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Dynamics of pattern formation in the Turing optical instability

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The transient dynamical evolution of spatial pattern formation in a nonlinear passive optical system in a Fabry-Pérot cavity driven by an external coherent field is studied. The associated decay of the unstable homogeneous state of the system is described analytically, with the inclusion of noise effects, in terms of a stochastic amplitude equation for the unstable mode in the direction of instability. As a result, the time scale on which the pattern is formed as well as the anomalous transient fluctuations of the field intensity are explicitly calculated. Our results show that the behavior of these fluctuations depends strongly on the position within the cavity. Finally, the symmetry-restoring effect of fluctuations is also discussed.

I. INTRODUCTION

Nonlinear optical systems exhibit a variety of nonequilibrium instabilities. Rather recently[1] a new type of instability leading to spontaneous pattern formation has been predicted and investigated. This contrasts with the more extensively studied temporal instabilities which lead to oscillations or pulsations.[2] The new type of instability is analogous in its mathematical formulation to the well-known Turing instability in nonlinear chemical systems. The difference is that the role played by diffusion in chemical systems is here played by diffraction associated with transverse effects. In its simplest form[1] the instability has been studied for a passive system in a Fabry-Pérot cavity driven by an external coherent field E_I. The cavity is filled with a Kerr medium or a two-level system in the pure dispersive limit. It has four mirrors orthogonal to the longitudinal and transverse coordinates. In this situation there is no temporal effect associated with the instability, and the coexisting cavity modes in the resulting nonhomogeneous stationary state have the same frequency as the driving field. The mathematical model used to study this situation is, in appropriate units,[1]

$$\partial_t E(x,t) = -E + E_I + iE(|E|^2 - \theta) + ia\partial_x^2 E \ . \tag{1.1}$$

E is the normalized envelope function of the field in the cavity. The coordinate x is transverse to the cavity, and in the units in (1.1), $0 \le x \le 1$. The incident field and the cavity are polarized in the direction perpendicular to the longitudinal coordinate and the x coordinate. The diffraction coefficient a is inversely proportional to the Fresnel number. The detuning parameter θ is[3,4]

$$\theta \equiv \theta' - \frac{2C\Delta}{1+\Delta^2} \ , \tag{1.2}$$

where θ' and Δ are the cavity and atomic detunings, respectively, and we choose the case $\Delta < 0$. The constant C is the bistability parameter. The derivation of the model (1.1) is discussed in Refs. 3 and 4. For this model, the instability has been identified by linear-stability analysis, and the stationary solution has been found, close to the instability point, using bifurcation theory.[1] The analysis of (1.1) has been extended by numerical solutions to the case of a cavity with spherical mirrors where spontaneous breaking of cylindrical symmetry is possible.[5] In this case the instability is accompanied by temporal oscillations. A related instability leading to a spatial pattern occurs in the laser.[6] In this case the competing spatial modes select a common oscillation frequency so that the laser intensity is stationary in time.

Our general purpose, in the context of these pattern-formation instabilities in optical systems, is to describe the transient dynamics of pattern formation. When the appropriate control parameter is suddenly changed, an initial homogeneous state becomes unstable and evolves to a final nonhomogeneous state. In the passive system described by (1.1) this occurs when the driving field changes from a value $E_I < E_{Ic}$ to a value $E_I > E_{Ic}$. The dynamical evolution consists of the decay of an unstable state during which a spatial pattern emerges. We aim to describe such processes by analytical methods of general validity which include noise effects. The consideration of fluctuations is essential, since we are dealing with a decay process which, properly speaking, is not possible in a deterministic framework. In this paper, we restrict ourselves to the simplest form of this type of instability, given by (1.1), in which no temporal effects appear. We will consider the Fabry-Pérot Cartesian geometry and parameter values for which the instability is supercritical.

The key idea to our development is the reduction of (1.1) to a normal-form amplitude equation for the amplitude of the unstable mode in the direction of the instability.[7,8] This can be done with the inclusion of noise terms. In this way, one reduces, close enough to the instability, the problem posed by a stochastic partial differential equation for a vector variable to an ordinary stochastic differential equation for a scalar variable. The method is of general validity and can be applied to a variety of similar problems. Previous studies[1,3-5] of the optical Turing instability based on Eq. (1.1) do not include analytical solutions for the dynamical evolution and do not take into account noise effects. In the absence of fluctuations the lifetime of the unstable homogeneous state is infinite.

The inclusion of fluctuations permits the calculation of the time scale on which the pattern is formed and the description of the important fluctuations occurring in the transient decay process. In addition, fluctuations restore the spontaneously broken symmetry associated with the different pattern-forming deterministic solutions. In this paper, we show the generality of the method and we calculate the transient dynamical evolution of the mean intensity associated with the field $E(x,t)$. This calculation describes the pattern formation and identifies the lifetime of the unstable homogeneous state. We also calculate the intensity fluctuations during the transient process. The behavior of these fluctuations is largely dependent on the space point in the cavity. Close to the walls of the cavity, the intensity fluctuations grow monotonically in time to a large value associated with a bimodal probability distribution for the intensity. At the center of the cavity such distribution evolves in time, remaining single-peaked, so that close to the center of the cavity, intensity fluctuations exhibit the characteristic peak associated with the anomalous transient fluctuations occurring in the decay of unstable states.[9] The symmetry-restoring effects of fluctuations are also discussed.

The paper is organized as follows. For clarity of presentation we first study, in Sec. II, the deterministic dynamics associated with Eq. (1.1). This includes a summary of the linear-stability analysis, the derivation of an amplitude equation, and the deterministic solution for the time- and space-dependent intensity. The discussion of the multiplicity of deterministic solutions goes under the name of space-dependent bistability. Section III is devoted to the stochastic dynamics. The starting original model is discussed in Sec. III A. It contains thermal-noise, phase, and intensity fluctuations of the driving field. Section III A also contains the derivation of a stochastic amplitude equation. In Sec. III B the qualitative stochastic behavior is discussed, and Sec. III C contains the stochastic transient dynamics calculation. In the Appendix we specify some of our results to lowest order in the parameter that measures the distance to the instability point.

II. DETERMINISTIC DYNAMICS

The homogeneous stationary solution of (1.1), $E_{\rm st}=E_{1,s}+iE_{2,s}$ satisfies the equation

$$E_I^2=|E_{\rm st}|^2[1+(|E_{\rm st}|^2-\theta)^2] \ . \tag{2.1}$$

Bistability, exhibiting two possible solutions of $|E_{\rm st}|^2$ as a function of E_I^2, occurs for $\theta>\theta_c=\sqrt{3}$. We denote by the complex variable

$$q(x,t)=q_1+iq_2 \ , \tag{2.2}$$

the deviation of $E(x,t)$ from the homogeneous stationary solution

$$E(x,t)=E_{\rm st}+q(x,t) \ . \tag{2.3}$$

The nonlinear equation satisfied by $q(x,t)$ can be written as

$$\partial_t q_\alpha(x,t)=\sum_\beta K_{\alpha\beta}q_\beta(x,t)+N_\alpha(q), \quad \alpha=1,2 \tag{2.4}$$

where $K_{\alpha\beta}$ determines the linear dynamics around E_s for a given E_I and $N_\alpha(q)$ includes the nonlinear contributions. Explicitly,

$$K_{\alpha\beta}=\begin{bmatrix} -1-2E_{1,s}E_{2,s} & \theta-E_{1,s}^2-3E_{2,s}^2-a\partial_x^2 \\ 3E_{1,s}^2+E_{2,s}^2-\theta+a\partial_x^2 & -1+2E_{1,s}E_{2,2} \end{bmatrix} \tag{2.5}$$

$$N_\alpha(q)=\sum_{\mu,\sigma=1} h_{\alpha\mu\sigma}q_\mu q_\sigma+\sum_{\mu,\nu,\sigma=1} g_{\alpha\mu\nu\sigma}q_\mu q_\nu q_\sigma \ , \tag{2.6}$$

where the coefficients h and g are given by

$$h_{1\mu\sigma}=\begin{bmatrix} -E_{2,s} & -2E_{1,s} \\ 0 & -3E_{2,s} \end{bmatrix}, \quad h_{2\mu\sigma}=\begin{bmatrix} 3E_{1,s} & 2E_{2,s} \\ 0 & E_{1,s} \end{bmatrix} , \tag{2.7}$$

$$g_{1112}=g_{1222}=-1, \quad g_{2122}=g_{2111}=1 \ , \tag{2.8}$$

and the remaining g coefficients vanish.

