Dynamics of pulsed SU(1,1) coherent states

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In this paper we consider the time evolution of SU(1,1) coherent states driven by a coherence-preserving Hamiltonian containing periodic or quasiperiodic pulsing terms. This is a generalization of a system consisting of a two-level atom subjected to quasiperiodic pulsing that was recently studied by Milonni, Ackerhalt, and Goggin [Phys. Rev. A **35**, 1714 (1987)]. The time-evolution operator in our case is given by a product of two finite group transformations of SU(1,1). Assuming an initial SU(1,1) coherent state, we determine the equivalent classical motion generated by a Poincaré map that is a Möbius transformation on the Lobachevski plane, the interior of the unit circle in the complex plane. The quantum-mechanical evolution of the state vector is calculated exactly and in closed form even though the Hilbert space is infinite dimensional. We also study the autocorrelation function which, as in the work of Milonni, Ackerhalt, and Goggin, is found to decay in the case of quasiperiodic pulsing that may possibly be associated with a manifestation of chaos in a quantum-mechanical system.

I. INTRODUCTION

In recent years there has been a great deal of interest in the study of the quantum dynamics of systems driven by time-periodic Hamiltonians. On the classical level, many of them give rise to the type of motion that has been described as "chaotic."[1] On the one hand, it is generally conceded that on the quantum level, the chaotic nature of the system in question becomes suppressed. For example, Hogg and Huberman[2] have proved a theorem that for bounded nonresonant systems, the state vector will reassemble itself infinitely often in time. In the problem of the quantum-kicked rotor, whose classical counterpart exhibits chaotic dynamics[3] and diffusive energy growth, one finds the suppression of the chaotic motion and of the energy growth; in fact, the energy becomes quasiperiodic in time.[1] This problem was also shown[4] to be related to that of Anderson's localization[5] of wave functions in a one-dimensional lattice in the presence of a static, random potential.

It has, however, been recently shown that some of the manifestations of chaos can make an appearance in quantum dynamics. Pomeau *et al.*[6] and Milonni *et al.*[7] considered quasiperiodically kicked two-level systems and showed that for incommensurate pulsing frequencies "quantum chaos" exists in the sense that (i) the autocorrelation function of the state vector decays, (ii) the power spectrum of the state vector is broadband, and (iii) the motion on the Bloch sphere is ergodic. The quantum localization effect for a kicked rotor is greatly weakened by the presence of the two incommensurate driving frequencies.[6] The system studied by Milonni *et al.*[7] consists of a two-level atom in the dipole approximation where the interacting electromagnetic field consists of a periodic sequence of δ functions modulated by a periodic function whose frequency is incommensurate with that of the δ-function sequence. If the atom is initially in the ground state, the Hamiltonian, which is linear in the Pauli matrices, generates a generalized coherent state associated with the Lie group SU(2).[8] In fact, the Hamiltonian will actually preserve the coherence of an arbitrary initial SU(2) coherent state and it is well known that the "classical" motion of that state takes place on the Bloch sphere.

In this paper we consider the noncompact analog of the SU(2) system studied in Ref. 7, namely, a Hamiltonian belonging to a general class[9] which preserves, under time evolution, coherent states (CS's) associated with the noncompact Lie group SU(1,1). Such states have been shown to be of considerable importance in the field of nonlinear optics as they provide an example of ideal squeezed states, in fact, squeezed vacuum states.[10,11] Such states are produced by the interaction of an intense coherent beam (laser light) with a nonlinear medium modeled as a degenerate or nondegenerate parametric amplifier.[12] Here, we examine quasiperiodic forcing when the Hamiltonian is linear in the generators of SU(1,1). Such a system could possibly be realized as a parametric amplifier with the pumping field being modulated by quasiperiodic pulsing. Assuming that the initial state is an SU(1,1) CS, as defined by Perelomov,[10] we employ group theory to determine the Poincaré maps which define the evolution of the CS from one pulse to the next. It should be emphasized that even though the Hilbert space representing the SU(1,1) dynamical group is infinite dimensional, the quantum-mechanical evolution of the relevant expectation values (energy, correlation functions, etc.) for the associated coherent states may be expressed in *closed form*. In the case of pulsing, this evolution is described by discrete "stroboscopic" or Poincaré-type evolution maps. (This is unlike the situation of the quantum rotor, where summations over an infinite basis of the relevant Hilbert space are *approximated* by truncations.) The phase space is the interior of the unit disk in the complex plane, with the non-Euclidean geometry of the Lobachevski plane. As in the case of the two-level atom, we find that even though coherence is preserved, the autocorrelation function of the state vector decays.

The paper is organized as follows. In Sec. II we describe the model Hamiltonian and determine its quantum dynamics. An expression for the autocorrelation function is also derived. In Sec. III, the case of quasiperiodic pulsing with commensurate frequencies, i.e., periodic pulsing, is treated. Section IV is concerned with the case of incommensurate frequencies or almost periodic pulsing. The results of some numerical calculations are discussed. We conclude with a brief summary in Sec. IV.

