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Spin and paramagnetism in classical stochastic electrodynamics

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We study the statistical properties of the spin **S** and magnetic dipole $\boldsymbol{\mu}$ of a particle with two constituents bounded by a harmonic force. We find the relation between $\boldsymbol{\mu}$ and **S** and also conclude that $\langle \mathbf{S}^2 \rangle \sim \hbar^2$. The gyromagnetic factor can assume any value depending on the charges and masses of the constituents. In another example, we consider the case of a permanent magnetic dipole moving in an external magnetic field, under the influence of the fluctuations associated with the zero-point and thermal radiations characteristic of stochastic electrodynamics (SED). We conclude that the system contains paramagnetic features, and comparison with the experimental data shows excellent agreement with SED.

I. INTRODUCTION

Quite recent studies have shown that stochastic electrodynamics (SED) has reached an important point in its development. We can see this not only in the reviews by Boyer[1] and de la Peña[2] but also in the new branches of the theory, as, for instance, the so-called "stochastic optics" recently developed by Marshall and Santos.[3]

As a classical theory able to describe microscopic phenomena, SED is penetrating the domain of quantum mechanics (QM). The main ingredient which allows this is the well-known zero-point radiation background.[1-4] It has electromagnetic fields that are random and considered as a superposition of plane waves with all frequencies. This is an hypothesis which perhaps could be justified if we admit that the zero-point radiation is that emitted by all the accelerated charges of the Universe.[5]

The electric field $\mathbf{E}(\mathbf{x},t)$ in a point $\mathbf{x}$ at the instant t is usually written as[1,4]

$$\mathbf{E}(\mathbf{x},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\varepsilon}(k,\lambda)H(\omega,T)\times\cos[\mathbf{k}\cdot\mathbf{x}-\omega t+\nu(\mathbf{k},\lambda)]\ , \tag{1.1}$$

where $\omega=c|\mathbf{k}|$ is the frequency of the plane wave characterized by the wave vector $\mathbf{k}$. The function $H(\omega,T)$ is associated with the wave amplitude and T is the temperature of the cavity in which some microscopic system is immersed. The phases $\nu(\mathbf{k},\lambda)$ are completely random,[1,4] that is, they are statistically independent and vary uniformly within the interval $0\le\nu\le2\pi$. The polarization of each wave is characterized by the unit vectors $\hat{\varepsilon}(\mathbf{k},\lambda)$ such that $\mathbf{k}\cdot\hat{\varepsilon}=0$ and

$$\sum_{\lambda=1}^{2}\varepsilon_i(\mathbf{k},\lambda)\varepsilon_j(\mathbf{k},\lambda)=\delta_{ij}-\frac{k_ik_j}{k^2}\ . \tag{1.2}$$

It is easy to see that the ensemble average gives

$$\langle\mathbf{E}\rangle=0 \tag{1.3}$$

and

$$\frac{\langle\mathbf{E}^2\rangle}{4\pi}=\int_0^\infty d\omega\,\omega^2\frac{H^2(\omega,T)}{c^3}\equiv\int_0^\infty d\omega\,\rho(\omega,t)\ , \tag{1.4}$$

where $\rho(\omega,T)$ is the spectral density. There is no difficulty in showing that $\rho(\omega,T)$ have the contributions from both the zero-point and thermal radiation, that is,[1,2]

$$\rho(\omega,T)=\frac{\hbar\omega^3}{\pi^2c^3}\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right]\ , \tag{1.5}$$

where $\hbar$ is Planck's constant divided by 2π. Indeed, Marshall,[6] Boyer,[7] and Jiménez[8] have shown that the expressions for $\rho(\omega,0)$ and $\rho(\omega,T>0)$ can be obtained through entirely classical arguments from SED. However, these calculations are not free from criticism.[6,8]

Before we start the description of our model of a composite particle, let us mention some of the main achievements of SED in the domain of microscopic phenomena.[1,2] One example is the correct description of diamagnetic properties of free and harmonically bounded charges.[4,6,9] Another example is the analysis of van der Walls forces.[10,11] The more studied system is the harmonic oscillator.[1,2,4] Many detailed discussions of this system were published since the pioneering work by Marshall[6] and others.[12-14] More recently a new branch of SED, called stochastic optics, started to be developed by Marshall and Santos.[3] The new phenomena described by this theory, for instance the anticorrelation observed in both channels of a beam splitter in the experiment by Grangier *et al.*,[15] has challenged physicists to revise the concept of the photon. In stochastic optics[3] photons are considered as short pulses of electromagnetic waves.

The success of SED in describing the above microscopic phenomena has encouraged us to investigate whether the theory is able to give reasonable results for the properties of the spin **S** in a simple model of a composite particle. The particle was assumed to have two constituents with charges q_1 and q_2 where $q_1+q_2=Q$ is the total charge of the particle and masses m_1 and m_2 with $m_1+m_2=M$ (the total mass). The dominant force between the constituents is a harmonic force with frequency

ω_0. With this model, we were able to describe the relative motion of the constituents as we will see in detail in Sec. II. We concluded that the magnetic dipole $\boldsymbol{\mu}$ of the system is such that

$$\boldsymbol{\mu}=g(Q/2Mc)\mathbf{S} , \tag{1.6}$$

where g is the gyromagnetic factor which depends, in a simple way, on the charges and masses of the constituents. The spin $\mathbf{S}$ is such that

$$\langle \mathbf{S}^2 \rangle = \tfrac{3}{2}\hbar^2(1-\xi)^4/(1+\xi^2)^2 , \tag{1.7}$$

where $\xi=q_1 m_2/q_2 m_1$. This result is independent of the temperature and also does not depend on the frequency ω_0 of the harmonic force.