A. Linear analysis

As a first step in studying the nonlinear dynamics of pattern formation through an amplitude equation, we first review the linear stability analysis of (1.1), introducing some more notation that will be used later. The linear-stability problem of (1.1) amounts to the analysis of the eigenvalue equation associated with $K_{\alpha\beta}$. Taking into account the reflecting boundary condition $\partial E/\partial x=0$ at $x=0,1$, the eigenvalue equation can be written as

$$\sum_\beta K_{\alpha\beta}(k^2)V_\beta^{k,j}(x)=\lambda_j(k^2)V_\alpha^{k,j}(x), \quad j=1,2 \ , \tag{2.9}$$

where the eigenvectors $V_\alpha^{k,j}(x)$ can be written as

$$V_\alpha^{k,j}(x)=O_\alpha^j(k)\cos(kx) \ , \tag{2.10}$$

with wave numbers $k=\pi n$, $n=0,1,\ldots$. The two eigenvalues $\lambda_j(k^2)$ for a given wave number have the form[1]

$$\lambda_j(k^2)=-1\pm(1-\zeta)^{1/2} \ , \tag{2.11}$$

where

$$\zeta=1+(|E_s|^2-\theta)(3|E_s|^2-\theta)+ak^2[ak^2-2(2|E_s|^2-\theta)] \ . \tag{2.12}$$

In the following, $j=1$ will be associated with the plus sign in (2.11). We will focus in this paper on the situation $\theta<\theta_c$ for which bistability does not occur. In this case (2.12) implies[1] that the homogeneous mode $n=0$ remains stable and that a range of unstable modes exists for $|E_s|^2\geq 1, 2|E_s|^2>\theta$. The growth of these unstable modes leads to the formation of a pattern in which the homogeneous solution coexists. The instability first occurs at $|E_s|^2=1$ for a critical wave number k_c, such that $ak_c^2=2-\theta$. We will assume to have parameter values such that a single-mode k_c becomes unstable.

There are several features of the linear dynamics worth noticing. The first thing to note is that, although $\lambda_j(k^2)$ only depends on $|E_s|^2$, the matrix $\underline{K}$ depends separately on $E_{1,s}$ and $E_{2,s}$, so that the problem is essentially a two-variable problem and cannot be reduced to a closed dynamical equation for the field intensity $I=|E|^2$. Second, it should be noted that the matrix $\underline{K}$ is real but nonsymmetric, a fact that has several consequences. The eigenvalues and eigenvectors are generally complex with $\lambda_1(k^2)=\lambda_2^*(k^2)$ and $O_\alpha^1(k)=[O_\alpha^2(k)]^*$. The eigenvectors $O_\alpha^j(k)$ are determined except for two real constants. They are chosen by a normalization condition

$$\sum_{\alpha=1,2} O_\alpha^1(k)O_\alpha^2(k)=1 , \quad (2.13)$$

and a global phase factor is fixed requiring the first component $O_1^j(k)$ to be real. Equation (2.13) makes it clear that the eigenvectors $O_\alpha^j(k)$ are not orthogonal for different j, but that they are linearly independent, except for accidental degeneracy of $\lambda_j(k^2)$.

In the following, we will also need to introduce the left eigenvectors $\bar{O}_\beta^j(k)$ of $\underline{K}$,

$$\sum_\beta \bar{O}_\beta^j(k)K_{\beta\alpha}=\lambda_j(k^2)\bar{O}_\alpha^j(k) . \quad (2.14)$$

The left eigenvectors are, by definition, orthogonal to right eigenvectors for different j. They are chosen here with the following normalization condition:

$$\sum_{\alpha=1}^{2} \bar{O}_\alpha^j(k)O_\alpha^l(k)=\delta_{jl} . \quad (2.15)$$

The explicit calculation for $\theta=1$ of eigenvectors and eigenvalues to first order in the distance $\nu=E_I-E_{Ic}$ from the instability point is given in the Appendix. The eigenvalues $\lambda_j(k_c^2)$ are real, while for $k=0,2k_c$, the eigenvalues are complex for finite ν, but $\lambda_j(k=0)$ at $\nu=0$ is real and degenerate. At $\nu=0$ the matrix $\underline{K}$ is symmetric.

B. Amplitude equation

A solution for the nonlinear Eq. (2.4) is sought in the form of an expansion in the complete set of functions associated with the linear problem[2,3]

$$q_\alpha(x,t)=\sum_{k,j}\zeta_{k,j}(t)O_\alpha^j(k)\cos(kx) . \quad (2.16)$$

The complex amplitudes $\zeta_{k,j}(t)$ carry the time dependence; the space dependence is given by the $\cos(kx)$ factor for different wave numbers, and the vector $O_\alpha^j(k)$ mixes in the solution the two eigenvectors with the same wave number. For complex eigenvalues $\lambda_j(k)$, $\zeta_{k,1}(t)=\zeta_{k,2}^*(t)$. The key idea is now to single out the dynamics of the amplitude $\zeta_{k_c,1}(t)$ of the unstable mode k_c in the direction of instability $j=1$, while the other amplitudes follow the dynamics of $\zeta_{k_c,1}(t)$. This greatly simplifies the problem passing from an equation for two space-dependent variables $q_\alpha(x,t)$ to an equation for a single space-independent variable amenable to analytical treatment. Other numerical approaches[5,6] to this and other similar problems are based on a straightforward Fourier analysis of $q_\alpha(x,t)$ which does not take advantage of the existence of a direction of instability $j=1$ for the unstable mode k_c. We make use here of this direction of instability by the explicit introduction of the vector $O_\alpha^j(k)$.

Equations for the amplitudes $\zeta_{kj}(t)$ are obtained by introducing (2.16) in (2.4) and using (2.15). We obtain for $k\neq 0$

$$\dot{\zeta}_{k,j}(t)=\lambda_j(k^2)\zeta_{k,j}+2\sum_{\substack{k'k''\\ j'j''}} J_{kk'k''}a_{kk'k'',jj'j''}\zeta_{k',j'}(t)\zeta_{k'',j''}(t)+2\sum_{\substack{k'k''k'''\\ j'j''j'''}} J_{kk'k''k'''}a_{kk'k''k''',jj'j''j'''}\zeta_{k',j'}(t)\zeta_{k'',j''}(t)\zeta_{k''',j'''}(t) , \quad (2.17)$$

where the coefficients a are complex numbers given by

$$a_{kk'k'',jj'j''}\equiv\sum_{\alpha,\mu,\sigma} h_{\alpha\mu\sigma}\bar{O}_\alpha^j(k)O_\mu^{j'}(k')O_\sigma^{j''}(k'') , \quad (2.18)$$

$$a_{kk'k''k''',jj'j''j'''} \equiv\sum_{\alpha,\mu,\sigma,\nu} g_{\alpha\mu\sigma\nu}\bar{O}_\alpha^j(k)O_\mu^{j'}(k')O_\sigma^{j''}(k'')O_\nu^{j'''}(k''') , \quad (2.19)$$

and the coefficients J introduce the selection rules for the possible mode couplings

$$J_{kk'k''}=\int_0^1 dx\,\cos(kx)\cos(k'x)\cos(k''x) , \quad (2.20)$$

$$J_{kk'k''k'''}=\int_0^1 dx\,\cos(kx)\cos(k'x)\cos(k''x)\cos(k'''x) . \quad (2.21)$$

For the homogeneous mode $k=0$, one finds slightly different coefficients:

$$\dot{\zeta}_{0,j}(t)=\lambda_j(0)\zeta_{0,j}(t)+\tfrac{1}{2}\sum_{j',j''}a_{000,jj'j''}\zeta_{0,j'}(t)\zeta_{0,j''}(t) +\tfrac{1}{2}\sum_{k',j'j''}a_{0k'k',jj'j''}\zeta_{k',j'}(t)\zeta_{k',j''}(t)+\mathcal{O}(\zeta^3) , \quad (2.22)$$

where the cubic nonlinearities are not written explicitly.

