Relation between Lagrangian and Eulerian spectra based upon a canonical statistical theory of geophysical fluid waves

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(Received 5 August 1987; revised manuscript received 23 January 1989)

A formulation of fluid dynamics in terms of Lagrangian variables allows one to make direct use of the standard methods of statistical mechanics. However, most observations and empirical studies of large geophysical fluid systems are in terms of Eulerian variables, and this raises the issue of how to relate quantities given in terms of one set of variables to the corresponding quantities given in terms of the other set. In a completely general treatment of fluids, one must consider both oscillatory and translational modes of motion. The oscillatory modes are wavelike and lead to correlations which are analogous to those for phonons in the solid. The translational modes are particlelike and lead to correlations which correspond to diffusion in a weakly interacting gas. In this paper we consider the class of fluids for which the translational modes can be neglected. Based upon the assumption that the statistical distribution of the canonically conjugate Lagrangian variables is of a Gaussian form, we obtain a tractable expression for the Eulerian spectra in terms of the Lagrangian spectra and show that, in general, the two types of spectra are significantly different. In particular, it is shown that the Eulerian wave-number spectrum exhibits a large wave-number power-law decay which is similar to that often observed in geophysical systems and further is independent of the detailed nature of the Lagrangian wave-number spectrum. The large wave-number decay of the Eulerian spectrum is due to advection and is strictly a kinematic effect. This also implies that experiments which focus on the large wave-number advective tail cannot yield information about the true dynamics of the system. The application of this result to the problem of explaining the observed distribution of oceanic internal waves is discussed.

I. INTRODUCTION

Most observations and empirical studies of large stochastic fluid systems are in terms of Eulerian variables. This is especially true of geophysical systems such as the earth's atmosphere and its oceans. For example, the empirical oceanic internal wave model of Garrett and Munk[1] (GM) is in terms of Eulerian variables. On the other hand, the methods of statistical mechanics which might be useful for understanding and interpreting these studies are usually formulated in terms of Lagrangian variables. This raises the issue of how to relate statistical quantities, such as spectra, which are given in terms of Lagrangian variables to the corresponding quantities given in terms of Eulerian variables. The difficulty is that, in general, the transformation between the two sets of variables is not tractable. The problem of relating Lagrangian and Eulerian aspects of fluid flow is an old one, going back at least to Taylor.[2] For ocean surface waves, a recent physical discussion is given by Longuet-Higgins.[2] In this paper it will be shown that there exists a class of systems for which the problem of implementing the transformation can be avoided, and still a tractable relation between Lagrangian and Eulerian spectra can be obtained. We will find that the kinematic distortion caused by the transformation results in Eulerian spectra which can be significantly different from the corresponding Lagrangian spectra.

In order to clarify this point we need first to discuss the important differences between Lagrangian and Eulerian variables. In a Lagrangian formulation the fluid is divided into a number of microscopically large but macroscopically small parcels which are identified by the various values of a three-dimensional parameter which we shall denote by χ. While it is not necessary, it is usually the case that χ corresponds to the position of the parcel under some reference condition, often taken to be the undisturbed condition. Once selected, a specific value for χ remains with the fluid parcel and does not change throughout the dynamic evolution of the system. We shall denote the Lagrangian displacements and velocities at the time t by $\eta_L(\chi,t)$ and $\mathbf{v}_L(\chi,t)$, respectively. The Eulerian displacements and velocities will be denoted by $\eta_E(\mathbf{x},t)$ and $\mathbf{v}_E(\mathbf{x},t)$, respectively, where a given value for the Eulerian label $\mathbf{x}$ corresponds to a specific point in space and it refers to the fluid parcel which happens to be at that point at the time t. Thus a given value for the Eulerian label $\mathbf{x}$ does not always refer to the same fluid parcel.

A formulation in terms of Lagrangian variables retains a faithful correspondence with the particles of Newtonian mechanics. Such a formulation can be given in terms of a Hamiltonian and Hamiltonian's canonical equations, or the Lagrangian variables can be used directly in

Hamilton's principle to obtain the well-known variational form of continuum mechanics. In the case of Eulerian variables no such straightforward Hamiltonian formulation is possible. While the Eulerian equations of motion have been cast into a variational form,[2] which is sometimes referred to as canonical, such a formulation requires the introduction of additional variables and constraints and is significantly different from the usual well-known Hamiltonian dynamics (for a discussion of this point see Seliger and Whitham[2]).