The most interesting situation appears when we study the motion of a rigid magnetic dipole $\boldsymbol{\mu}$, with $g=2$, in an external magnetic field $\mathbf{B}_0$. This analysis was done before by Boyer[16] and we simply give a summary of the calculations and approximations used. Boyer was able to find a stationary solution of a Fokker-Planck equation for the probability distribution $P(\theta)$, where θ is the angle between $\mathbf{S}$ (or $\boldsymbol{\mu}$) and $\mathbf{B}_0$. Taking $\mathbf{B}_0$ in the z direction, Boyer showed that

$$\frac{\langle S_z \rangle}{S} = \coth\left[\frac{2S}{\hbar\coth(\hbar\mu B_0/2SkT)}\right] - \frac{\hbar}{2S}\coth\left[\frac{\hbar\mu B_0}{2SkT}\right] , \tag{1.8}$$

where $S=|\mathbf{S}|$ and $\mu=|\boldsymbol{\mu}|$. Next we assume that $g=2$, that is, $\mu=QS/Mc$, and also $S=N\hbar$, where N is the only free parameter that can depend on the internal structure of the particle or atom. By doing a comparison between $\langle \mu_z \rangle = (Q/Mc)\langle S_z \rangle$ and some experimental data,[17] we conclude that there is a very good agreement with SED predictions in the case where $N=2$, 3, and 4. A confrontation with the corresponding QM predictions is done.[17] We present a detailed comparison between the Brillouin functions, characteristic of the observed paramagnetism, and the corresponding Boyer functions defined in Sec. III. Our conclusions are left to Sec. IV of this paper.

II. SIMPLE MODEL OF A COMPOSITE PARTICLE

Despite the success in describing many properties of the above-mentioned systems, there are few attempts to understand properties of microscopic particles like electrons and nucleons, for instance. Boyer[4,16] and de la Peña and co-worker[2,18] tried, independently, to describe some properties of the spin and magnetic dipole associated with the electron. We have to recognize that the present work was inspired by the quite interesting efforts of these and other authors.[19,20]

We are going to study here some properties of spin and magnetic dipole of very simple composite systems. The first example is a particle composed of two constituents. This is the simplest model of an "elementary" particle. We keep ourselves to this model because in this case the results are more easily interpreted and the conclusions more definitive.

We shall assume in the first example that the constituents are pointlike charges q_1 and q_2, such that $Q=q_1+q_2$ is the total charge of the composite particle. Besides this, the masses will be m_1 and m_2 in such a way that $M=m_1+m_2$ is the total mass. We also assume that the internal motion is nonrelativistic, as we have mentioned in Sec. I. We shall consider that the constituents do not have intrinsic angular momentum. With this the *spin* of the composite particle can only be attributed to the *orbital motion* of the constituents.

Another approximation is that the dominant force is the harmonic attraction (frequency ω_0) between the constituents. This means that the average distance between them is large enough to consider the Coulomb force negligible. In other words, the centrifugal force is much larger than the Coulomb attraction or repulsion. This approximation will introduce restrictions to the parameters (charges, masses, frequency ω_0) of the system. These restrictions will be discussed below.

Previous experience[1,2,6] with harmonic oscillators shows that a resonance with radiation at frequency ω_0 will be reached in the stationary regimen. Since the motion is nonrelativistic we can imagine the charges at "rest" but separated by the distance x. If the wavelength is large as compared to x we can have $x\omega_0/2\pi c \ll 1$. This is true for the harmonic oscillator within SED, so that the approximation $\mathbf{E}(\mathbf{x},t)\simeq\mathbf{E}(0,t)$, for the random electric field, is fine.[21,22]

In this way the constituent particles, located at the points $\mathbf{r}_1$ and $\mathbf{r}_2$, will obey the following equations of motion:

$$m_1\ddot{\mathbf{r}}_1 = m\omega_0^2(\mathbf{r}_2-\mathbf{r}_1)+\frac{2}{3}\frac{q_1^2}{c^3}\dddot{\mathbf{r}}_1+q_1\mathbf{E}(t) \tag{2.1}$$

and

$$m_2\ddot{\mathbf{r}}_2 = -m\omega_0^2(\mathbf{r}_2-\mathbf{r}_1)+\frac{2}{3}\frac{q_2^2}{c^3}\dddot{\mathbf{r}}_2+q_2\mathbf{E}(t) . \tag{2.2}$$

Here $m=m_1m_2/M$ is the reduced mass, $q_i\mathbf{E}(t)$ are the random forces, and $\frac{2}{3}(q_i^2/c^3)\dddot{\mathbf{r}}_i$ is an *approximation*[22] for the radiation reaction force over the particles 1 or 2. In writing (2.1) and (2.2) we have assumed that the particles irradiate independently.[23] Another hypothesis is that $m_1\ddot{\mathbf{r}}_1 \simeq m\omega_0\mathbf{r}$, where $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, which implies that the radiation is emitted mainly in the direction perpendicular to $\mathbf{r}$. This is in accordance with our assumption that the charges are emitting independently. The above approximations and hypothesis deserve more comments. Equations (2.1) and (2.2) are not exact. The radiation reaction force above is the first approximation of a more complicated expression.[22] Another point is that (2.1) and (2.2) are justified only if we have $\mathbf{E}(t)$. We *cannot eliminate* the external interaction taking $\mathbf{E}(t)\equiv 0$. The reason for this is that we have neglected the Coulomb force and *the noise is needed* to justify this approximation (the random field $\mathbf{E}$ shakes the system and gives the energy necessary to maintain the charges far from each other).