The unstable amplitude is associated with the single positive eigenvalue $\lambda_1(k_c^2)$. To make clear the difference between stable and unstable modes we introduce the notation

$$u=\zeta_1(k_c) ,$$

and (2.23)

$$S_{k,j}=\zeta_j(k) ,$$

with

$k \neq k_c, \quad j=1,2$ or $k=k_c, \; j=2$.

Close enough to the instability, the unstable amplitude is a small quantity and the stable amplitudes are at least one order smaller than u. On these grounds (2.17) can be simplified as follows. In the equation for u, terms of order s^2, u^2s, us^2, and s^3 are neglected in front of u^3. The coupling among the unstable mode and the stable modes in the equations for u is kept by terms of order us. In the same way, in the equations for s, terms of order us, u^3, s^2, u^2s, us^2, and s^3 are neglected in front of u^2. The resulting simplified form of Eqs. (2.17) is

$$\dot{u}(t)=\lambda_1(k_c^2)u + A_u^{(2)}u^2 + A_u^{(3)}u^3 + \sum_{k,j} A_{us}^{(2)}S_{k,j}u \;, \tag{2.24}$$

$$\dot{S}_{k,j}(t)=\lambda_j(k^2)S_{k,j} + A_{s_{k,j}}^{(2)}u^2 \;, \tag{2.25}$$

where $A_u^{(3)}=\frac{3}{4}a_{k_c k_c k_c k_c,1111}$. Selection rules (2.20) and (2.21) are such that $A_u^{(2)}=0$ and u only couples in (2.24) to the homogeneous mode $k=0$ and the first harmonic $k=2k_c$. Explicitly,

$$\sum_{k,j} A_{us}^{(2)}S_{k,j}u = 4u \,\mathrm{Re} \sum_{k=0,2k_c} (1-\tfrac{1}{2}\delta_{k,2k_c})a_{k_c k_c k,111}S_{k,1} \;. \tag{2.26}$$

In addition, $A_{s_k,j}^{(2)}$ vanish except for $k=0,2k_c$

$$A_{s_k,j}^{(2)} = \tfrac{1}{2}a_{kk_c k_c,j11}, \quad k=0,2k_c \;. \tag{2.27}$$

In summary, for $k=0,2k_c$, $S_{k,j}$ relaxes to zero, since $\lambda_j(k^2)<0$, and the stable modes $k=0,2k_c$ are coupled to the dynamics of the unstable amplitude. Invoking the adiabatic elimination[8] principle based on the different time scales determined by $\lambda_1(k_c^2)$ and $\lambda_1(k=0,2k_c)$, we express $S_{k,j}$, $k=0,2k_c$, in terms of u by setting $\dot{S}_{k,j}=0$ in (2.25). This gives the final closed-amplitude equation for u

$$\dot{u}(t)=\lambda_1(k_c^2)u + Cu^3 \;, \tag{2.28}$$

where

$$C \equiv \tfrac{3}{4}a_{k_c k_c k_c k_c,1111} - 4\,\mathrm{Re}\left[\frac{1}{2\lambda_1(0)}a_{0k_c k_c,111}a_{k_c k_c 0,111} - \frac{1}{4\lambda_1(2k_c)}a_{2k_c k_c k_c,111}a_{k_c k_c 2k_c,111}\right] . \tag{2.29}$$

The parameters $\lambda_1(k_c^2)$ and C in (2.28) can be calculated for a given E_I in terms of the linear dynamics associated with the matrix $\underline{K}$. The explicit calculation to lowest order in $\nu = E_I - E_{I_c}$ is given in the Appendix. From the solution of (2.28) and going back to (2.16) the solution $E(x,t)$ to the original problem can be constructed. In (2.16) only $k=0,k_c,2k_c$ will contribute, since other modes have zero amplitude in this approximation.

C. Transient dynamics and space-dependent bistability

The time development of the spatial pattern associated with the field amplitude $E(x,t)$ is determined by the solution of (2.28). We are interested in the transient dynamics which follow an instantaneous change of E_I from a value $E_I < E_{I_c}$ to a value $E_I > E_{I_c}$. The first E_I value determines the initial condition. In the present deterministic context, $u=0$ in the steady state corresponding to $E_I < E_{I_c}$. The solution of (2.28) with such an initial condition is $u(t)=0$ for all times. This just reflects that the decay of the unstable state created by the change of E_I is not possible in the absence of fluctuations. Such fluctuations will be introduced in Sec. III. Here we allow for a small initial condition u_0 and solve (2.28) with $u(t=0)=u_0$. The choice of the sign of u_0 gives rise to two different solutions $u_\pm(t)$ with the same absolute value of the initial condition

$$u_\pm(t)=\pm|u_0|\exp[\lambda_1(k_c)t]/\{1-[Cu_0^2/\lambda_1(k_c)][\exp 2\lambda_1(k_c)t]-1\}^{1/2} \;. \tag{2.30}$$

The simple solution (2.30) is shown in Fig. 1 for different values of E_I and u_0. The important point to notice is that the time needed to reach the final stationary value $u_\pm(\infty)=\pm[\lambda_1(k_c)/-C]^{1/2}$ and the rate at which it is approached depends on E_I, but the time scale in which the decay process starts is determined in this context by the chosen initial condition. It is this last time scale that is associated with the lifetime of the unstable state.

Associated with the two solutions $u_\pm(t)$ there are two solutions for the field amplitude obtained from (2.16):

$$\begin{aligned} E_\alpha^\pm(x,t) = {} & E_{\mathrm{st},\alpha} + u_\pm(t)O_\alpha^1(k_c)\cos(k_c x) \\ & + 2\,\mathrm{Re}[S_{0,1}(t)O_\alpha^1(k=0)] \\ & + 2\,\mathrm{Re}[S_{2k_c,1}(t)O_\alpha^1(2k_c)\cos(2k_c x)] \;. \end{aligned} \tag{2.31}$$

Using the adiabatic elimination of the linearly stable modes,

$$E_\alpha^\pm(x,t)=E_{\mathrm{st},\alpha} + u_\pm(t)O_\alpha^1(k_c)\cos k_c x + u_\pm^2(t)A_\alpha(x) \;, \tag{2.32}$$

where

$$A_\alpha(x) = -\mathrm{Re}\left[\frac{1}{\lambda_1(0)}a_{0k_c k_c,111}O_\alpha^1(k=0)\right] - \mathrm{Re}\left[\frac{1}{\lambda_1(2k_c)}a_{2k_c k_c k_c,111}O_\alpha^1(2k_c)\right]\cos(2k_c x) \;. \tag{2.33}$$