A. Lagrangian variables

We shall first discuss, in some detail, the Lagrangian formulation and then briefly contrast this to the Eulerian formulation. In our canonical formulation the Cartesian components of the Lagrangian displacement $\eta_{L\alpha}(\chi,t)$ and velocity $v_{L\alpha}(\chi,t)$, where $1\leq\alpha\leq 3$, are expressed in terms of the modal expansions

$$\eta_{L\alpha}(\chi,t)=\sum_{j=1}^{N}\phi_{j\alpha}(\chi)q_j(t)+\sum_{j=1}^{\bar{N}}\bar{\phi}_{j\alpha}(\chi)\bar{q}_j(t) \tag{1a}$$

and

$$v_{L\alpha}(\chi,t)=\sum_{j=1}^{N}\phi_{j\alpha}(\chi)\omega_j p_j(t)+\sum_{j=1}^{\bar{N}}\bar{\phi}_{j\alpha}(\chi)\frac{1}{m_j}\bar{p}_j(t)\ , \tag{1b}$$

where the $q_j(t)$ and $\bar{q}_j(t)$ are real independent generalized displacements, the $p_j(t)$ and $\bar{p}_j(t)$ are the corresponding canonically conjugate momenta, $N+\bar{N}$ is the number of degrees of freedom which later will be allowed to become arbitrarily large, and ω_j and m_j are constants the significance of which will become clear later. In (1) the $\phi_{j\alpha}(\chi)$ and $\bar{\phi}_{j\alpha}(\chi)$ are real eigenfunctions associated with writing the Hamiltonian in the form

$$\begin{aligned}H(p,q)&=\sum_{j=1}^{N}H_{0j}(p,q)+\sum_{j=1}^{\bar{N}}\bar{H}_{0j}(\bar{p},\bar{q})+\lambda V_I(p,\bar{p},q,\bar{q})\\&=\sum_{j=1}^{N}\frac{\omega_j}{2}[p_j^2(t)+q_j^2(t)]+\sum_{j=1}^{\bar{N}}\frac{1}{2m_j}\bar{p}_j^{\,2}(t)\\&\quad+\lambda V_I(p,\bar{p},q,\bar{q})\ ,\end{aligned} \tag{2}$$

where $(p,\bar{p},q,\bar{q})$ denotes the set $\{\ldots,p_j,\bar{p}_j,q_j,\bar{q}_j,\ldots\}$ of canonically conjugate dynamical variables. In (2) the terms $H_{0j}(p,q)$ and $\bar{H}_{0j}(\bar{p},\bar{q})$ are quadratic in the dynamical variables and lead to separated linear equations of motion. The term $\lambda V_I(p,\bar{p},q,\bar{q})$ is of cubic and higher order in the dynamical variables and leads to nonlinear interactions between the otherwise independent linear modes. We shall refer to $\lambda V_I(p,\bar{p},q,\bar{q})$ as the interaction potential. The expression for the Hamiltonian given by (2) describes the system in terms of deviations from some state of static equilibrium which can be characterized by the static density profile $\rho(z)$. The specific choice for $\rho(z)$ will determine the dispersion relation ω_j, the eigenfunctions $\phi_{j\alpha}(\chi)$ and $\bar{\phi}_{j\alpha}(\chi)$, as well as the nonlinear coupling coefficients in $\lambda V_I(p,\bar{p},q,\bar{q})$, but the form of (2) is general.

The Lagrangian equations of motion are obtained by using the Hamiltonian given by (2) and Hamilton's canonical equations. In a general case they are nonlinear and cannot be solved exactly. The linear approximation is obtained by setting $\lambda V_I(p,\bar{p},q,\bar{q})$ equal to zero in which case it is easy to show that