At this point we can make an estimate of the Coulomb force $q_1q_2/r^2 \sim Q^2/r^2$ between the constituents. The harmonic force is of the order $m\omega_0^2\langle r^2\rangle^{1/2}$. Since

$\langle r^2 \rangle \sim \hbar/m\omega_0$, the inequality

$$\left[\frac{Q^2}{\langle r^2 \rangle}\right]^2 (m^2\omega_0^4\langle r^2 \rangle)^{-1} = \frac{Q^4}{m^2\omega_0^4\langle r^2 \rangle^3} \ll 1 \tag{2.3}$$

should be valid if the Coulomb interaction is less important than the harmonic force. The above expression can be rewritten as

$$\frac{Q^4}{\hbar^2 c^2} \ll \frac{\hbar\omega_0}{mc^2} \ll 1 \ , \tag{2.4}$$

where the last inequality was imposed based on the assumption that the motion is nonrelativistic. The above relation (2.4) must be satisfied by the parameters of our model in order that the approximations we used be valid. We shall return to this point later on.

The equations of motion (2.1) and (2.2) are more conveniently expressed in terms of the center of mass and relative coordinates, namely,

$$\begin{aligned} M\mathbf{R} &\equiv m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \ , \\ \mathbf{r} &\equiv \mathbf{r}_2 - \mathbf{r}_1 \ . \end{aligned} \tag{2.5}$$

In terms of $\mathbf{r}$ and $\mathbf{R}$ the equations of motion (2.1) and (2.2) take the form

$$\ddot{\mathbf{r}} = -\omega_0^2\mathbf{r} + \frac{\Gamma_1}{\omega_0^2}\dddot{\mathbf{r}} + \frac{\Gamma_2}{\omega_0^2}\dddot{\mathbf{R}} + \frac{\tilde{Q}}{M}\mathbf{E} \tag{2.6}$$

and

$$\ddot{\mathbf{R}} = \frac{\Gamma_3}{\omega_0^2}\dddot{\mathbf{R}} + \frac{\Gamma_4}{\omega_0^2}\dddot{\mathbf{r}} + \frac{Q}{M}\mathbf{E}(t) \ , \tag{2.7}$$

where

$$\begin{aligned} \Gamma_1 &\equiv \frac{2}{3}\frac{m}{c^3}\omega_0^2\left[\frac{q_1^2}{m_1^2} + \frac{q_2^2}{m_2^2}\right] \ , \\ \Gamma_2 &\equiv \frac{2}{3}\frac{\omega_0^2}{c^3}\left[\frac{q_2^2}{m_2} - \frac{q_1^2}{m_1}\right] \equiv \frac{M}{m}\Gamma_4 \ , \\ \Gamma_3 &\equiv \frac{2}{3}\frac{\omega_0^2}{Mc^3}(q_1^2+q_2^2) \ , \\ \tilde{Q} &\equiv M\left[\frac{q_2}{m_2} - \frac{q_1}{m_1}\right] \ . \end{aligned} \tag{2.8}$$

In (2.7) we see that the center-of-mass motion depends on the internal coordinate $\mathbf{r}$. This is not strange for us, since we know that composite systems have more complicated motion due to the effect of radiation reaction.[22]

It is easy to see from (2.4) that the constants Γ_i above are such that

$$\Gamma_i/\omega_0 \sim \frac{Q^2}{\hbar c}\frac{\hbar\omega_0}{Mc^2} \lesssim 10^{-3} \ll 1 \ . \tag{2.9}$$

This means that Eq. (2.6) and (2.7) decouple approximately if the motion is nonrelativistic. These equations, which are linear in $\mathbf{R}$ and $\mathbf{r}$, could be solved exactly in the case in which $\mathbf{E}(t)$ is given by (1.1). Here, however, we prefer to discuss an approximate solution using the fact that the ratios Γ_i/ω_0 are very small as well as the assumption that the harmonic force is dominant in (2.6). This means that

$$\ddot{\mathbf{r}} \simeq -\omega_0^2\mathbf{r}, \quad \dddot{\mathbf{r}} \simeq -\omega_0^2\dot{\mathbf{r}} \tag{2.10}$$

and Eqs. (2.6) and (2.7) can be decoupled. In this way the resulting equation for the relative coordinate $\mathbf{r}$ is

$$\begin{aligned} \ddot{\mathbf{r}} \simeq -\omega_0^2\mathbf{r}\left[1 + \frac{\Gamma_1\Gamma_3 - \Gamma_2\Gamma_4}{\omega_0^2}\right] - \Gamma_1\dot{\mathbf{r}} + \frac{\tilde{Q}}{M}\mathbf{E}(t) \\ + \left[\frac{\tilde{Q}\Gamma_3 - Q\Gamma_2}{M\omega_0^2}\right]\dot{\mathbf{E}}(t) \ . \end{aligned} \tag{2.11}$$

In the stationary (resonant) regime we have $\dot{\mathbf{E}} \sim \omega_0\mathbf{E}$ and, since $\Gamma_i/\omega_0 < 10^{-3}$, we can simplify the above equation to

$$\ddot{\mathbf{r}} \simeq -\omega_0^2\mathbf{r} - \Gamma_1\dot{\mathbf{r}} + \frac{\tilde{Q}}{M}\mathbf{E}(t) \ . \tag{2.12}$$

This equation governs the relative motion that is the most interesting for us. We shall ignore the center-of-mass motion in what follows.

The stationary solution of Eq. (2.12) is known from many other works[1,2,4] in SED, namely,

$$\mathbf{r} = \mathrm{Re}\sum_{\lambda=1}^{2}\frac{\tilde{Q}}{M}\int d^3k\frac{\hat{\boldsymbol{\varepsilon}}(\mathbf{k},\lambda)H(\omega,T)\exp[-i\omega t + i\nu(\mathbf{k},\lambda)]}{\omega_0^2 - \omega^2 + i\omega\Gamma_1} \ . \tag{2.13}$$

From the above result we can extract many properties of the spin $\mathbf{S}$ (orbital angular momentum) and magnetic dipole $\boldsymbol{\mu}$ associated to the composite particle. The first conclusion is that, as far as the internal or relative motion is concerned, the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ is such that

$$\begin{aligned} \langle \mathbf{p} \rangle &= \langle \mathbf{r} \rangle = \langle r_i p_j \rangle = 0 \ , \\ \langle \mathbf{p}^2 \rangle &= m^2\langle \dot{\mathbf{r}}^2 \rangle = m^2\omega_0^2\langle \mathbf{r}^2 \rangle \ . \end{aligned} \tag{2.14}$$