II. MODEL

The most general Hamiltonian which preserves SU(1,1) CS is of the form[9]

$$H(t)=A(t)K_0+f(t)K_+ +f^*(t)K_- +B(t)\ , \tag{2.1}$$

where $A(t)$ and $B(t)$ are arbitrary functions of time and $f(t)$ is an arbitrary complex function of time. The operators K_0, $K_\pm = K_1 \pm K_2$ close as an su(1,1) Lie algebra:

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=2K_0\ . \tag{2.2}$$

The Perelomov SU(1,1) CS's are defined by the action[10]

$$|\xi,k\rangle=\exp(\alpha K_+ -\alpha^* K_-)|0,k\rangle\ , \tag{2.3}$$

where $\alpha=-(\theta/2)e^{-i\phi}$, $\xi=-\tanh(\theta/2)e^{-i\phi}$, with ϕ and θ being group parameters with ranges $-\infty<\theta<\infty$, $0\le\phi\le 2\pi$. The constant k is the Bargmann index related to the eigenvalue $k(k-1)$ of the Casimir operator $C=K_0^2-K_1^2-K_2^2$. As usual we consider only the unitary irreducible representations denoted as $\mathcal{D}^\dagger(k)$, whose basis states $|m,k\rangle$ diagonalize the compact generator K_0 as follows: $K_0|m,k\rangle=(m+k)|m,k\rangle$, $m=0,1,2,\ldots$, with $k>0$. The parameter ξ defines the phase space, the Lobachevski plane,[8,9,13] and $|\xi|<1$. In terms of the basis vectors, the SU(1,1) CS becomes

$$|\xi,k\rangle=(1-|\xi|^2)^k\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right]^{1/2}\xi^m|m,k\rangle\ . \tag{2.4}$$

Specific realizations and representations of the Lie algebra are required for any application to quantum systems. We first consider the single-mode case, where the algebra is realized in terms of a single set of Bose operators

$$K_0=\tfrac{1}{4}(a^\dagger a+aa^\dagger),\quad K_+=\tfrac{1}{2}(a^\dagger)^2,\quad K_-=\tfrac{1}{2}a^2\ . \tag{2.5}$$

The Bargmann index becomes $k=\frac{1}{4}$ (even photon number) or $k=\frac{3}{4}$ (odd photon number). These operators are sufficient for the case of a degenerate parametric amplifier.[10,11] For a two-mode (nondegenerate) parametric amplifier an appropriate realization is given by[10]

$$K_0=a_1^\dagger a_1+a_2^\dagger a_2+1,\quad K_+=a_1^\dagger a_2^\dagger,\quad K_-=a_1a_2\ , \tag{2.6}$$

and the corresponding Bargmann index is $k=n_1-n_2$, the difference between the number of photons in mode 1 and in mode 2. If the Hamiltonian H is linear in the generators [of Eq. (2.6)], as is the case in Eq. (2.1), then k is a fixed number since the Casimir operator commutes with H. In most of what follows, the results are independent of the index k. However, when we require a specific system we shall use the realization for the degenerate parametric amplifier with the $k=\frac{1}{4}$ representation, which includes the vacuum state.

With no loss of generality, we set $B(t)=0$ in Eq. (2.1) and $A(t)=2\omega_0$, which is appropriate for the degenerate parametric amplifier. Furthermore, we assume that $f(t)$ is real and has the form

$$f(t)=F(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\ , \tag{2.7}$$

where $T>0$ and $F(t)$ is a real valued periodic function of time to be specified later. The Hamiltonian thus becomes

$$H=2\omega_0K_0+2K_1F(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\ , \tag{2.8}$$

where we have set $K_1=\frac{1}{2}(K_++K_-)$. If the state vector just prior to nth δ-function pulse is designated as $|\psi(n)\rangle$, then immediately after the pulse it is given by

$$|\psi(n)\rangle'=e^{-i2K_1F(nT)}|\psi(n)\rangle\ . \tag{2.9}$$

Between pulses the evolution is governed by the free field Hamiltonian $H_0=2\omega_0K_0$, so that the state vector just prior to the $(n+1)$th pulse is

$$|\psi(n+1)\rangle=U(n)|\psi(n)\rangle\ , \tag{2.10}$$

where $U(n)$ is the evolution operator

$$U(n)=e^{-2i\omega_0TK_0}e^{-2iF(nT)K_1}\ . \tag{2.11}$$

This evolution operator constitutes a product of finite SU(1,1) group transformations. Using the non-Hermitian 2×2 representation of the su(1,1) Lie algebra, where $K_0=\sigma_3/2$ and $K_1=i\sigma_2/2$ (σ_i denote the usual Pauli matrices), we obtain the corresponding 2×2 group elements

$$[e^{-2i\omega_0TK_0}]_{2\times2}=\begin{bmatrix}e^{-i\omega_0T} & 0\\ 0 & e^{i\omega_0T}\end{bmatrix}\ , \tag{2.12}$$

$$[e^{-2iF(nT)K_1}]_{2\times2}=\begin{bmatrix}\cosh[F(nT)] & -i\sinh[F(nT)]\\ i\sinh[F(nT)] & \cosh[F(nT)]\end{bmatrix}\ . \tag{2.13}$$

Thus the 2×2 group element corresponding to the evolution operator U is

$$[U(n)]_{2\times2}=\begin{bmatrix}a_n & b_n\\ b_n^* & a_n^*\end{bmatrix}\ , \tag{2.14}$$

where

$$a_n = e^{-i\omega_0 T}\cosh[F(nT)] ,$$
$$b_n = -ie^{-i\omega_0 T}\sinh[F(nT)] , \tag{2.15a}$$

and

$$|a_n|^2 - |b_n|^2 = 1 . \tag{2.15b}$$

We now let the initial state at $n=0$ be an SU(1,1) CS, to be denoted as $|\xi_0,k,0\rangle = |\xi_0,k\rangle$. The action of the evolution operator of Eq. (2.11) produces the time evolved SU(1,1) CS, that is,