$$p_j(t)=p_j(0)\cos(\omega_j t)-q_j(0)\sin(\omega_j t)\ , \tag{3a}$$

$$q_j(t)=q_j(0)\cos(\omega_j t)+p_j(0)\sin(\omega_j t)\ , \tag{3b}$$

$$\bar{p}_j(t)=\bar{p}_j(0)\ , \tag{3c}$$

and

$$\bar{q}_j(t)=\frac{1}{m_j}\bar{p}_j(0)t+\bar{q}_j(0)\ , \tag{3d}$$

where the $p_j(0)$, $\bar{p}_j(0)$, $q_j(0)$, and $\bar{q}_j(0)$ are initial values. From the form of (3) it is clear that in the linear approximation the terms which involve the $p_j(t)$ and $q_j(t)$ are oscillatory with frequency ω_j and correspond to waves. The modes which involve the $\bar{p}_j(t)$ and $\bar{q}_j(t)$ are translational and, in the case of a vertically stratified fluid, correspond to horizontal flows. Later we will neglect the translational modes, but we retain them for now because there are some potentially important issues concerning them yet to be discussed. For large many-body systems the precise specification of the initial values is not possible and we must resort to the use of statistical methods. In order to deal with the nonlinear interactions it is necessary to use perturbation or other approximation techniques. We shall refer to the nonlinear interactions which arise from $\lambda V_I(p,\bar{p},q,\bar{q})$ as dynamic nonlinearities and even though they can sometimes be treated as weak, they play an important role in the time evolution of the relevant statistical quantities.

In a perturbation treatment the initial values are replaced by amplitudes which exhibit a slow time dependence due to the nonlinear interactions. In this paper we shall be concerned primarily with the weak-interaction approximation. In this approximation the nonlinear interactions cause a transfer of energy between the linear modes but are not large enough to significantly alter the fundamental character of the linear modes. The statistical treatment of the problem is concerned with describing the statistical distribution of these amplitudes and usually proceeds via a study of correlations. For example, the development given by Prigogine[3] introduces the phase-space density function $g(p,\bar{p},q,\bar{q})$ which satisfies the Liouville equation and uses perturbation methods to investigate its time evolution. By making the weak-interaction approximation and the random-phase assumption as an initial condition Prigogine obtains a master equation and shows that its long-time solution is of the Gaussian form given by

$$g(p,\bar{p},q,\bar{q})=\frac{1}{Z}\exp\left[-\sum_{j=1}^{N}\frac{H_{0j}(p,q)}{\Lambda_j}-\sum_{j=1}^{\bar{N}}\frac{\bar{H}_{0j}(\bar{p},\bar{q})}{\bar{\Lambda}_j}\right], \tag{4}$$

where Z is the partition function and Λ_j and $\bar{\Lambda}_j$ are the average energies of the corresponding modes. In the case of canonical equilibrium, which is the case actually discussed by Prigogine $\Lambda_j=\bar{\Lambda}_j=E_0$, where E_0 is a constant. The slightly more general form given by (4) will be useful for later discussion.

The wavelike modes are analogous to phonons in the solid and by using methods such as those developed by Prigogine[3] and the weak-interaction approximation it can be shown that, in the limit of large reference time t, the displacement $C_{qjk}(t,\tau)$ and momentum $C_{pjk}(t,\tau)$ correlation functions are given by

$$C_{qjk}(t,\tau)=E[q_j(t+\tau)q_k(t)]$$
$$=C_{pjk}(t,\tau)=E[p_j(t+\tau)p_k(t)]$$
$$=\delta_{jk}\frac{\Lambda_j}{\omega_j}\cos(\omega_j\tau)\exp(-\lambda\Gamma_j|\tau|)\ , \tag{5a}$$

where $E[\cdot]$ denotes the expectation value of the bracketed quantity and $\lambda\Gamma_j$ is the single mode relaxation rate. Thus, in the case of the wavelike modes the correlation functions are stationary (i.e., independent of the reference time t). The spectra associated with (5a) are given by

$$S_{qjk}(\omega)=\int_{-\infty}^{\infty}d\tau\,C_{qjk}(\tau)\exp(i\omega\tau)=S_{pjk}(\omega)=\int_{-\infty}^{\infty}d\tau\,C_{pjk}(\tau)\exp(i\omega\tau)$$
$$=\delta_{jk}\frac{\Lambda_j}{\omega_j}\left[\frac{\lambda\Gamma_j}{[(\omega+\omega_j)^2+(\lambda\Gamma_j)^2]}+\frac{\lambda\Gamma_j}{[(\omega-\omega_j)^2+(\lambda\Gamma_j)^2]}\right], \tag{5b}$$