The probability distribution $W(\mathbf{r},\mathbf{p})$ in phase space can be obtained from (2.14) and the central limit theorem, and is given by[1,2,4]

$$W(\mathbf{r},\mathbf{p}) = \frac{\exp[-\frac{3}{2}(\mathbf{r}^2/\langle \mathbf{r}^2 \rangle + \mathbf{p}^2/\langle \mathbf{p}^2 \rangle)]}{[(2\pi/3)^2\langle \mathbf{r}^2 \rangle\langle \mathbf{p}^2 \rangle]^{3/2}} \ , \tag{2.15}$$

where we only need the expression for $\langle \mathbf{r}^2 \rangle$ in order to complete the calculation of $W(\mathbf{r},\mathbf{p})$. The mean-square value $\langle \mathbf{r}^2 \rangle$ of the relative coordinate can be obtained from (2.13) by taking the average over the random phases $\nu(\mathbf{k},\lambda)$. The results is well known, since similar calculation was done by many authors.[1,2,4] The only difference is the presence of the constants $\tilde{Q}$ and Γ_i, defined in (2.8), which implies that

$$\langle \mathbf{r}^2 \rangle = \frac{3}{2}\frac{\hbar}{m\omega_0}\frac{(1-\xi)^2}{(1+\xi^2)}\coth\left[\frac{\hbar\omega_0}{2kT}\right] \ . \tag{2.16}$$

The new factor depending on

$$\xi \equiv \frac{q_1 m_2}{q_2 m_1} \tag{2.17}$$

introduces new features, since the usual result (see Boyer[4]) is obtained by putting $\xi=0$, that is, $q_1=0$ or $m_1\to\infty$, which corresponds to the traditional analysis of the harmonic oscillator in SED.

The spin, or *intrinsic angular momentum* of the composite particle, that is, referred to a frame in which the center of mass is instantaneously at rest, is defined by

$$\mathbf{S}=\mathbf{r}\times\mathbf{p}=m\mathbf{r}\times\dot{\mathbf{r}}\ , \tag{2.18}$$

where m is the reduced mass as we said above. The variance associated to $\mathbf{S}$ is such that

$$\langle \mathbf{S}^2\rangle=\tfrac{2}{3}\langle \mathbf{r}^2\rangle\langle \mathbf{p}^2\rangle=\tfrac{2}{3}m^2\omega_0^2\langle \mathbf{r}^2\rangle^2\ , \tag{2.19}$$

since $W(\mathbf{r},\mathbf{p})$ is given by (2.15).

Taking into account (2.16) we get

$$\langle \mathbf{S}^2\rangle=\tfrac{3}{2}\hbar^2\frac{(1-\xi)^4}{(1+\xi^2)^2}\coth^2\left[\frac{\hbar\omega_0}{2kT}\right]\ . \tag{2.20}$$

Here we expect that $\hbar\omega_0\gg 1$ eV for any microscopic particle (like a nucleon for instance), since $\hbar\omega_0$ represents the energy which characterizes the internal motion of the constituents. This means that we can take $\coth(\hbar\omega_0/2kT)\simeq 1$ even for very high values of the temperature T because $\hbar\omega_0\gg kT$. Considering this approximation we obtain

$$\langle \mathbf{S}^2\rangle=\tfrac{3}{2}\hbar^2\frac{(1-\xi)^4}{(1+\xi^2)^2}\ . \tag{2.21}$$

This is a quite interesting result in our model. As have stated, there is in the literature[1,2,4] many analyses in which $m_1\gg m_2$ or $q_1=0$, that is, $\xi=0$, and consequently

$$\langle \mathbf{S}^2\rangle_{\xi=0}=\tfrac{3}{2}\hbar^2 \tag{2.22}$$

is the only possibility for $\langle \mathbf{S}^2\rangle$, as was discussed many times. Here it is seen that we have more flexibility in this model of a composite particle in which both constituents are charged and have finite masses. As the value of ξ varies we obtain different results for $\langle \mathbf{S}^2\rangle$, that is,

$$0<\langle \mathbf{S}^2\rangle\le 6\hbar^2 \quad\text{or}\quad 0<(\langle \mathbf{S}^2\rangle)^{1/2}<5\hbar/2\ . \tag{2.23}$$

It is interesting to remark that the quantum-mechanical version of an elementary particle with only two pointlike (spinless) constituents *cannot* generate half integer spins, due to the lack of one of the rotational degrees of freedom. Our model does not have this restriction and predicts a maximum (average) spin of less than $5\hbar/2$. The variation of $\langle \mathbf{S}^2\rangle$ with ξ can be appreciated in Fig. 1. We have to exclude the neighborhood of $\xi=1$, because in this case $\langle \mathbf{r}^2\rangle=0$ and so the Coulomb interaction becomes important.

The fact that our analysis does not apply to the case $\xi=q_1m_2/q_2m_1=1$ is easily understood because, if $q_1/m_1=q_2/m_2$, both particles have the same random acceleration $(q_1/m_1)\mathbf{E}$. This means that the attractive harmonic force will bring the constituents more and more close to each other. This will cause the breakdown of this study, since we have neglected the Coulomb interaction

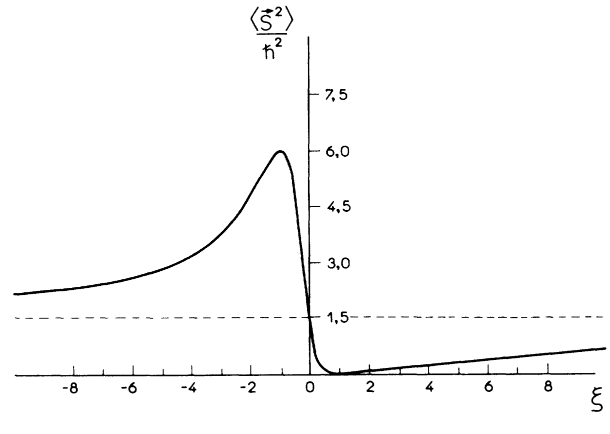

FIG. 1. $\langle \mathbf{S}^2\rangle$ as a function of the variable $\xi=q_1m_2/q_2m_1$.

between the charges.