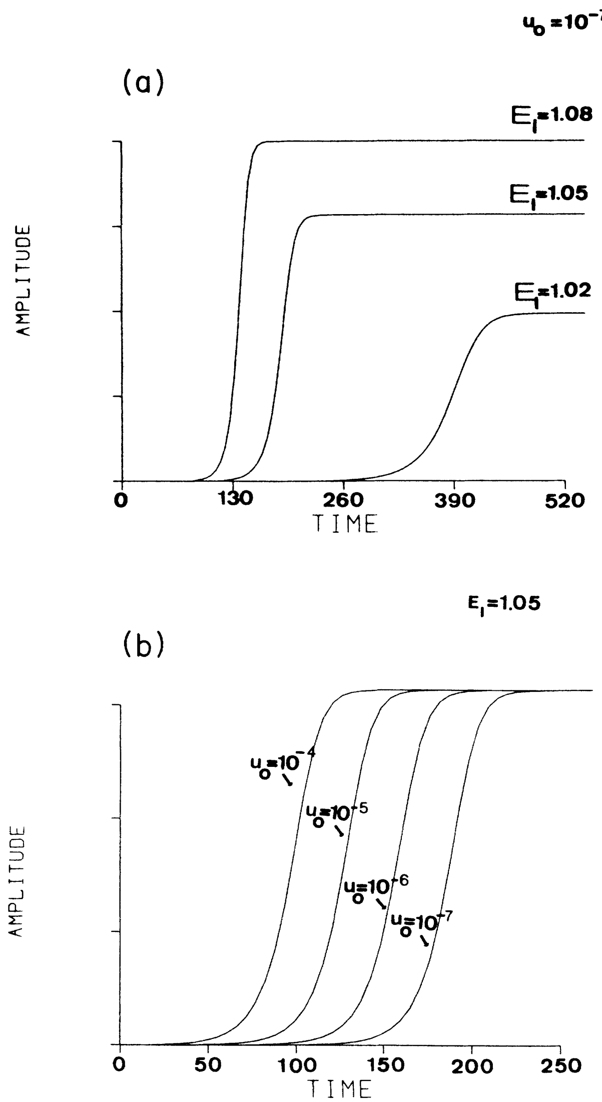

FIG. 1. Solution given by (2.30) for different values of E_I and of the initial amplitude as indicated. The constant C in (2.30) is calculated from (2.29) for those parameter values and $\theta=1$, $k_c=\pi$. (Arbitrary scale for the amplitude.)

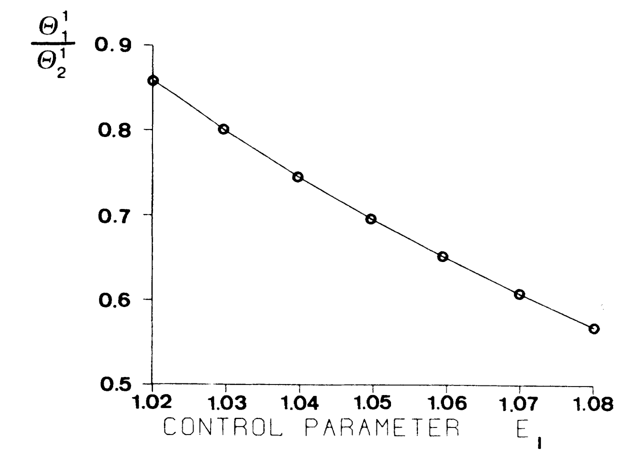

FIG. 2. Ratio of the real and imaginary parts of the field in the cavity as a function of E_I ($\theta=1$, $k_c=\pi$).

Equation (2.32) gives the solution for $E(x,t)$ in terms of quantities which can be calculated explicitly for any E_I. The explicit form of this solution to first order in $\nu=E_1-E_{I_c}$ is given in the Appendix.

The structure of (2.32) makes it clear that the dominant contribution to $E(x,t)-E_{\text{st}}$ comes from the unstable mode k_c. In addition, the mode $k=0$ gives a new contribution to the homogeneous part of the field. In the dominant contribution the eigenvector $O_\alpha^1(k_c)$ weights the different contributions of the amplitude $u_\pm(t)$ in the two components of the electric field for $\alpha=1,2$. At $E_I=E_{I_c}$, $O_1^1(k_c)=O_2^1(k_c)$ (see the Appendix), giving the same weight in the two components. However, as we move away from the instability point the weight of the real component becomes larger. The relative weight of the real and imaginary parts of the field is shown in Fig. 2.

The emergent structure pattern for the relevant physical quantity which is the field intensity $I(x,t)=|E(x,t)|^2$ is obtained from (2.32)

$$I_\pm(x,t)=I_{\text{st}}+a_1(x)u_\pm(t)+a_2(x)u_\pm^2(t)+a_3(x)u_\pm^3(t)+a_4(x)u_\pm^4(t)\ , \qquad (2.34)$$

where

$$\begin{aligned} a_1(x)&\equiv 2\cos(k_cx)\sum_\alpha E_{\text{st},\alpha}O_2^1(k_c)\ ,\\ a_2(x)&\equiv \sum_\alpha \{[O_\alpha^1(k_c)]^2\cos^2k_cx+2E_{\text{st},\alpha}A_\alpha(x)\}\ ,\\ a_3(x)&\equiv 2\cos(k_cx)\sum_\alpha O_\alpha^1(k_c)A_\alpha(x)\ ,\\ a_4(x)&\equiv \sum_\alpha A_\alpha^2\ . \end{aligned} \qquad (2.35)$$

Associated with the two solutions $u_\pm(t)$ we find two solutions for the intensity. This fact means that (2.34) implies symmetry breaking in two senses. First, the emergence of the pattern breaks the continuous-space translational symmetry. The pattern formation starting from the homogeneous solution is shown in Fig. 3. Second, the solutions $I_\pm(x)$ have no definite parity when $\Delta\to-\Delta$, being $x=\frac{1}{2}+\Delta$. The choice of the $\pm$ solution by the choice of the sign of the initial condition breaks the reflection symmetry. The I_+ solution in Fig. 3 gives a stronger intensity to the left-hand side of the cavity, while the I_- solution would give more light intensity in right-hand side of the cavity. For example, for $\theta=1$, $n_c=1$, we have

$$a_{1,3}(\Delta)=-a_{1,3}(-\Delta),\quad a_{2,4}(\Delta)=a_{2,4}(-\Delta)\ , \qquad (2.36)$$

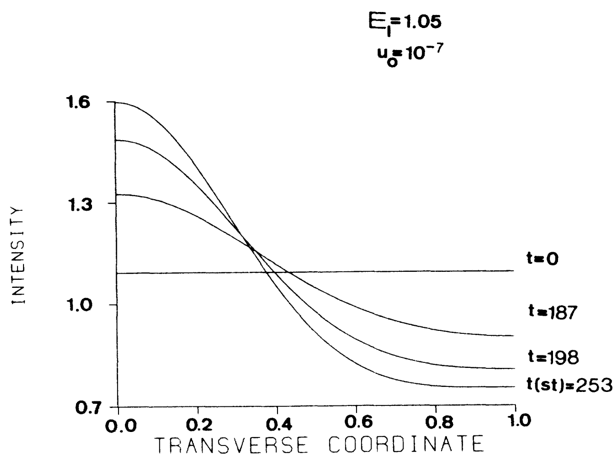

FIG. 3. Field intensity I_+ as given by Eq. (2.34) vs transverse coordinate for different times ($\theta=1$, $k_c=\pi$). t(st) indicates the time at which, numerically, the system reaches the steady state.

so that the complete set of solutions has reflection symmetry since $I_+(\Delta)=I(-\Delta)$.

The existence of these two solutions for the intensity can be understood as a space-dependent bistability in the sense that for each space point x two solutions are possible. But the difference between these two solutions depends on the space point. For $n_c=1$ the two solutions coincide at $x=\frac{1}{2}$ and their largest difference occurs at the sides of the cavity. An important point to note is that these two solutions do not grow symmetrically from I_{st} so that for a general point x, $I_+-I_{st}\neq-(I_{st}-I_-)$. The two solutions cannot be made to coincide by a trivial redefinition of variables, as in (2.28), where bistability does not exist for $\gamma=u^2$.