where we have suppressed display of the reference time t. In obtaining (5a) the nonlinear interactions have contributed in two ways. First, they have been important in establishing the phase-space density function given by (4). Second, they play a role in the time evolution from t to $t+\tau$ and are responsible for the exponentially decaying factor in (5a). If the relaxation rate $\lambda\Gamma_j$ is small in comparison to the frequency ω_j, then in (5b) we can make the replacement

$$\frac{\lambda\Gamma_j}{[(\omega\pm\omega_j)^2+(\lambda\Gamma_j)^2]}\Longrightarrow\pi\delta(\omega\pm\omega_j) \tag{5c}$$

in which case the system is confined to the dispersion surface described by ω_j. This approximation is equivalent to using the linear expressions given by (3a) and (3b) to describe the time evolution for the relatively short times τ, and is entirely adequate for our purposes in this paper.

While the oscillatory modes produce bounded motions, the translational modes produce unbounded motions. If these modes are occupied, then two initially nearby fluid parcels may diverge from each other and, given enough time, be arbitrarily far apart. There are two ways in which the translational modes can give rise to this behavior. The first way is direct and occurs when there are significant mean currents present. If there exists a mean flow which exhibits velocity variations in a direction perpendicular to the flow (i.e., mean shear), then fluid parcels which were once near neighbors may have different mean velocities and will therefore separate with increasing time. In fluids with substantial mean currents one might expect some difficulties if the translational modes are neglected.

The second way in which the translational modes contribute to these effects is indirect but more ubiquitous. Even if there are no mean currents, nonlinear interactions randomly transfer energy into, out of, and between the various translational modes in a way which is analogous to that which, for the oscillatory modes, lead to (5). The statistical treatment of the translational modes requires consideration of some subtle issues which were not of concern in the treatment of the bounded oscillatory modes. By using well-known methods (see, e.g., Uhlenbeck and Ornstein; Wang and Uhlenbeck[3]) it can be shown that, at large reference times t, the momentum correlation functions are stationary and given by

$$C_{\bar{p}jk}(t,\tau)=E[\bar{p}_j(t+\tau)\bar{p}_k(t)]$$
$$=\delta_{jk}m_j\bar{\Lambda}_j\exp(-\lambda\bar{\Gamma}_j|\tau|)\ . \tag{6a}$$

The corresponding spectra are given by

$$S_{\bar{p}jk}(\omega)=\delta_{jk}m_j\bar{\Lambda}_j\frac{2\lambda\bar{\Gamma}_j}{[\omega^2+(\lambda\bar{\Gamma}_j)^2]}\ . \tag{6b}$$

We note that for the translational modes there are no natural frequencies ω_j and thus the system is not oscillatory. It is important to realize that the $\bar{p}_j$ are normal mode momenta and not the momenta associated with individual fluid parcels. Fox[3] has shown that spectra of the form given by (6b) can, depending upon the nature of $\lambda\bar{\Gamma}_j$, lead to long-time tails in the fluid parcel velocity correlation functions [see Eqs. (I.7.65) and (I.7.69) of Fox[3]]. Thus, the exponentially decaying form of (6a) does not preclude the possibility of long-time tails which are known to occur in hydrodynamic systems.

It can also be shown that, in the approximation which leads to (6a) and at large reference time t, the lead term in the displacement correlation function is given by

$$C_{\bar{q}jk}(t,\tau)=E[\bar{q}_j(t+\tau)\bar{q}_k(t)]$$
$$=\delta_{jk}\frac{2\bar{\Lambda}_j}{m_j\lambda\bar{\Gamma}_j}t=4\delta_{jk}D_jt\ , \tag{7}$$

where D_j is a diffusion coefficient. It is apparent that the displacement correlation functions are not stationary, but instead exhibit a linear growth with time which is characteristic of diffusion. We point out here that the physical dimensions of the $\bar{q}_j(t)$ depend upon the specific normalization of the $\bar{\phi}_{j\alpha}(\chi)$ in (1). We have formally chosen the normalization such that

$$\sum_{\alpha=1}^{3}\int d^3\chi\,\rho\bar{\phi}_{j\alpha}(\chi)\bar{\phi}_{k\alpha}(\chi)=m_j\delta_{jk}\ ,$$

where ρ is the fluid density. However, this is only symbolic and has been adopted to obtain the familiar form for the Hamiltonian given by (2). It is clear that specific values for the m_j are arbitrary and will in turn affect the scaling of the $\bar{q}_j(t)$. In order to make comparisons with other physical quantities with units of length, (7) must be used in conjunction with (1a) to obtain physical displacement correlation functions with units of length squared. This will affect numerical values but the functional form of (7) is unchanged, and it is that functional form rather than any specific value which is of concern to us in this paper.