Now we pass to discuss the properties of the intrinsic magnetic moment $\boldsymbol{\mu}$ of the system. From the definition we have

$$\boldsymbol{\mu}=\frac{1}{2c}(q_1\mathbf{r}_1\times\dot{\mathbf{r}}_1+q_2\mathbf{r}_2\times\dot{\mathbf{r}}_2)\ . \tag{2.24}$$

Since $\boldsymbol{\mu}$ is referred to the center of mass, we must have $\mathbf{r}_1=-(m_2/M)\mathbf{r}$ and $\mathbf{r}_2=+(m_1/M)\mathbf{r}$. With this it is easy to show that

$$\boldsymbol{\mu}=g\frac{Q}{2Mc}\mathbf{S}\ , \tag{2.25}$$

where g is called gyromagnetic factor because Q is the total charge, M is the total mass, and $\mathbf{S}$ is the intrinsic orbital angular momentum or spin. The expression for g is

$$g\equiv\frac{g_1m_2}{Qm_1}+\frac{g_2m_1}{Qm_2} \tag{2.26}$$

and was derived here within the context of purely classical mechanics and electromagnetism, without the necessity of introducing random fields. From expression (2.26) it is easy to see that g can take any value, depending on which values we attribute to m_1, m_2, q_1, and q_2, even with constraints $m_1+m_2=M$ and $q_1+q_2=Q$.

The results (2.21) for $\langle \mathbf{S}^2\rangle$ and (2.25) for $\boldsymbol{\mu}$ are the main achievements of this section. Here we have shown that under some circumstances (for instance, two charged particles, radiating independently, under the action of a harmonic force and the random radiation of SED) a simple system can have intrinsic angular momentum such that $\langle \mathbf{S}^2\rangle\sim\hbar^2$. Another finding, which is in fact a very early result[24] of nonrelativistic classical physics, was that the relation $\boldsymbol{\mu}=g(Q/2Mc)\mathbf{S}$ is valid for the magnetic dipole $\boldsymbol{\mu}$ and the gyromagnetic factor g can take small as well as very large values, negative or positive.

III. MOTION OF A RIGID MAGNETIC DIPOLE

In this section we shall consider the second simple system we mentioned in Sec. I, that is, a classical spinning magnetic dipole $\boldsymbol{\mu}$ in an external (constant) magnetic field

$\mathbf{B}_0$. This system was treated in 1979 by Sachidanandam[25] and with more details by Boyer[16] in 1984. Therefore we give here only a sketch of Boyer's calculation, that is, the starting point, the approximations, and the final result for $\langle \mu \rangle = (ge/2mc)\langle S_z \rangle$, where z is the direction of $\mathbf{B}_0$. Here e is the charge of a particle (mass m) which has spin $\mathbf{S}$ and gyromagnetic factor g. We are assuming that $\boldsymbol{\mu} = g(e/2mc)\mathbf{S}$, but we also consider the additional hypothesis (as Boyer) of rigidity, that is, $|\mathbf{S}|$ is constant.[16]

However, the direction of the vector $\mathbf{S}$ can change due to the external torque applied by $\mathbf{B}_0$ and $\mathbf{B}(0,t)$, where $\mathbf{B}(\mathbf{r},t)$ is the random magnetic field of SED. A convenient expression for $\mathbf{B}(\mathbf{r},t)$ is[16]

$$\mathbf{B}(\mathbf{r},t) = \sum_{\lambda=1}^{2} \int d^3K\, \hat{\varepsilon}(\mathbf{K},\lambda) H(\omega,t) \times \cos[\mathbf{K}\cdot\mathbf{r} - \omega t + \nu(\mathbf{K},\lambda)] \; . \tag{3.1}$$

Here $\hat{\varepsilon}(\mathbf{K},\lambda)$ are the polarization vectors defined in the Introduction and $\nu(\mathbf{K},\lambda)$ are, as before, random phases. The function $H(\omega,T)$ is also the same, that is,

$$\pi^2 H^2(\omega,T) = \tfrac{1}{2}\hbar\omega \coth(\hbar\omega/2kT) \; . \tag{3.2}$$

Since the magnetic dipole emits radiation, the equation of motion for $\mathbf{S}$ must contain also the self-torque due to radiation reaction. Then, according to Boyer,[16] the equation of motion for our spinning magnetic dipole is that given by Bhabha[26]

$$\dot{\mathbf{S}} = \boldsymbol{\mu} \times [\mathbf{B}_0 + \mathbf{B}(0,t)] - \tfrac{2}{3}\frac{\boldsymbol{\mu}\times\ddot{\boldsymbol{\mu}}}{c^3} \; , \tag{3.3}$$

where the last term is the radiation reaction torque (which can be obtained from energy conservation). This term will be considered small, as well as $\boldsymbol{\mu}\times\mathbf{B}(t)$, which is the random torque, when compared with $\boldsymbol{\mu}\times\mathbf{B}_0$.

If the equation of motion were simply

$$\dot{\mathbf{S}} = \boldsymbol{\mu}\times\mathbf{B}_0 \; , \tag{3.4}$$

the solution would be a precession, with frequency

$$\eta = \mu B_0 / S \tag{3.5}$$

around the z axis. The angle θ that the spin $\mathbf{S}$ makes with the z axis (direction of $\mathbf{B}_0$) should be constant in this approximation. However, the complete equation of motion (3.3) is a nonlinear-stochastic differential equation. The fluctuating torque, contained in (3.3), must be considered and the effect is that we expect to find a probability distribution $P(\theta)$ for the spin orientation.