$$|\xi_0,k,1\rangle = U(0)|\xi_0,k,0\rangle \tag{2.16}$$

and

$$|\xi_0,k,j+1\rangle = U(j)|\xi_0,k,j\rangle, \quad j=1,2,..,n . \tag{2.17}$$

However, the action of a finite SU(1,1) transformation U on an SU(1,1) CS $|\xi,k\rangle$ is given as[14]

$$U|\xi,k\rangle = e^{i\Phi}|\xi',k\rangle , \tag{2.18}$$

where

$$\xi' = \frac{a\xi - b}{-b^*\xi + a^*}, \quad \Phi = 2k \arg(a - b\xi) , \tag{2.19a}$$

and

$$|a|^2 - |b|^2 = 1 . \tag{2.19b}$$

Using Eqs. (2.14) and (2.15), we obtain

$$\begin{aligned} |\xi_0,k,1\rangle &= U(0)|\xi_0,k\rangle = e^{i\Phi_0}|\xi_1,k\rangle , \\ |\xi_0,k,2\rangle &= U(1)|\xi_0,k,1\rangle = e^{i(\Phi_0+\Phi_1)}|\xi_2,k\rangle , \\ &\vdots \\ |\xi_0,k,n+1\rangle &= U(n)|\xi_0,k,n\rangle \\ &= e^{i(\Phi_0+\Phi_1+\cdots+\Phi_n)}|\xi_{n+1},k\rangle . \end{aligned} \tag{2.20}$$

Thus we have the evolution equations

$$\begin{aligned} \xi_{n+1} &= e^{-2i\omega_0 T}\left[\frac{\xi_n + i\tanh[F(nT)]}{1 - i\xi_n\tanh[F(nT)]}\right] , \\ &= \frac{a_n\xi_n - b_n}{-b_n^*\xi_n + a_n^*} , \end{aligned} \tag{2.21a}$$

where the a_n and b_n are given in Eq. (2.15), and

$$\Phi_n - = 2k \arg(a_n - b_n\xi_n) . \tag{2.21b}$$

These equations essentially define the Poincaré maps which relate the two state vectors which exist just prior to two consecutive δ-function pulses. We shall write Eq. (2.21a) in the form

$$\xi_{n+1} = R_n(\xi_n) . \tag{2.22}$$

The functions R_n belong to a special class of Möbius transformations which map the unit circle $|\xi|=1$ and its interior $|\xi|<1$ one-to-one and onto themselves, respectively.[15] The latter region corresponds to the Lobachevski phase space associated with the SU(1,1) Lie algebra. Note that in the trivial case $\lambda=0$, i.e., zero pulsing, the maps in Eq. (2.21) reduce to the relations

$$\xi_{n+1} = e^{-2i\omega_0 T}\xi_n, \quad \Phi_n = -2k\omega_0 T ,$$

implying a rotation of the ξ state vector in phase space with constant angular frequency $-2\omega_0$, consistent with the assumed form of the free-field Hamiltonian. (The Poincaré maps take "snapshots" of the time evolution at intervals of the pulsing period T.) The dynamics associated with nontrivial pulsing will be discussed in the Secs. III and IV.

The energies E_n associated with each state vector $|\psi(n)\rangle$ (between pulses) will be given by[9] [cf. Eq. (2.8)]

$$E_n = 2\omega_0\langle \xi_n,k|K_0|\xi_n,k\rangle = 2\omega_0 k\left[\frac{1+|\xi_n|^2}{1-|\xi_n|^2}\right] . \tag{2.23}$$

Clearly, the energies E_n become unbounded if $|\xi_n| \to 1$.

We now return to the form of the pulsing function $f(t)$ in Eq. (2.7) which will be adopted in this study. If $F(t)$ is periodic, with period T', then $f(t)$ is a quasiperiodic function, as defined by Pomeau and co-workers.[16] Following Ref. 7, we have chosen the periodic amplitude function $F(t)$ in Eq. (2.7) to be

$$F(t) = \lambda\cos(\omega' t) . \tag{2.24}$$

Since the angular frequency of the δ-function sequence is $\omega = 2\pi/T$, we have

$$F(nT) = \lambda\cos(2\pi\chi n), \quad \chi = \omega'/\omega . \tag{2.25}$$

When ω and ω' are commensurate, i.e., when χ is rational, the next pulsing function $f(t)$ in Eq. (2.7) is periodic in time. When χ is irrational, the pulsing is *almost periodic*. (The definitions of quasiperiodic and almost periodic functions are presented briefly in the Appendix.)

We now derive an expression for the autocorrelation function of the state vector, defined as[6]

$$C(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\langle\psi(t)|\psi(t+\tau)\rangle . \tag{2.26}$$

For our pulsed system this expression reduces to the discretized form

$$C(l) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\langle \xi_0,k,n|\xi_0,k,n+l\rangle . \tag{2.27}$$

[Clearly, $C(0)=1$.] From Eqs. (2.20), however, we have

$$C(l) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N} e^{i(\Phi_n+\cdots+\Phi_{n+l})}\langle \xi_n,k|\xi_{n+l},k\rangle . \tag{2.28}$$

The inner product in Eq. (2.28) is simply expressed in closed form as

$$\langle \xi_n,k|\xi_{n+l},k\rangle = (1-|\xi_n|^2)^k(1-|\xi_{n+l}|^2)^k(1-\xi_n^*\xi_{n+l})^{-2k} , \tag{2.29}$$

III. PERIODIC PULSING

We first consider the special case $\chi=0,+1,+2,\ldots,$ in Eq. (2.24), which gives a constant pulsing amplitude $F(nT)=\lambda$ in Eq. (2.8). The evolution of the SU(1,1) CS reduces to the dynamics with iteration of a Möbius transformation, i.e.,

$$\xi_{n+1}=R(\xi_n)\ , \tag{3.1}$$

where R has the form in Eq. (2.19) and

$$a=e^{-i\omega_0 T}\cosh\lambda\ ,\qquad b=-ie^{-i\omega_0 T}\sinh\lambda\ . \tag{3.2}$$

The dynamics are relatively uncomplicated and determined from a knowledge of the fixed points of $R(z)$. To summarize, we let $C=\{z,\in C,|z|=1\}$ denote the unit circle in the complex plane and $S=\{z\in C,|z|<1\}$ its interior. As stated earlier, $R(C)=C$ and $R(S)=S$.