From this point on we shall consider only the oscillatory or wavelike modes. By neglecting the translational modes we are ignoring the direct effects of mean shear. We must, therefore, expect that in certain locations our treatments will be incomplete. In the oceanic case, for example, near the edge of the gulf stream might be one such location. In some other locations such neglect is reasonable. We point out that there is considerable instantaneous shear due to the internal wave modes which is included in our treatment.

A potentially more serious issue is the neglect of the diffusion processes which are always present at some level. These diffusion processes result in Lagrangian correlation functions which are not stationary. In general the nonlinear interactions also couple the oscillatory modes with the translational modes, and therefore even the wavelike modes will develop some degree of nonstationarity. The neglect of these processes is not serious provided the rate of decay of the Lagrangian correlations due to diffusion is small in comparison to the frequency of the wave we wish to resolve. Thus, in order to neglect the translational modes we must assume that in the time $T_j=1/\omega_j$ that it takes to resolve the frequency ω_j the distance two nearby fluid parcels diffuse apart is negligible.

While some concern about these issues in the oceanic case has been expressed (see, e.g., Holloway[4]), so far as we are aware there have been no investigations which are sufficiently detailed to prove or disprove the above assumption. Nevertheless, the practice of ignoring the translational modes (these modes are sometimes called geostrophic or vortical modes) is almost universal in the treatment of oceanic internal waves. Additionally, there is some concern that in the presence of interactions the translational and wavelike modes will be mixed so that the existence of the translational modes is obscured. This issue has been considered by Holloway[4] and Müller[4] and, while it may ultimately be of some importance, we will not consider it further here. In some cases we might expect our treatment to be incomplete, but in others it should be adequate. In any case, it is reasonable to examine the problem to be considered here in isolation before attempting to deal with the more complicated problem of combined processes.

B. Eulerian variables

We shall now briefly discuss the Eulerian formulation and contrast it with the Lagrangian formulation. The Cartesian components of the Eulerian displacement $\eta_{E\alpha}(\mathbf{x},t)$ and velocity $v_{E\alpha}(\mathbf{x},t)$ can also be written in the form

$$\eta_{E\alpha}(\mathbf{x},t)=\sum_{l=1}^{N}\phi_{j\alpha}(\mathbf{x})a_j(t) \tag{8a}$$

and

$$v_{E\alpha}(\mathbf{x},t)=\sum_{l=1}^{N}\phi_{j\alpha}(\mathbf{x})\omega_j b_j(t)\ , \tag{8b}$$

where we have neglected the translational modes. While the form of (8) is identical to that of (1), in general the $a_j(t)$ and $b_j(t)$ are not related in any simple way to the $q_j(t)$ and $p_j(t)$. The linearized equations of motion are the same, so that for small enough amplitude disturbances we may write $a_j(t)=q_j(t)$ and $b_j(t)=p_j(t)$. However, the nonlinear terms associated with the two sets of variables are different. The dynamic nonlinearities also contribute to the Eulerian equations of motion, but because individual fluid parcels are continually flowing into and out of the region of interest there is an additional nonlinear flow term given by $(\mathbf{v}_E\cdot\nabla)\mathbf{v}_E$ which we shall call the advective nonlinearity. Thus, for larger amplitude disturbances the Eulerian amplitudes cannot be equated to the $q_j(t)$ and $p_j(t)$ and the exact transformation between the variables becomes intractable. It is important to realize that the two types of nonlinearity are fundamentally different. The dynamic nonlinearities are associated with the details of the forces between collections of fluid parcels. The advective nonlinearity is associated with the flow of fluid parcels into and out of a fixed region of space and is strictly an Eulerian frame concept. From a Lagrangian frame point of view the advective nonlinearity is a kinematic effect. It is, however, important to account for this effect when making comparisons between theory and experiment.