Boyer[16] was able to obtain a Fokker-Planck equation for $P(\theta)$ by using a perturbative quasi-Markovian approximation of the equation of motion (3.3). We address the reader to Boyer's 1984 paper, which contains all the details of the calculations required to obtain such an equation.

In the stationary regime the Fokker-Planck equation obtained was

$$-P(\theta)\frac{\langle\Delta\theta\rangle}{\tau} + \frac{1}{2}\frac{\partial}{\partial\theta}\left[P(\theta)\frac{\langle(\Delta\theta)^2\rangle}{\tau}\right] = 0 \; , \tag{3.6}$$

where $\langle\Delta\theta\rangle$ is the average first moment and $\langle(\Delta\theta)^2\rangle$ is the average second moment, for the change $\Delta\theta$ in θ, due to the perturbating torques during a small time τ. At this point we must say a few words about what we mean by small τ.

Boyer's approximations[16] are valid if

$$\frac{1}{\eta} \ll \tau \ll 1 \text{ sec} \; , \tag{3.7}$$

where $2\pi/\eta$ is the period of the unperturbed precession. Since $\eta = eB_0/mc$ we have $1/\eta \simeq 10^{-11}$ sec if $B_0 \sim 10^4$ G (a magnetic field attainable in the laboratories) and m is the electron mass.

According to Boyer[16] we can use the following expression for $\langle\Delta\theta\rangle/\tau$:

$$\frac{\langle\Delta\theta\rangle}{\tau} = -\tfrac{2}{3}\eta^3\frac{\sin\theta}{S^4} + \frac{2\pi^2\mu^2}{3c^3S^2}\cot(\theta)\eta^2H^2(\eta,T) \; , \tag{3.8}$$

where the first term is due to the contribution of radiation reaction torque and the second is due to the fluctuating torque. Consistently with this, Boyer[16] obtained for $\langle(\Delta\theta)^2\rangle$ the result

$$\frac{\langle(\Delta\theta)^2\rangle}{\tau} = \frac{4}{3}\frac{\pi^2}{c^3}\left[\frac{\mu}{S}\right]^2\eta^2H^2(\eta,T) \; , \tag{3.9}$$

where $H^2(\eta,T)$ is given by (3.2). Thus the Fokker-Planck equation becomes

$$\frac{dP(\theta)}{d\theta} + \left[\frac{S^2\eta^2}{\mu B_0\pi^2H^2(\eta,T)}\sin\theta - \cot\theta\right]P(\theta) = 0 \tag{3.10}$$

in the stationary regimen.

The exact solution of the above equation gives for the probability distribution,

$$P(\theta) = \text{const}\,\sin\theta\,\exp\left[\frac{S\eta}{\pi^2H^2(\eta,T)}\cos\theta\right] . \tag{3.11}$$

In the limit $\hbar\to 0$ (or $\hbar\eta \ll kT$) we have $\pi^2H^2(\eta,T)\simeq kT$ and since $\eta=\mu B_0/S$ we get

$$\frac{S\eta}{\pi^2H^2(\eta,T)} = \frac{\mu B_0}{kT}$$

and $P(\theta)$ becomes the Boltzmann distribution

$$P_B(\theta) = \text{const}\,\sin\theta\,\exp\left[\frac{\mu B_0}{kT}\right] , \tag{3.12}$$

as is expected[16] to be valid in the high-temperature limit.

The probability distribution $P(\theta)$ can be used to calculate the average component of the magnetic dipole $\boldsymbol{\mu}$ along the z direction of the magnetic field $\mathbf{B}_0$, that is,

$$\langle\mu_z\rangle = \frac{ge}{2mc}\langle S_z\rangle = \frac{geS}{2mc}\langle\cos\theta\rangle \; . \tag{3.13}$$

Using (3.11), (3.5), and (3.2) we get

$$\frac{2mc}{geS}\langle \mu_z \rangle = \coth\left[\frac{2S}{\hbar\coth(\hbar\mu B_0/2SkT)}\right] - \frac{\hbar}{2S}\coth\left[\frac{\hbar\mu B_0}{2SkT}\right], \quad (3.14)$$

which is Boyer's 1984 final result.[16]

Now, instead of discussing only the limits of high and low temperatures, as was done before by Boyer, we are going to make a detailed comparison between $\langle \mu_z \rangle$ and experimental data measured in a large range of B_0/T. In order to do this we introduce two additional hypotheses. Firstly, we assume that $g=2$, which is the experimental value of the electron gyromagnetic factor (we also take m as being the electron mass). Secondly, we introduce the hypothesis that $S=N\hbar$, where N is an unknown number (this is the only free parameter in what follows). These hypotheses deserve some comment. To assume that $g=2$ is not a quantum hypotheses because we have shown in Sec. II that g can take *any value* in classical electromagnetism. Also, to assume that $S=N\hbar$ does not embarass us because we have seen, at least in the simple example of Sec. II, that $S\sim\hbar$ is possible within the realm of SED. The only thing we have not discussed is the rigidity assumption (we left this point for a future project). We only want to mention that a rigid magnetic dipole is not a quantum concept in our opinion.

Returning to the expression (3.14) for $\langle \mu_z \rangle$, introducing $S=N\hbar$ and $g=2$, we obtain

$$\frac{\langle \mu_z \rangle}{g\mu_0} = N\coth\left[\frac{2N}{\coth(\mu_0 B_0/kT)}\right] - \tfrac{1}{2}\coth\left[\frac{\mu_0 B_0}{kT}\right], \quad (3.15)$$

where $\mu_0 = e\hbar/2mc$ is the bohr magneton.