The two fixed points of $R(z)$, say, p_1 and p_2, satisfy the quadratic equation

$$b^*z^2+(a-a^*)z-b=0\ . \tag{3.3}$$

Note that $p_1p_2=-b/b^*$ so that $|p_1||p_2|=1$. There are three classifications according to the location and nature of the fixed points.

(i) *Hyperbolic case:* p_1 and p_2 are distinct and both lie on C. One point, say, p_1, is *attractive,* i.e., $|R'(p_1)|<1$ and the other is *repulsive,* $|R'(p_2)|>1$.

(ii) *Parabolic case:* $p_1=p_2=p$ a degenerate fixed point lying on C, which is *indifferent,* $|R'(p)|=1$.

(iii) *Elliptic case:* p_1 and p_2 are distinct, inverse to each other with respect to C, and we let $p_1\in S$. Both fixed points are *indifferent.*

If ω_0 and T are fixed and λ is considered a variable parameter, all being real, then the fixed points p_1 and p_2 migrate as λ is varied. The parabolic case occurs at a critical value $\lambda=\lambda_p$, for which the discriminant D associated with the quadratic equation (3.3) vanishes. We may use Eq. (2.19b) to write

$$D^2=4[\mathrm{Re}(a)]^2-4\ , \tag{3.4}$$

so that $D=0$ implies $\mathrm{Re}(a)=+1$. Thus λ_p satisfies the transcendental equation

$$\cosh\lambda=\pm\sec(\omega_0 T)\ . \tag{3.5}$$

The system is hyperbolic for $\lambda>\lambda_p$ and elliptic for $\lambda<\lambda_p$. The dynamics of the iteration sequence $\{\xi_n\}$ may then be summarized as follows.[15]

(i) $\lambda>\lambda_p$: $\xi_n\to p_1$ as $n\to\infty$ for all $|\xi_0|\le 1$ except $\xi_0=p_2$, in which case $\xi_n=p_2$, $n=0,1,2,\ldots$. (The iterates approach the attractive fixed point along geodesics in the hyperbolic geometry.[15]) Thus, for any (physical) starting value $|\xi_0|<1$, we have $E_n\to\infty$ as $n\to\infty$, i.e., the energy is unbounded.

(ii) $\lambda=\lambda_p$: Let A be an open neighborhood of ξ_0 such that $\overline{A}$ (closure of A) does not contain the fixed point p, and let V be any neighborhood of p. Then there exists an integer $N>0$, such that for all $n>N$, $R^n(A)\supset V$. Since V may be made arbitrarily small, the states ξ_n must approach the unit circle arbitrarily closely. Hence the energies E_n form an unbounded sequence.

(iii) $\lambda<\lambda_p$: For $\xi_0\in S$, the iterates ξ_n lie on an invariant "circle" (in hyperbolic geometry) containing ξ_0. For $\xi_0=p_1$, this circle degenerates to a point. Thus the energies E_n will form a bounded sequence which is either periodic or quasiperiodic.

In Fig. 1 energy sequences that represent the three categories listed above are shown. The initial state was $\xi_0=0.5$, and we chose $\omega_0=0.126$, $T=1$. A solution to Eq. (3.5) occurs at $\lambda_p\simeq 0.126\,33$. In the hyperbolic case $\lambda=0.15$ the energies E_n are seen to grow exponentially, which is a consequence of their geometric approach to the attractive fixed point on the unit circle, and the energy formula, Eq. (2.23).

Up to now, we have considered only the special case, $\chi=$an integer, in Eq. (2.24) for periodic kicking. In the more general case of commensurate frequencies, i.e., $\chi=\omega/\omega'$ rational, the pulsing is also periodic. The dynamics can also be reduced to the study of a single Möbius transformation, as we now show. Let $\chi=p/q$, where p and q are relatively prime integers, $q\neq 0,1$. Then $F(nT)=F((n+K)T)$, where $K\ge 2$ is the least common multiple of p and q. It then follows that the rational maps R_n in Eq. (2.21) are periodic, i.e.,