III. STOCHASTIC DYNAMICS

We now address the question of giving a stochastic formulation of the dynamics of pattern formation associated with (1.1). The inclusion of noise terms in (1.1) is essential in describing the transient process. In connection with the choice of initial condition for (2.28), to give a deterministic description of the decay of the unstable state, we have already seen that fluctuations are needed to initiate the decay. They determine the time scale in which the relaxation occurs and they are also responsible for the transient fluctuations which are generally anomalously large.[9] In addition, fluctuations have in these problems an important effect in the final steady state; the broken reflection symmetry discussed above is restored when a small noise term is included because fluctuations mix the two solutions $I_\pm(x,t)$. This symmetry-restoring effect is preserved if the noise intensity is taken to zero once the steady state is reached. This fact reflects the noncommutativity of the deterministic limit and the limit $t\to\infty$.

A. Stochastic model and amplitude equation

Possible sources of noise to be included in (1.1) are quantum noise, parametric or external noise, and thermal noise. Quantum noise[10] will not be considered here. In order to introduce parametric noise[11] associated with phase and intensity fluctuations of the incident field, a frame of reference selecting the phase difference between E and E_I is introduced:

$$E_I(t)=|E_I(t)|\exp[i\psi(t)]\ ,\qquad \bar{E}(x,t)=E(x,t)\exp[-i\psi(t)]\ . \tag{3.1}$$

The chosen stochastic model for $\bar{E}(x,t)$ is

$$\partial_t\bar{E}(x,t)=-\bar{E}+|E_I|+i(|E|^2-\theta)\bar{E}+ia\partial_x^2\bar{E}-i\dot{\psi}(t)\bar{E}+\omega(t)+\eta(x,t)\ , \tag{3.2}$$

where $\dot{\psi}(t)$ and $\omega(t)$ are real space-independent stochastic processes. $\dot{\psi}(t)$ is associated with frequency fluctuations. Following the ordinary-phase diffusion model, $\dot{\psi}(t)$ is taken as a Gaussian white noise of zero mean and correlation

$$\langle\dot{\psi}(t)\dot{\psi}(t')\rangle=\epsilon_p\delta(t-t')\ . \tag{3.3}$$

$\dot{\psi}(t)$ models equally well the fluctuations of the detuning parameter θ considered in Ref. 12. The process $\omega(t)$ is associated with amplitude fluctuations introduced when replacing $|E_I|$ by $|E_I|+\omega(t)$; it is also taken as a Gaussian white noise of zero mean and correlation

$$\langle\omega(t)\omega(t')\rangle=\epsilon_I\delta(t-t)\ . \tag{3.4}$$

The complex space-dependent process $\eta=\eta_1+i\eta_2$ models thermal fluctuations. It is taken to be Gaussian white noise with correlation

$$\langle\eta_\alpha(x,t)\eta_\beta(x,t')\rangle=\epsilon_T\delta(x-x')\delta(t-t')\delta_{\alpha\beta}\ ,\quad \alpha,\beta=1,2\ . \tag{3.5}$$

The stochastic nonlinear equation for $q(x,t)=\bar{E}(x,t)-E_{st}$ becomes (2.4) supplemented by noise terms. Amplitude and thermal noise appear as new independent terms in the right-hand side of (2.4), while frequency noise contributes to the linear term. Explicitly,

$$\partial_t q_\alpha(x,t)=\sum_\beta K_{\alpha\beta}q_\beta(x,t)+N_\alpha(q)+\sum_\beta M_{\alpha\beta}q_\beta\dot{\psi}(t)+\delta_{\alpha,1}\omega(t)+\eta_\alpha(x,t)\ , \tag{3.6}$$

where $M_{\alpha\beta}$ is the unit antisymmetric matrix, $M_{11}=M_{22}=0$, $M_{12}=1$, $M_{21}=-1$.

Our strategy is now to reduce (3.6) to its normal form, given by a stochastic amplitude equation much in the same way that (2.4) was reduced to (2.28). We still seek a solution of the form (2.16) for $q_\alpha(x,t)$, where, now, $S_{k,j}(t)$ will become stochastic processes. A similar decomposition is introduced for $\eta_\alpha(x,t)$:

$$\eta_\alpha(x,t)=\sum_{k,j}\eta_{k,j}(t)O^j_\alpha(k)\cos(kx)\ , \tag{3.7}$$

so that

$$\langle \eta_{k,j}(t)\eta_{k',j'}(t')\rangle = 2(1-\tfrac{1}{2}\delta_{k,0})\epsilon_T\delta(t-t')\delta_{k,k'} \times \sum_\alpha \bar{O}\,^j_\alpha(k)\bar{O}\,^{j'}_\alpha(k') \ . \quad (3.8)$$

Following the same steps as in Sec. II the new terms that appear on the right-hand side of (2.17) are

$$\sum_{j'} a_{kk,jj'}\dot{\psi}(t)\zeta_{k,j'} + \eta_{k,j}(t) + \delta_{k,0}\bar{O}\,^j_1(k=0)\omega(t) \ , \quad (3.9)$$

$$a_{kk',jj'} \equiv \sum_{\alpha,\beta} M_{\alpha\beta}\bar{O}\,^j_\alpha(k)O^{j'}_\beta(k') \ , \quad (3.10)$$

and Eqs. (2.24) and (2.25) become

$$\dot{u}(t) = \lambda_1(k_c)u + A_u^{(3)}u^3 + \sum_{k,j} A_{u,s}^{(2)}S_{k,j}u + \dot{\psi}\sum_j B_u\zeta_j(k_c) + \eta_{k_c,1}(t) \ , \quad (3.11)$$

$$\dot{S}_{k,j}(t) = \lambda_j(k)S_{k,j} + A^{(2)}_{s_{k,j}}u^2 + \dot{\psi}(t)\sum_{k,j} B_s s_{k,j} + \eta_{k,j}(t) + \delta_{k,0}\bar{O}\,^j_1(k=0)\omega(t) \ , \quad (3.12)$$

where

$$B_u \equiv a_{k_c k_c,1j} \ , \quad (3.13)$$

$$B_s \equiv a_{kk,1j} + a_{kk,2j} + a_{kk,j2} \ . \quad (3.14)$$

Equations (3.11) and (3.12) indicate that amplitude noise only couples to the homogeneous mode, but frequency noise couples in a multiplicative way to all the modes and introduces a nondiagonal linear term in (3.11). This is so because the expansion (2.16) is based on the linearization of only the deterministic part of (3.6).

The multiplicative character of frequency noise makes the adiabatic elimination of the stable modes more difficult than in the deterministic case. We will consider, in this paper, a first approximation of the noise terms in (3.11)–(3.14) introduced as follows: The important fluctuations are those connected with the growth of the unstable mode and, therefore, noise terms in (3.12) are neglected. Also, since the main interest here is in the early stages of decay, multiplicative frequency noise in (3.11) can be neglected in front of $\eta_{k_c,1}$. In fact, B_u is such that strictly at the instability point $E_I = E_{I_c}$, $a_{k_c k_c,11} = 0$, so that $\dot{\psi}(t)$ does not couple to u [see (3.13)]. Neglecting multiplicative noise is known to be generally a good approximation when studying the first stages of decay of an unstable state[13–15] because multiplicative-noise terms vanish for $u=0$. We note that within the approximation scheme used here, the most relevant source of noise in (3.11) influencing the decay process is the k_c component of $\eta(x,t)$. Following the same adiabatic elimination scheme of Sec. II, the stochastic amplitude equation becomes

$$\dot{u} = \lambda_1(k_c^2)u + Cu^3 + \eta_{k_c,1}(t) \ , \quad (3.15)$$

where

$$\langle \eta_{k_c,1}(t)\eta_{k_c,1}(t')\rangle = \epsilon\delta(t-t') \quad (3.16)$$

and ϵ defined through (3.8). It can be seen from the results in the Appendix for $\bar{O}\,^j_\alpha(k)$ that $\epsilon = \epsilon_T$ for $E_I = E_{I_c}$.