The comparison of this expression with the experimental data is shown in Fig. 2. The only free parameter is $N=S/\hbar$, which is adjusted to fit the experimental data measured for the paramagnetic ions Cr^{3+}, Fe^{3+}, and Gd^{3+}, respectively. In our model N is a parameter which should depend on the detailed structure of the paramagnetic particle (or atom). We have no means to calculate N in the Boyer simple model of a rigid magnetic dipole. However, it is quite surprising to see in Fig. 2 the impressive agreement between the experimental data and Boyer's theory modified with our additional hypothesis, namely, $S=N\hbar$. The surprise is that within Boyer's calculation the angle θ is a *continuous* variable. There is no quantization of S_z (which takes discrete values in quantum mechanics). According to QM,

$$\langle \mathbf{S}^2 \rangle = \hbar^2 J(J+1) \quad (3.16)$$

and S_z assume *discrete* values within $-J\hbar \le S_z \le J\hbar$. The magnetic dipole is $\boldsymbol{\mu} = (eg/2mc)\mathbf{S}$ as in our calculation based on SED. The quantum theory of Brillouin[27] is more successful for explaining the data presented in Fig. 2. The reason is that there is a quantum theory for the internal structure of the ion and therefore no free parameters.

It is interesting to compare the numerical predictions

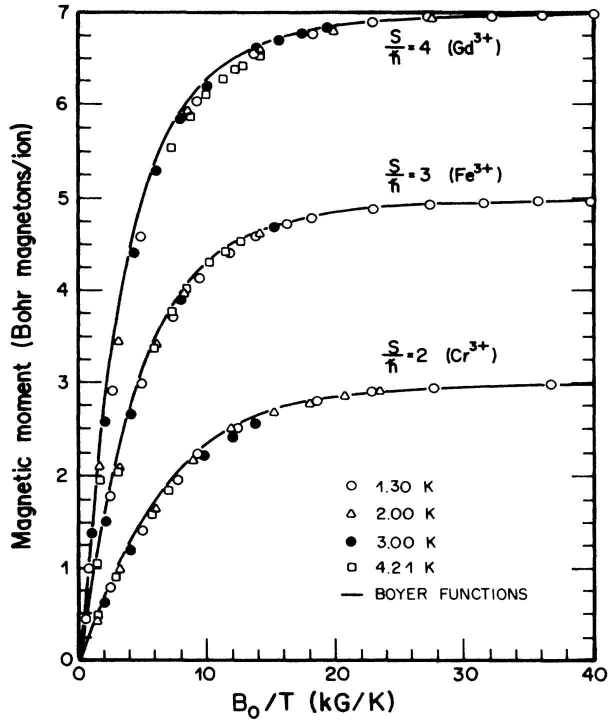

FIG. 2. Magnetic dipole variation as a function of B_0/T (after Henry, Ref. 17).

of SED and QM. In QM the formula which corresponds to (3.15) is[27]

$$\frac{\langle \mu_z \rangle}{g\mu_0} = (J+\tfrac{1}{2})\coth\left[2(J+\tfrac{1}{2})\frac{\mu_0 B_0}{kT}\right] - \tfrac{1}{2}\coth\left[\frac{\mu_0 B_0}{kT}\right] \quad (3.17)$$

which is the well-known Brillouin function.

For the paramagnetic ions of Fig. 2 we have $g=2$ and $J=\frac{3}{2}$ for Cr^{3+}, $J=\frac{5}{2}$ for Fe^{3+}, and $J=\frac{7}{2}$ for Gd^{3+}. These correspond to $(\langle \mathbf{S}^2 \rangle/\hbar^2)^{1/2} = 1.94$, 2.95, and 3.97, respectively[17,27] In the rigid spin model of SED the parameter $N=S/\hbar$ is to be identified with $(\langle \mathbf{S}^2 \rangle/\hbar^2)^{1/2}$ of QM. It was not a surprise that, in fitting the data by adjusting the free parameter N, we have found $N=2$, 3, and 4 for the paramagnetic ions Cr^{3+}, Fe^{3+}, and Gd^{3+}, respectively, as we can see from Fig. 2.

In order to understand this more easily let us assume that $\mu_0 B_0/kT$ is small enough so that $\tanh(\mu_0 B_0/kT) \simeq \mu_0 B_0/kT$. Only an elementary calculation is required to show that Eqs. (3.15) and (3.17) take the same form, namely,

$$\frac{\langle \mu_z \rangle}{g\mu_0} = C\coth\left[\frac{2C\mu_0 B_0}{kT}\right] - \tfrac{1}{2}\coth\left[\frac{\mu_0 B_0}{kT}\right], \quad (3.18)$$

where $C=J+\frac{1}{2}$ for the quantum theory and $C=N$ for SED.

Even for not so small $\mu_0 B_0/kT$ the agreement between

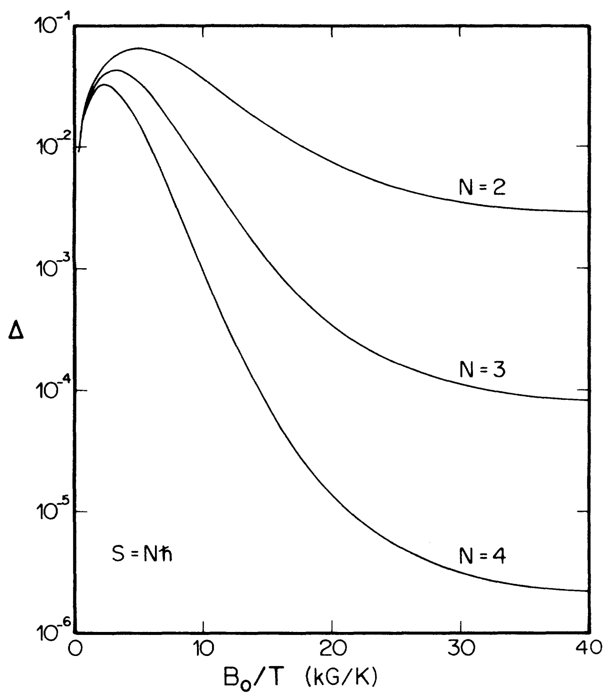

FIG. 3. Variation of $\mu_0\Delta \equiv \langle \mu_z \rangle_{\text{Boyer}} - \langle \mu_z \rangle_{\text{Brillouin}}$ as a function of B_0/T (see text).