C. Statistical distribution

In Sec. II of this paper we will introduce an exact transformation which expresses the Eulerian variables in terms of the Lagrangian variables. We will show that if the statistical distribution of Lagrangian variables is described by the phase-space density function

$$g(p,q)=\frac{1}{Z}\exp\left[-\sum_{j=1}^{N}\frac{\omega_j}{2}(p_j^2+q_j^2)/\Lambda_j\right]\ , \tag{9}$$

then without further approximation tractable expressions for the Eulerian correlation functions and spectra can be obtained. We will find that in general the Eulerian spectra are significantly different from the corresponding Lagrangian spectra. In the remainder of this section we shall provide some discussion concerning the use of (9).

During the past two decades there have been a number of attempts to obtain a fundamental statistical theory of oceanic internal waves. The goal has usually been to justify (9) and the specific form for Λ_j proposed by GM.[1] Early efforts[5] were directed at obtaining an action rate equation which describes the nonlinear transfer of energy between the linear modes. Subsequently, these rate equations have been used in conjunction with the empirical

GM action spectrum to argue that the GM distribution corresponds, in some sense, to a minimum in the energy-transfer rates. While these studies have helped to establish a better understanding of the details of the nonlinear interactions, they have not provided a fundamental explanation for the observed GM action spectrum.

It is significant to note that the Hasselmann[5] action rate equation can be obtained in a straightforward way from the weak-interaction master equation developed by Prigogine.[3] It is well known that the long-time solution to the Prigogine master equation yields the phase-space density function given by (9) with $\Lambda_j = E_0$. This result corresponds to canonical equilibrium and it is straightforward to show that the action spectrum (i.e., average wave action A_j) is given by $A_j = E_0/\omega_j$. This action spectrum is entirely different from that proposed by GM and this disagreement has often been interpreted to mean that the oceanic internal wave system must be far from canonical equilibrium. For example, McComas and Müller[6] propose that if generation and dissipation terms are added to the rate equation, then the GM spectrum will be a stationary solution to the altered equation. Thus, the problem is viewed as a cascade of energy via weak interactions from large scales where sources are believed to be active to small scales where the dissipation is believed to be active.

The Prigogine[3] formulation is for a closed system, but there exist other treatments which allow the inclusion of additional degrees of freedom. For example, Mori[3] has used Zwanzig's[3] projection operator technique to cast the problem into the form of a generalized Lagevin equation which can be used to include the effects of generation and dissipation. The application of Langevin methods to the oceanic internal wave system has been considered by Pomphrey, Meiss, and Watson[7] who also argue that the GM spectrum corresponds to a local minimum in the energy-transfer rates.

Unfortunately, these studies have not necessarily resulted in an improved understanding of the appropriate generation and dissipation mechanisms. Indeed, it seems clear that the reason for this is that none of these studies has been sufficiently detailed to have been able to produce such an improved understanding. This shortcoming has been partially addressed by West[8] who considers a set of test waves which interact among themselves and with an additional set of waves which act as a heat bath. He obtains a generalized Langevin equation, the associated Fokker-Planck equation, and shows that the steady-state solution for the distribution of test waves depends upon the assumed distribution of heat bath waves. Specifically it is shown that the steady-state phase-space density function is given by

$$g(p,q)=\frac{1}{Z}\exp\left[-\sum_{j=1}^{N}\left[\frac{\omega_j}{2}(p_j^2+q_j^2)+\lambda V_{Ij}(p,q)\right]\Big/\Lambda_j\right], \quad (10)$$

where we have suppressed explicit display of the reference time t, the interaction potential in (3) has been written in the form

$$\lambda V_I(p,q)=\sum_{j=1}^{N}\lambda V_{Ij}(p,q)\ ,$$

and, in general, Λ_j depends upon the distribution of heat bath variables.

In the weak-interaction approximation the nonlinear interactions described by $\lambda V_I(p,q)$ play an important role in the time evolution of the phase-space density function but can otherwise be neglected. Thus, it is only $H_0(p,q)$ which occurs in the phase-space density function for canonical equilibrium. Prigogine and Henin[9] have considered the case of strong nonlinear interactions in closed systems. They obtain a generalized master equation and show that its long-time solutions correspond to canonical equilibrium so that the phase-space density function is given by (10) with Λ_j independent of the mode index j. Thus, (10) may be viewed as a generalization of canonical equilibrium which includes the possibility of an interaction with a generalized heat bath.