$$R_n(\xi)=R_{n+K}(\xi)\ . \tag{3.6}$$

This implies that we may focus on a subset of the iterates ξ_n, defined by

$$\zeta_m=\xi_{mK},\quad m=0,1,2,\ldots\ . \tag{3.7}$$

This subset is generated by the iteration procedure

$$\zeta_{m+1}=S(\zeta_m),\quad m=0,1,2,\ldots\ , \tag{3.8}$$

where

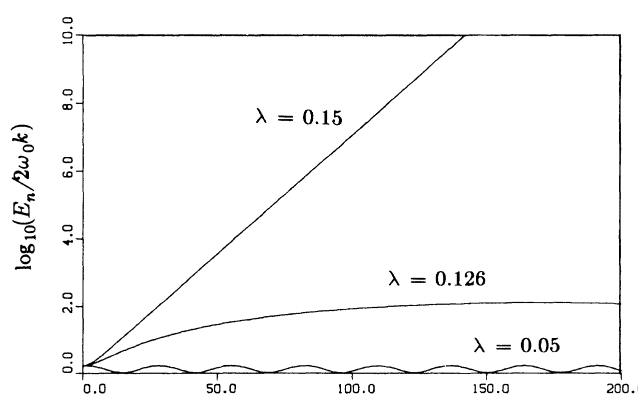

FIG. 1. Evolution of scaled energies $E_n/2\omega_0 k$ under periodic kicking, cf. Eqs. (3.1) and (2.23), for three representative cases: (i) $\lambda=0.5$ (hyperbolic), (ii) $\lambda=0.126$ (parabolic) (the oscillatory divergence is not evident during this time scale), (iii) $\lambda=0.15$ (elliptic). In all cases, $\xi_0=0.5$, so that $E_0/2\omega_0 k=\frac{5}{3}$. Also, $\omega_0=0.126$, $T=1.0$.

$$S(z)=R_{K-1}\circ R_{K-2}\circ \cdots \circ R_1\circ R_0(z)$$
$$=\frac{rz-s}{-s^*z+r^*}\ , \tag{3.9}$$

and the convolution symbol denotes composition of functions. The map $S(z)$ is necessarily a Möbius transformation of the same class as the $R_i(z)$. Its coefficients r and s are determined from the matrix product $U(K-1)U(K-2)\cdots U(1)U(0)$ where the $U(n)$ matrices are defined in Eq. (2.14).

Thus the dynamics of the nontrivial commensurate cases has been reduced to the iteration procedure, Eq. (3.7). A fixed point p_0 of $S(z)$ corresponds to a K cycle of the ξ iterates, i.e., $S(p_0)=p_0$ implies the existence of the cycle $\{p_0,p_1,\ldots,p_{K-1}\}$, where

$$p_1=R_0(p_0),\ \ p_2=R_1(p_1),\ldots\ ,$$
$$p_K=R_{K-1}(p_{K-1})=p_0\ . \tag{3.10}$$

Since all maps R_i have the unit circle C invariant it also follows that if $p_0\in C$, then $p_i\in C$, $i=1,\ldots,K-1$. Similarly, if $p_0\in S$, then $p_i\in S$, $i=1,\ldots,K-1$. The K cycle in (3.9) will be attractive, indifferent, or repulsive, according to whether $|S'(p_0)|$ is less than, equal to, or greater than 1, respectively. The dynamics is then classified as