SED, QM, and the experimental data is so great for the cases presented above that is almost impossible to distinguish the Boyer[16] predictions given by (3.15) from the Brillouin[17,27] curves given by (3.17). For this reason we decided to make another numerical comparison of the two theories. In order to do this we defined an adimensional number which is the difference between the Brillouin and the Boyer predictions for $\langle \mu_z \rangle / \mu_0$. Namely,

$$\mu_0\Delta(y) \equiv \langle \mu_z \rangle_{\text{Boyer}} - \langle \mu_z \rangle_{\text{Brillouin}} , \tag{3.19}$$

where $y = \mu_0 B_0 / kT$.

This is presented in Fig. 3 for the values $N=2$ $(J=\frac{3}{2})$, $N=3$ $(J=\frac{5}{2})$, and $N=4$ $(J=\frac{7}{2})$. We see that $\Delta(y)$ is small and becomes smaller for increasing values of S, as was predicted by Boyer[16] in 1984.

IV. SUMMARY OF CONCLUSIONS

As many works based on the ideas of SED, this paper represents another small step towards the understanding of microscopic phenomena within classical electron theory including classical electromagnetic zero-point radiation. This classical theory (SED) has produced an increasing number of results[1–4] usually thought to be obtainable only within quantum mechanics.

In the first part of the present work we have shown that the magnetic dipole $\boldsymbol{\mu}$ and the intrinsic orbital angular momentum (or spin) $\mathbf{S}$ are related by $\boldsymbol{\mu} = gQ\mathbf{S}/2Mc$. This relation is observed experimentally in many elementary particles such as the electron and the nucleons. The above results were obtained by assuming that the particle is composed of two constituents in nonrelativistic motion. Due to the calculational difficulties of classical electromagnetism and of stochastic processes, we have considered a very simple model in which the constituents are maintained together by a harmonic force that is dominant over the other ones acting on the charges. Despite the simplicity of the model, we were able to show that the gyromagnetic factor g can take any value. This is a general result of classical physics which is independent of the presence of the zero-point random electromagnetic radiation. These stochastic electromagnetic fields are, however, responsible for the result $\langle \mathbf{S}^2 \rangle \sim \hbar^2$ we have obtained for the spin in our simple composite particle model. Another characteristic of this model is that $\boldsymbol{\mu}$ and $\mathbf{S}$ are not rigid, but fluctuating quantities. Therefore we decided to study another example.

We considered a simple model of a rigid classical spinning magnetic dipole proposed by Sachidanandam[25] in 1979 and discussed with great detail by Boyer[16] in 1984. When the magnetic dipole is in an external magnetic field $\mathbf{B}_0$, the spin precesses and the system loses energy by emitting radiation. However, due to the presence of the random (zero-point and thermal) electromagnetic fields, the system absorbs energy and there is no complete alignment with the magnetic field $\mathbf{B}_0$. By using an entirely classical treatment, that is, the angle θ between $\mathbf{S}$ and $\mathbf{B}_0$ being a *continuous variable*, Boyer[16] was able to calculate the probability distribution $P(\theta)$ for any orientation of $\boldsymbol{\mu}$ or $\mathbf{S}$, with respect to the z (or $\mathbf{B}_0$) direction. The main points of the calculation were sketched in Sec. III.

With the additional assumptions, namely, taking $g=2$ and $S=|\mathbf{S}|=N\hbar$, we introduce the only free parameter N into the Sachidanandam-Boyer model.[16] In this way we were able to use the explicit analytical expressions for $\langle \mu_z \rangle = (eg/2mc)\langle S_z \rangle$ obtained by Boyer in 1984.

The comparison of the theoretical results with experimental data was shown in Sec. III for a wide range of the variable B_0/T, that is, mainly for *high* B_0 and *low* temperature T. The agreement with the experiments was quite impressive. SED predictions were almost indistinguishable from the QM ones and quite different from the usual (without zero-point radiation) Langevin theory.[17,27]

However, we must recognize that the QM theory of the observed paramagnetism is superior when compared with the SED explanation presented here. As we explained in Sec. III we have introduced the free parameter $N=S/\hbar$ into SED calculations. This was unavoidable in the actual development of SED because, due to mathematical (or perhaps other more profound) difficulties,[1,2] we do not have a detailed stochastic explanation to the internal structure of atoms and elementary particles. Nevertheless, some progress was obtained by SED as far as the paramagnetic properties of matter are concerned. We hope that this should encourage other researchers in this direction. These additional efforts are important to physics because the concept of spin is both intriguing[28] and extremely difficult (according to QM the angular momentum is related to the *classical* concept of rotation and to the *space quantization*).

The first person to propose the possibility of the spin seemed to be Compton[29] in 1921. He considered the electron as an extended charge spinning around an axis. The goal of Compton was to explain the departure of x-ray scattering data from Thomson's predictions, using only classical electromagnetism.[30] Many people believe that the last word was said by Dirac,[31] in 1928, with the introduction of his efficient relativistic wave equation for the electron. According to Dirac, the electron is structureless but has spin, that is, "internal" quantum numbers. To make the situation more puzzling,[32] nowadays physicists have discovered the muon and tau leptons. All these particles are considered *pointlike,* have the same charge, the same spin, and the same main interaction (electromagnetic) but have *quite different masses.* We do not believe that we have heard the final word about the spin and magnetic dipole distributions[32–34] of elementary particles

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