The simple stochastic equation (3.15) can be studied by standard techniques. The stochastic behavior of $\bar{E}(x,t)$ can be constructed from $u(t)$ in the same manner as the deterministic case in Sec. II C. It is important to note that the method followed here to reduce a complicated stochastic space-dependent dynamics is rather general. In fact, the main steps of the method and general features of our results remain the same for any general nonlinear equation like (2.4) with quadratic and cubic nonlinearities.

B. Qualitative stochastic picture: symmetry-restoring and transient fluctuations

Important qualitative facts concerning noise effects can be discussed without further explicit calculation. As previously discussed, for $\epsilon=0$, $u(t)$ reaches at $t=\infty$, either of two values giving rise to two solutions $I_\pm(\Delta)$, which only coincide at the center of the cavity $\Delta=0$. For small ϵ the qualitative stochastic picture is then that for $\Delta\neq 0$; an intensity probability distribution that initially peaks around I_{st} will evolve splitting into two peaks that will approach positions centered around $I_\pm(\Delta)$. The peak splitting will be more noticeable close to the walls of the cavity, while in the middle of the cavity the probability distribution remains single peaked for all times, the position of the peak moving from I_{st} at $t=0$ to $I(\Delta=0)$ for $t=\infty$. The symmetry restoring mentioned before is now obvious and it is due to the mixing of the two deterministic solutions by fluctuations. Indeed, $u(t)$ in (2.34) is now a stochastic process such that $\langle u^{2n+1}(t)\rangle = 0$ for all times. Then $\langle I(\Delta,t)\rangle = \langle I(-\Delta,t)\rangle$, because only the a_2 and a_4 terms in (2.34) remain after averaging. Of course, a given realization approaches at intermediate times one of the two deterministic solutions, but at late times the two solutions are mixed by an escape mechanism activated by fluctuations.

The time scale in which a solution I_+ escapes to a solution I_- is given by Kramers time $T \simeq e^{\Delta v/\epsilon}$, where Δv is the potential barrier associated with the two stationary solutions of (3.15) as $\epsilon\to 0$. However, we note that, in these late-stage processes, noise terms neglected in (3.15) may become important, changing the actual value of the escape time T. A second important noise effect concerns transient fluctuations. It is generally known that in the decay of an unstable state the time-dependent variance of the relevant variable exhibits anomalously large fluctuations.[9] This means that fluctuations are relatively small around the initial and final stationary values at $t=0$ and $t=\infty$, but that in the transient a large maximum exists. The phenomenon is understood as the amplification by deterministic nonlinear dynamics of the initial fluctuations that triggers the decay process. The maximum of the time-dependent variance occurs after the system has left the vicinity of the unstable state. This time scale is associated with the lifetime of the unstable state given by a first-passage time. Anomalously large transient fluctuations are well known during laser switch on.[16,17] In our problem, transient intensity fluctuations are obtained from (2.34) and (3.15) as

$$\delta(x,t)=\langle (I-\langle I\rangle)^2\rangle = a_1^2(x)\langle u^2(t)\rangle+2a_1(x)a_3(x)\langle u^4(t)\rangle + a_3^2(x)\langle u^6(t)\rangle+\Sigma(x,t) , \quad (3.17)$$

where

$$\Sigma(x,t)= a_2^2(x)\langle [u^2(t)-\langle u^2(t)\rangle]^2\rangle + a_4^2(x)\langle [u^4(t)-\langle u^4(t)\rangle]^2\rangle + 2a_2(x)a_4(x)\langle [u^2(t)-\langle u^2(t)\rangle] \times[u^4(t)-\langle u^4(t)\rangle]\rangle . \quad (3.18)$$

$\delta(x,t)$ has two contributions: $\Sigma(x,t)$ involves variances, and at steady state ($t=\infty$), Σ vanishes as $\epsilon\to 0$. The remaining part $\delta-\Sigma$ grows with time and reaches a final value which remains finite for $\epsilon=0$. In fact,

$$\lim_{\epsilon\to 0}\delta(x,t=\infty)=a_1^2(x)u_\infty^2+2a_1(x)a_3(x)u_\infty^4+a_3^2(x)u^6 , \quad (3.19)$$

where $u_\infty^n=[\lambda_1(k_c)/(-C)]^{n/2}$. This result makes clear the noncommutativity of the limits $\epsilon\to 0$ and $t\to\infty$, since for $\epsilon=0$, $\delta\equiv 0$ for all times. The above result is valid for all points x, but in particular for $x=\frac{1}{2}$ and $\theta=1$, $a_1(x)=a_3(x)=0$, so that $\delta(x=\frac{1}{2},t)=\Sigma(x=\frac{1}{2},t)$ and $\lim_{\epsilon\to 0}\delta(x=\frac{1}{2},\, t=\infty)=0$. The essential physical picture is the following: $\delta-\Sigma$ is associated with fluctuations occurring in two-peaked probability distributions. These remain in the limit of vanishing fluctuations in which the probability distribution becomes a superposition of two δ functions. On the other hand, Σ is associated with the transient anomalous fluctuations, which already occur in the evolution of a single-peaked distribution. Those are the anomalous fluctuations mentioned before which depend here on the space point x. They are masked by the part of δ which grows monotonically with time, except close to the center of the cavity ($\Delta=0$) in which $\delta-\Sigma$ becomes small because no peak splitting occurs. Only in these space points will the characteristic peak of the transient anomalous fluctuations become apparent in $\delta(t)$. The intensity fluctuations $\delta(x,t)$ relax to a small value for points close to $x\simeq\frac{1}{2}$ and to a large value given by (3.19) close to the walls of the cavity. Another way of understanding (3.19) is to notice that

$$\lim_{\epsilon\to 0}\delta(x,t=\infty)=\lim_{\epsilon\to 0}[\langle I^2(t=\infty)\rangle-\langle I(t=\infty)\rangle^2] =\tfrac{1}{4}[I_+(t=\infty)-I_-(t=\infty)]^2 , \quad (3.20)$$

so that this quantity grows with the square of the difference between the two stationary deterministic solutions for the intensity.

C. Transient-dynamics calculation

The above discussion can be made quantitative by solving the stochastic theory[18] (QDT) version of the dynamical scaling theory,[9] as reviewed elsewhere.[17] The basic idea is to approximate the process $u(t)$ by a process $\bar{u}(t)$ obtained from the nonlinear deterministic solution (2.30) replacing the initial condition u_0 by a stochastic process $h(t)$:

$$\bar{u}(t)\simeq\{h(t)\exp[\lambda_1(k_c^2)t]\}/(1-[Ch^2(t)/\lambda_1(k_c^2)]\{\exp[2\lambda_1(k_c)t]-1\})^{1/2} . \quad (3.21)$$

The stochastic process $h(t)$ appears in the solution of the linear approximation to (3.15) as an effective stochastic initial condition when $u_0=0$,

$$u_l(t)=h(t)e^{\lambda_1(k_c^2)t} , \quad (3.22)$$

$$h(t)=\int_0^t dt' e^{-\lambda_1(k_c^2)t'}\eta_{k_c,1}(t') . \quad (3.23)$$

Equation (3.21) gives $u(t)$ as a functional of $h(t)$ in which the initial fluctuating regime around $u_0=0$ is dominated by noise and the linear term is propagated on time by the nonlinear deterministic mapping. The process $h(t)$ is Gaussian of zero mean and with variance

$$\sigma^2(t)=\frac{\epsilon}{2\lambda_1(k_c)}\{1-\exp[-2\lambda_1(k_c)t]\} . \quad (3.24)$$

In this scheme the calculation of any time-dependent average of a function $f(u(t))$ of the process $u(t)$ reduces to a Gaussian average over the probability distribution $P(h,t)$ of the process $h(t)$:

$$\langle f(u(t))\rangle=\int_{-\infty}^{\infty} dh\, P(h,t)f[u=\bar{u}(h(t),t)] . \quad (3.25)$$

In particular, the moments of $u(t)$ needed to calculate the mean intensity and intensity fluctuations from (2.34) and (3.17), respectively, are easily calculated. Analytic expressions are available for the moments in terms of confluent hypergeometric functions.[18] The process $h(t)$ saturates on time because, for times $\lambda_1(k_c)t\gg 1$, $\sigma^2(t)$ becomes time independent. As a consequence, the approximation (3.21) for $\bar{u}(t)$ takes into account the essential fluctuations in the early stages of decay of the unstable state, but neglects fluctuations occurring in the late stages of evolution around the final equilibrium state. This means that, as $t\to\infty$, a calculation based on (3.21) reproduces the limit $\epsilon\to 0$ discussed in Sec. III B. It also means that late-stage fluctuations connecting the deterministic solutions $I_\pm(x,t)$ are not included here.