(i) *Hyperbolic case:* both fixed points of $S(z)$, say, p_0

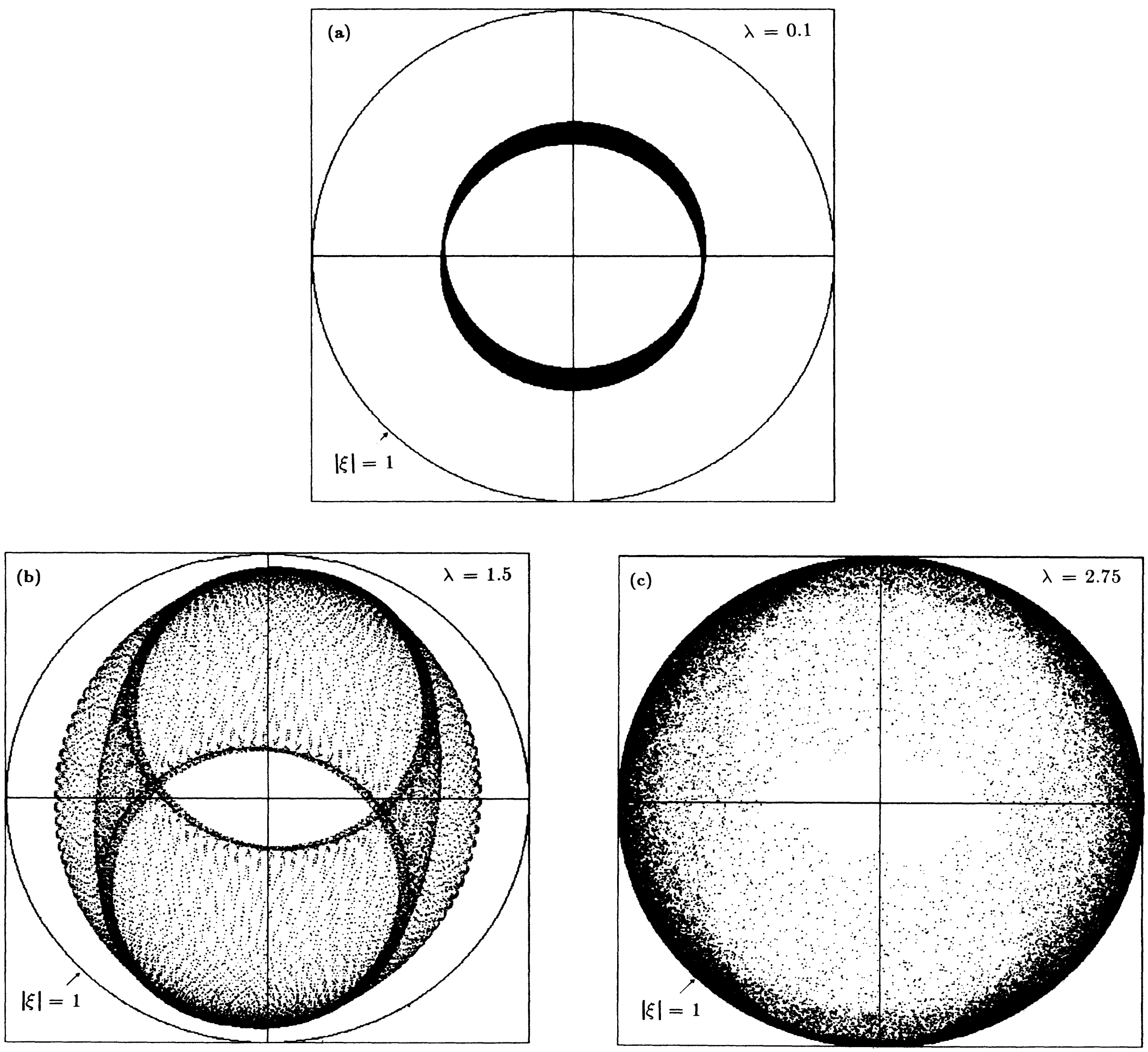

FIG. 2. Some representative phase portraits of ξ_n values assumed during almost-periodic pulsing; $\xi_0=0.5$, $\omega_0=0.126$, $T=1.0$, $\chi=4637/13\,313$: (a) $\lambda=0.1$, (b) $\lambda=1.5$, (c) 2.75. For $\lambda>2.8$, the ξ_n quickly settle near the unit circle; at the resolution of these graphs, essentially on it.

(attractive) and q_0 (repulsive), lie on C. The iteration procedure $\xi_{n+1}=f_n(\xi_n)$ will then produce an attractive K cycle $\{p_0,\ldots,p_{K-1}\}$ and repulsive K cycle $\{q_0,\ldots,q_{K-1}\}$, both lying on C.

(ii) *Parabolic case:* $p_0=q_0$, a degenerate fixed point of $S(z)$ on C, yielding an indifferent K cycle on C.

(iii) *Elliptic case:* attractive K cycle in S.

As an illustrative example, we consider the case $\chi=\frac{1}{2}$, so that $F((2k+1)T)=-\lambda$, $F(2kT)=\lambda$, $k=0,1,2,\ldots$. A little algebra reveals that the map $S(z)=R_1 \circ R_0(z)$ in Eq. (3.9) is defined by the parameters

$$a=e^{-2i\omega_0 T}\cosh^2\lambda-\sinh^2\lambda \ , \qquad b=i\sinh\lambda\cosh\lambda(e^{-2i\omega_0 T}-1) \ . \tag{3.11}$$

From Eq. (3.4), the critical value λ_p satisfies one of the equations

$$\cos(2\omega_0 T)\cosh^2\lambda-\sinh^2\lambda\pm 1=0 \ . \tag{3.12}$$

For our usual case $\omega_0=0.126$, $T=1$, we find that $\lambda_p\simeq 2.7633$ [plus sign in (3.12)], which is consistent with observed phase-space trajectories.

IV. ALMOST-PERIODIC PULSING

As mentioned in Sec. II, the case of incommensurate frequencies, i.e., χ irrational, produces almost-periodic pulsing. The coefficients a_n and b_n in Eq. (2.14b) are almost periodic. At present, our study is limited to a series of numerical experiments, the results of which are discussed below. A number of "irrational" values of χ were employed and found to yield similar behavior. The results cited below correspond to the particular value $\chi=4637/13\,313$, which has been used in Refs. 6 and 7 to approximate an irrational frequency ratio. For a given modulation amplitude λ, the ξ_n sequence was calculated to $N\sim 10^4$ terms. The general qualitative behavior of the sequences (e.g., autocorrelation vector) is found to be independent of the starting values ξ_0, as well as the parameters ω_0 and T_0. For the results shown below, we chose $\xi_0=0.5$, $\omega_0=0.126$, $T=1$. From the ξ_n, the autocorrelation (AC) coefficients $C(l)$ were calculated, usually to $l=200$.

Firstly, for $\lambda>0$, the iterates ξ_n are observed to fill regions in phase space which may be donut shaped. The sizes of these regions are dependent upon the initial conditions. Nevertheless, as λ increases, the outer boundaries of these regions migrate toward the unit circle C. Some representative portraits are presented in Fig. 2 for $\lambda=0.1$, 1.5, and 2.75. At $\lambda\cong 2.8$ there is a sudden transition, and the ξ_n quickly settle on an annulus whose inner diameter lies very close to C. For example, when $\lambda=3.0$, the ξ_n for $n>40$ are found within 1 part in 10^{14} of C. The energies E_n corresponding to the trajectories in Fig. 10 are plotted in Fig. 3. Recalling Eq. (2.23), we see that the energy quickly increases as the ξ_n approach the circle C.

As λ increases to 2.8, the autocorrelation $C(l)$ decreases in norm. In Fig. 4 sample AC vectors for $\lambda=0.5$,

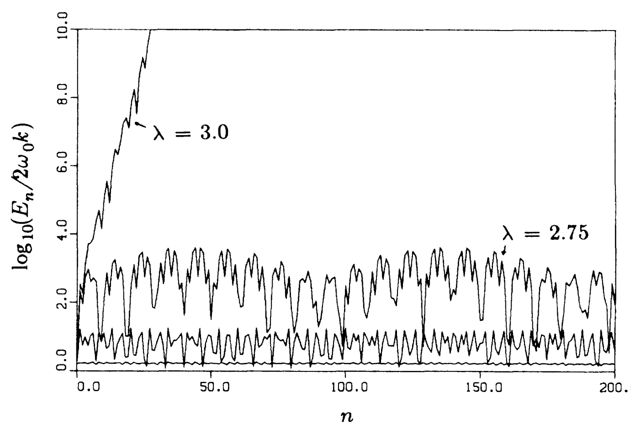

FIG. 3. Scaled energies $E_n/2\omega_0 k$ of state vectors ξ_n corresponding to the trajectories plotted in Figs. 2(a)–2(c), as well as for the case $\lambda=3.0$.

