] might help improve the agreement between theory and experiment especially at values of N_{ca} removed from the regime of constant λ/b. Unfortunately, such a theory does not exist at present, but one hopes that the present experiments might provide some impetus to generate one.

ACKNOWLEDGMENTS

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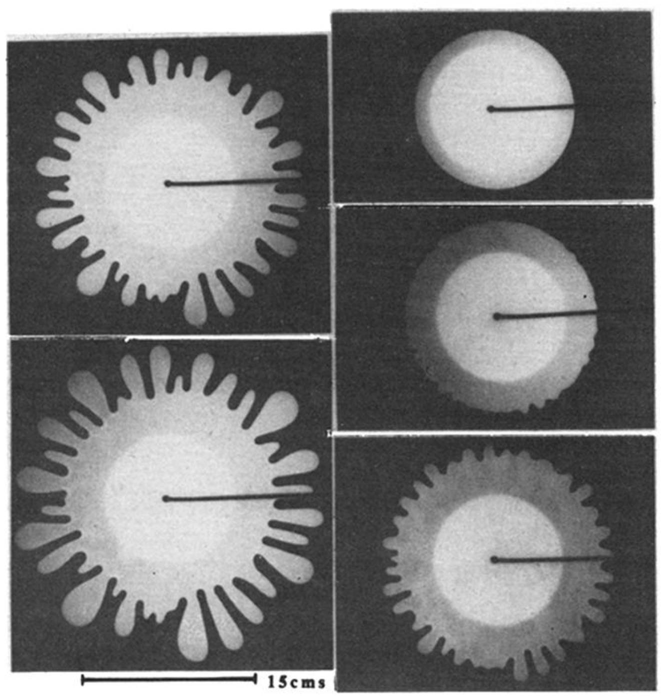

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