The lifetime of the unstable state is determined by the initial fluctuating regime.[13,14,17–19] It can be defined as the time T that the process $u^2(t)$ takes to reach a given reference value u_R^2. This can be calculated inverting (3.22) and averaging over h. One finds[19,20] for small ϵ,

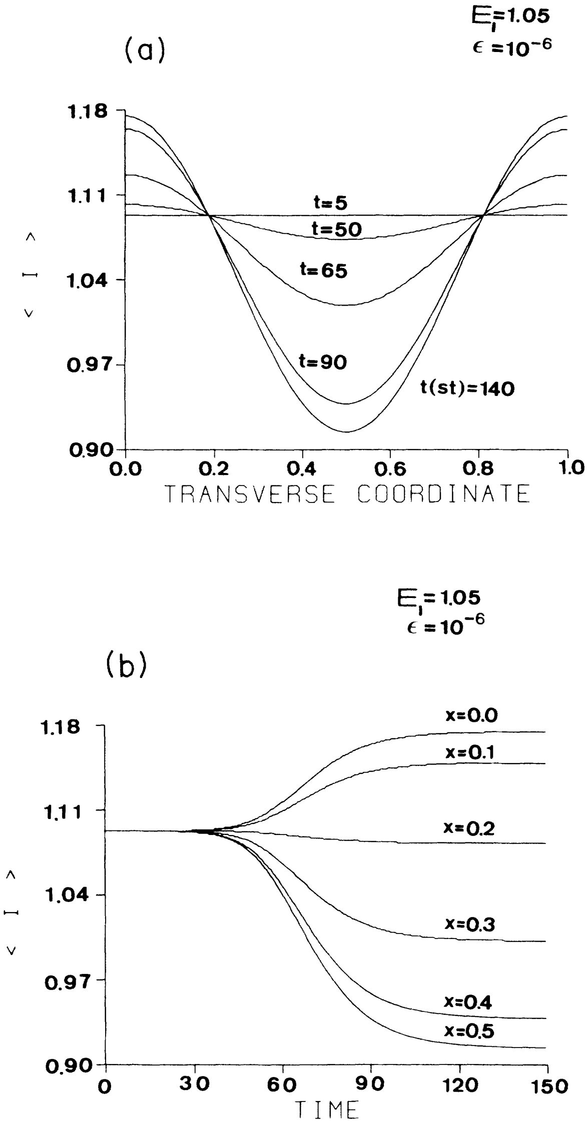

FIG. 4. Mean intensity as a function of time and transverse position. Notice that the lifetime of the homogeneous state is about $T \approx 60$ ($\theta=1$, $k_c=\pi$).

$$T \simeq \frac{1}{2\lambda_1(k_c)} \ln \frac{2u_R^2 \lambda_1(k_c)}{\epsilon} - \frac{\psi(1)}{2\lambda_1(k_c)} , \tag{3.26}$$

where ψ is the digamma function.[21]

Results of a calculation of $\langle I(x,t)\rangle$ and $\delta(x,t)$ based on (3.21) are shown in Figs. 4–6. The evolution of the mean intensity is seen in Fig. 4. The pattern formation for the mean intensity is shown in Fig. 4(a) where $\langle I(x,t)\rangle$ goes to a final spatial structure starting from a homogeneous situation. Comparing Fig. 4(a) with Fig. 3 the symmetry-restoring effect of fluctuations is explicitly seen. In fact, it is found that

$$\begin{aligned}\langle I(x,t=\infty)\rangle &= \tfrac{1}{2}[I_+(x,t=\infty)+I_-(x,t=\infty)]\\ &= I_{\rm st} - a_2(x)\frac{\lambda_1(k_c^2)}{C} + a_4(x)\frac{\lambda_1^2(k_c^2)}{C^2} .\end{aligned} \tag{3.27}$$

The mean intensity evolves in time, taking values smaller or larger than the original $I_{\rm st}$ value depending on the space position, as seen in Fig. 4(b). The time scale in which the pattern forms is the time at which $\langle I\rangle$ becomes essentially different from $I_{\rm st}$ at all points simultaneously. This time scale is given by the first-passage time (3.26). For the parameter values in Fig. 4(b) and taking $u_R^2 = u^2(\infty)/2$, one finds $T \simeq 60$, which agrees with the time scale that one can associate with the lifetime of the homogeneous state in Fig. 4(b).

Figure 5 shows the intensity fluctuations $\delta(x,t)$. The symmetry around $x=\frac{1}{2}$ due to fluctuations is apparent. It is also clear that fluctuations are much larger at the sides than in the center of the cavity. This agrees with our previous qualitative discussion. The large intensity fluctuations at the sides of the cavity are due to peak splitting of the probability distribution. The space dependence of the intensity fluctuations at $t=\infty$ is obtained from (3.19) $[\Sigma(x,t=\infty)=0]$,

$$\delta(x,t=\infty) = \tfrac{1}{2}[1+\cos(2k_c x)][d_1 + d_2\cos(2k_c x) + d_3\cos^2(2k_c x)] , \tag{3.28}$$

where d, d_1, d_2, and d_3 are given in terms of a_i and u_∞^n. It predicts a maximum of fluctuations at $x=0,1$ and a minimum at $x=\frac{1}{2}$. The numerical values shown in Fig. 5 coincide with (3.20). At the sides of the cavity $\delta(x,t)$ is seen to grow monotonically with time, while for $x \simeq \frac{1}{2}$ the crossing of lines indicates the existence of a maximum as

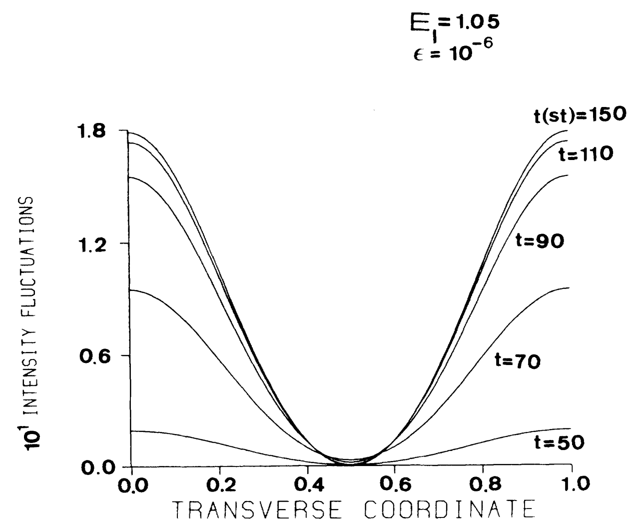

FIG. 5. Intensity fluctuations as calculated from Eq. (3.20) vs position ($\theta=1$, $k_c=\pi$).