2.0, and 3.0 are shown. For $\lambda=3.0$, we find that $C(l)<0.001$ for $l>0$ and are thus indistinguishable from zero in the graph. The decrease in the norm of $C(k)$ as λ increases is revealed in Fig. 5, where in analogy to Fig. 5 of Ref. 6, we plot the following norm of the AC function:

$$\|C\|=\max_{1\le n\le N}|C(n)| \ , \tag{4.1}$$

as a function of λ. This decrease, which follows the pattern observed in Ref. 6, has been used as a fingerprint to characterize (or *define*) quantum chaos. (Some comments on this point are made in Sec. V.)

We should mention that the critical pulsing amplitude λ_c varies with the parameters ω_0 (free-field frequency) and T (pulsing period). The quantitative nature of this dependence has not been investigated in detail. However, from the form of the Poincaré maps, e.g., Eq. (2.21), it follows that for fixed T, λ_c is periodic with respect to ω_0.

For λ sufficiently large, the ξ_n iterates lie arbitrarily

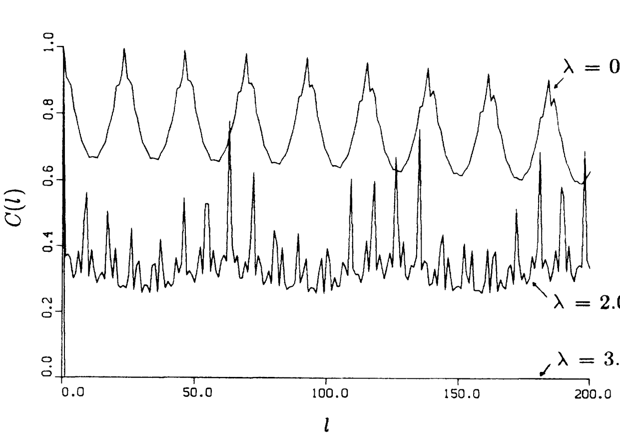

FIG. 4. Autocorrelation coefficients $C(k)$, $0\le k\le 200$ for almost-periodic pulsing, $\lambda=0$, 2.0, and 3.0. For $\lambda=3.0$, $C(k)<0.005$, hence indistinguishable from zero on the graph.

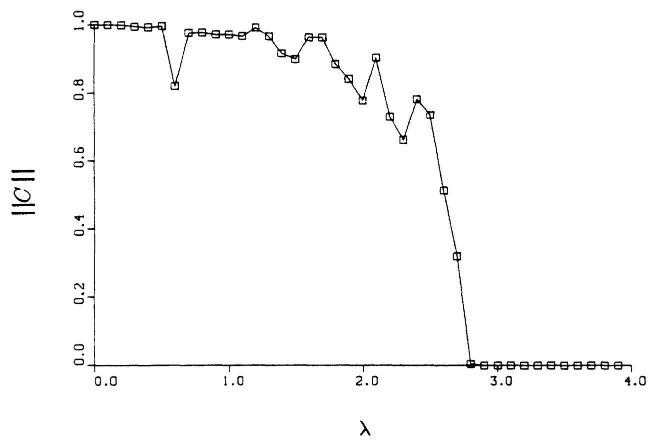

FIG. 5. Norm of autocorrelation vectors $\|C\|$, defined in Eq. (4.1), for almost-periodic pulsing, $0.0 \leq \lambda \leq 4.0$.

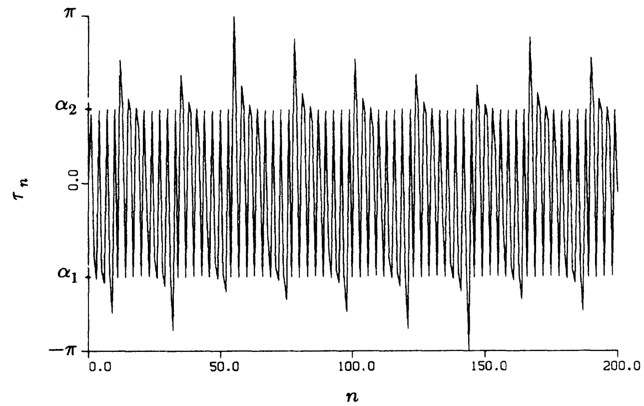

FIG. 6. Plot of $\tau_n = \arg(\xi_n)$ for almost-periodic kicking, $\lambda = 3.0$, where $|\xi_n| = 1$ to one part in 10^{14} for $n \geq 40$. The plot reveals that the iterates are concentrated near two τ values, α_1 and $\alpha_2 = \alpha_1 + \pi$.

close to the unit circle C. This can be seen, at least partially, as follows. From Eq. (2.21a), setting $r_n = |\xi_n|$ and $t_n = \tanh[F(nT)]$ for simplicity,

$$r_{n+1}^2 = \frac{r_n^2 + t_n^2 + 2r_n t_n \sin\tau_n}{1 + r_n^2 t_n^2 + 2r_n t_n \sin\tau_n}, \quad \tau_n = \arg(\xi_n) . \tag{4.2}$$

For $|t_n| \sim 1$, we have $r_{n+1} \sim 1$. As λ increases, more of the quasiperiodically varying t_n will assume values sufficiently close to unity to keep the iterates near the unit circle. To investigate the dynamics in this regime more closely we consider the angular distribution of the ξ_n in these annuli. A plot of the angles $\tau_n = \arg(\xi_n)$ versus n for the case $\lambda = 3.0$ is shown in Fig. 6, revealing that most iterates are found near two angles, α_1 and $\alpha_2 = \alpha_1 + \pi$. In Fig. 7 we obtain an understanding of the return map $\tau_{n+1} = P(\tau_n)$ generated by plotting consecutive pairs of angles $(\tau_n, \tau_n + 1)$. The rather singular distribution of iterates is evident. Based on these and other numerical results, we would conjecture that in the limit $\lambda \to \infty$, the invariant measure (assuming it exists) associated with the return map P is composed of two point masses located at α_1 and $\alpha_1 + \pi$ ($r = |\xi| = 1$ in both cases).

V. SUMMARY

In this paper, the Poincaré maps defining the evolution of SU(1,1) coherent states under (periodic and quasiperiodic) pulsing have been derived *in closed form*. In the periodic case, this evolution reduces to the iteration of a Möbius transformation on the phase space, the interior of the unit circle. Depending upon the pulsing amplitude, the energies E_n of the state vectors between pulses may form periodic or quasiperiodic sequences which are either bounded (elliptic case) or unbounded (parabolic and hyperbolic cases). In the quasiperiodic case, the norm of the autocorrelation vector is found to decrease with increasing pulsing amplitude, as found with other systems. There appears to be a critical pulsing amplitude beyond which all state vector parameters ξ_n are attracted to the unit circle, implying unboundedness of the E_n.

The decay of the autocorrelation vector has been used as one possible characterization of quantum-mechanical chaos.[6,7] However, there are some questions about such an *a priori* classification, since the autocorrelation vector of a (time) series is, in essence, a reflection of its Fourier transform. In the case of almost-periodic pulsing, its decay would be expected as a natural consequence of the application of almost-periodic maps. This has previously been noted in Ref. 7 and also in Ref. 17. We also mention that Ford and Mantica[18] have come to the same conclusion with a study of the pulsed quantum-mechanical

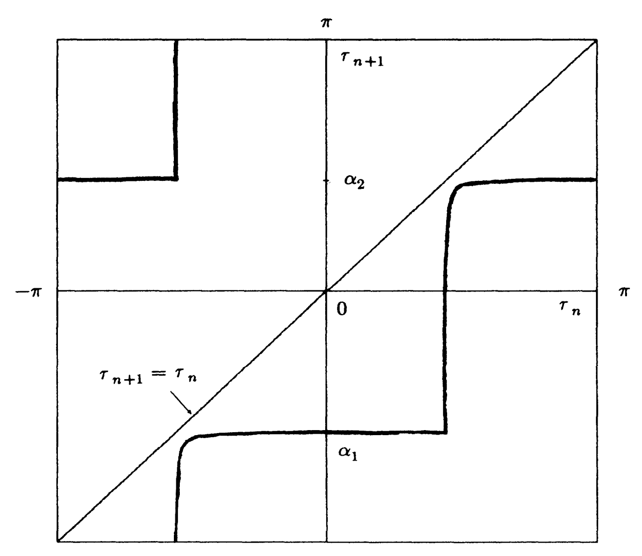

FIG. 7. Return map $\tau_{n+1} = P(\tau_n)$ corresponding to τ_n vs n graph of Fig. 6, obtained by plotting continuous pairs (τ_n, τ_{n+1}). $\lambda = 3.0$.

rotor and the quantum cat map. An advantage of these SU(1,1) model problems is that the transformations involved in the time evolution may be written in closed form.

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APPENDIX

For purposes of clarity and distinction, we present brief definitions of quasiperiodic and almost-periodic functions, following Refs. 16 and 19. Let y be a function of n independent variables $t_1, \ldots, t_n$, and periodic, of period 2π in each argument, i.e.,

$$y(t_1, \ldots, t_j, \ldots, t_n) = y(t_1, \ldots, t_j + 2\pi, \ldots, t_n), \quad j = 1, \ldots, n \ . \qquad \text{(A1)}$$

If the n variables t_j are all proportional to the time t,

$$t_j = \omega_j t, \quad j = 1, \ldots, n \ , \qquad \text{(A2)}$$

then y is said to be *quasiperiodic* in time.[16]

Let f be a function of a single (real) variable t, such that for any given $\epsilon > 0$, the inequality

$$|f(t+\tau) - f(t)| < \epsilon, \quad \text{for all } t \qquad \text{(A3)}$$

is satisfied by infinitely many values of τ on the real line, located over the line in such a way that empty intervals of arbitrarily great length are not left. (The set of τ values satisfying the inequality is said to be *relatively dense* on R.) Then $f(t)$ is said to be *almost periodic* in t.[19] An example of an almost-periodic function is

$$f(t) = \sin(2\pi t) + \sin(2\sqrt{2}\pi t) \ . \qquad \text{(A4)}$$

(See Ref. 19, p. 1, for proof.)

With the periodic amplitude function $F(t)$ given in Eq. (2.24), the net pulsing function $f(t)$ in Eq. (2.7) is seen to be quasiperiodic in t with angular frequencies ω' and $\omega = 2\pi/T$. If the frequencies are *commensurate,* i.e., if the ratio ω'/ω is rational, then $f(t)$ is periodic. If the ratio is irrational, then $f(t)$ is almost periodic in t.

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