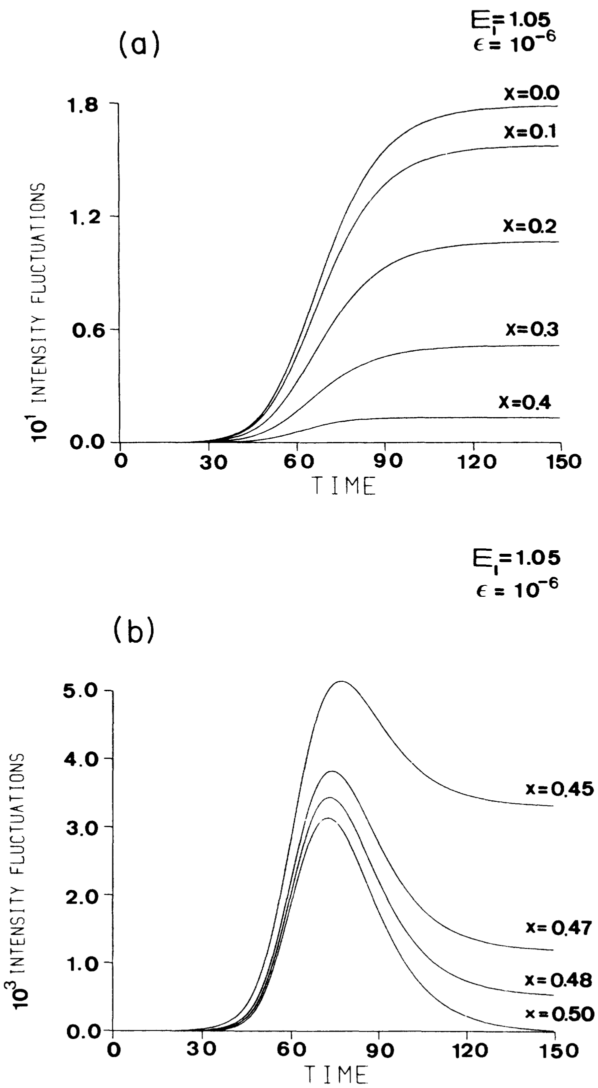

FIG. 6. Time evolution of the intensity fluctuations for different values of the transverse coordinate.

a function of time. This is more clearly depicted on Fig. 6, which makes quantitative our previous discussion of the two contributions to $\delta(x,t)$. At $x=0.5$ only Σ survives in δ and a clear peak associated with transient anomalous fluctuations occurs. The peak becomes less defined when moving away from the center of the cavity because the Σ contribution due to peak splitting of the intensity probability distribution comes in. For $x<0.4$ or $x>0.6$, $\delta(t)$ grows monotonically with time to its final value, which is larger the closer it is to the cavity walls.

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APPENDIX

In this appendix we give explicit results for some of our previous formulas through an explicit calculation to lowest order in a small parameter $\nu=E_I-E_{I_c}$, which measures the distance in parameter space to the point in which the homogeneous solution becomes unstable. Our results are specified for the value of the detuning parameter $\theta=1$ used in the calculations shown in the figures.

For $\theta=1$, $E_{I_c}=1$ and the homogeneous stationary solution is

$$E_{1,s}=1+\nu,\quad E_{2,s}=2\nu\ , \tag{A1}$$

so that the eigenvalues (2.11) becomes

$$\lambda_j(k^2)=-1\pm[\nu(8ak^2-4)+2ak^2-a^2k^4]^{1/2}\ . \tag{A2}$$

The eigenvalues used in our calculation are

$$\lambda_1(k_c^2)=2\nu,\quad \lambda_2(k_c^2)=-2-2\nu\ , \tag{A3}$$

$$\lambda_j(k=0)=-1\pm 2i\nu^{1/2}\ , \tag{A4}$$

$$\lambda_j(k=2k_c)=-1\pm i2\sqrt{2}(1-\tfrac{7}{4}\nu)\ . \tag{A5}$$

For k_c we have real eigenvalues, $\lambda_1(k_c^2)$, indicating the soft-mode instability, and $\lambda_2(k_c^2)$ remaining negative beyond the instability point. For $k=0,2k_c$ we have complex eigenvalues that come in pairs. An important point to notice is that $\lambda_j(k=0)$ becomes real and degenerate as $\nu\to 0$. The expansion (2.16) is based on independent eigenvectors obtained for $\nu\neq 0$. The eigenvalues associated with $k=2k_c$ remain complex as $\nu\to 0$.

The associated eigenvectors are calculated with the normalization conditions discussed in Sec. II A. Our aim is to calculate $E_\alpha(x,t)$ in (2.33) to order ν. Since $\lambda_1(k_c^2)$ is already of order ν, it is enough to find C in (2.29) and $O_\alpha^1(k=0)$ and $A_\alpha(x)$ in (2.32) to order zero in ν. For $k=k_c$ and $k=2k_c$, it is then enough to calculate the eigenvectors at $\nu=0$. For $k=k_c$, the matrix $K_{\alpha\beta}$ is symmetric at $\nu=0$ and

$$\begin{aligned} O_\alpha^1(k_c)&=\bar{O}_\alpha^{\,1}(k_c)=\frac{1}{\sqrt{2}}(1,1)\ ,\\ O_\alpha^2(k_c)&=\bar{O}_\alpha^{\,2}(k_c)=\frac{1}{\sqrt{2}}(1,-1)\ . \end{aligned} \tag{A6}$$

For $k=2k_c$,

$$\begin{aligned} O_\alpha^1(k)&=\frac{1}{\sqrt{3}}(\sqrt{2},i)\ ,\\ \bar{O}_\alpha^{\,1}(k)&=\sqrt{3}\left[\frac{1}{2\sqrt{2}},-i/2\right]\ , \end{aligned} \tag{A7}$$

and $O_\alpha^2(k)=[O_\alpha^1(k)]^*$, $\bar{O}_\alpha^{\,2}(k)=[\bar{O}_\alpha^{\,1}(k)]^*$. The situation for $k=0$ is different since there are no independent eigenvectors at $\nu=0$. We keep the dominant contribution in ν, and the final constants involved in the solution (2.32) are well defined as $\nu\to 0$:

$$O^1_\alpha(k=0)=(\nu^{1/2},-2\nu^{1/2}-i)\ ,$$
$$\bar{O}^{\,1}_\alpha(k=0)=\left[\frac{\nu^{-1/2}}{2}+i,i/2\right]\ , \tag{A8}$$

and $O^2_\alpha(k=0)=[O^1_\alpha(k=0)]^*$, $\bar{O}^{\,2}_\alpha(k=0)=[\bar{O}^{\,1}_\alpha(k=0)]^*$. The fact that the first component of $O^1_\alpha(k=0)$ diverges as $\nu\to 0$ reflects the degeneracy of $\lambda_j(k=0)$ at $\nu=0$. The products of left and right eigenvectors appearing in the coefficients $a_{kk'k'',jj'j''}$ and $a_{kk'k''k''',jj'j''j'''}$ [(2.18) and (2.19)] remain finite as $\nu\to 0$.

Substituting the above eigenvectors in (2.18) and (2.19) we find, for C in (2.29) as $\nu\to 0$, $C=-\frac{19}{36}$ (Ref. 22) and the solution (2.32) becomes

$$E^{\pm}_\alpha(x,t)=\begin{bmatrix}1+(E_I-E_{I,C})\\ 2(E_I-E_{I,C})\end{bmatrix}+\frac{1}{\sqrt{2}}\cos k_c x\begin{bmatrix}1\\1\end{bmatrix}u_\pm(t) + u^2_\pm(t)\left[\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix}-\cos 2k_c x\begin{bmatrix}\frac{7}{18}\\ \frac{2}{9}\end{bmatrix}\right]\ . \tag{A9}$$

For small ν the dominant term is the one corresponding to the unstable mode and proportional to $u_\pm(t)$, which is of order $\nu^{1/2}$. Both components of this term have the same amplitude. This is no longer so when ν becomes larger, as seen in Fig. 2.

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