The important difference between canonical equilibrium and the more general distribution given by (10) is that (10) includes an interaction with a generalized heat bath. We shall refer to interactions with a generalized heat bath as external interactions and to interactions among the system or test waves as internal interactions. In all realistic situations both types of interactions are present. The crucial issue is which type of interaction dominates the time evolution of the phase-space density function. If internal interactions dominate, then we would expect the phase-space density function to evolve near to the canonical distribution [i.e., (10) with $\Lambda_j = E_0$]. If external interactions dominate, then we would expect the more general distribution given by (10). We point out that the inclusion of external interactions does not preclude the possibility of obtaining the canonical distribution. Indeed, the canonical distribution is a special case which is included within the distributions encompassed by (10). While it is never explicitly stated by West,[8] it is clear from his results [see his Eq. (3.18)] that if the heat bath waves are distributed in accordance with the canonical distribution, then the test waves are also distributed in accordance with the canonical distribution. Thus, we may view the canonical distribution as a special case of (10) which is obtained if either the external interactions are negligible in comparison to the internal interactions or if the heat bath is distributed in accordance with the canonical distribution.

In general the full expression given by (10) is too complicated to be of practical value. The difficulty is that the interaction potential is a complicated function of the dynamical variables and its inclusion in the full Hamiltonian used in (10) leads to intractable expressions. The weak-interaction approximation consists of neglecting the interaction potential in (10). This results in a phase-space density function which is of the Gaussian form given by (9) and leads to many interesting calculations which are mathematically tractable. However, it is important to realize that for the weak-interaction approximation to be valid we must limit not only the average energy per mode but also the wave-number bandwidth. If the modes are occupied out to arbitrarily small length

scales, then the contributions from the interaction potential will be arbitrarily large. Thus, the number of degrees of freedom cannot be extended to include arbitrarily small length scales. For example, if the average energy per mode is the constant E_0 out to some length scale μ and the modes which correspond to smaller length scales are excluded, then it can be shown that the ratio E_0/μ^3 (i.e., the average energy per unit volume) must be limited if the weak-interaction approximation is to be valid. Under these conditions the phase-space density function is given by (9) with $\Lambda_j = E_0$.

In geophysical fluids, such as the ocean, molecular viscosity provides a fundamental basis for excluding the small scale modes. In a typical situation the modes which correspond to length scales shorter than about a millimeter are strongly damped and are, thus, ineffective for storing energy. We can, therefore, think of a minimum lower bound for a length scale cutoff μ_m which is provided by molecular viscosity. If we now consider a system for which the internal interactions dominate, then we would expect the phase-space density function to be given by (9) with $\Lambda_j = E_0$ and the number of degrees of freedom determined by the length scale μ_m. The factor E_0/μ_m^3 must be smaller than some fixed value for the weak-interaction approximation to be valid (a more precise statement of this condition will be given later) but if this condition is met, then (9) can be used for the calculation of statistical averages. We shall refer to this scenario as case I. We believe that other processes are more important for limiting the participation of small length-scale oceanic internal waves, but include this case for completeness and because it is the simplest and most familiar case for which the use of (9) is justified.

If we consider a system for which external interactions dominate, then we would expect the phase-space density function to be given by (9) with Λ_j determined by the heat bath. In this case the heat bath provides a length-scale cutoff μ_h such that length scales smaller than μ_h are not significantly populated. If, as a simple example, $\Lambda_j \approx E_0$ for length scales larger than μ_h and decreases to zero for smaller length scales, then E_0/μ_h^3 must be limited for the weak-interaction approximation to be valid. If this condition is satisfied, then (9) can be used for the calculation of statistical averages. It is this scenario which has been proposed by McComas and Müller.[6] It is tempting to propose that generation and dissipation mechanisms provide a heat bath which establishes Λ_j such that the GM action spectrum is obtained. However, this proposal transfers our lack of understanding to the heat bath, the detailed nature of which must eventually be explained. In the case of oceanic internal waves the precise details of generation and dissipation are not completely known, but what is known does not seem to lead to an explanation for the quasiuniversal character of the GM spectrum (cf. Holloway[10]). Further, existing estimates[11] of the evolution rates due to internal interactions seem to suggest that at most length scales of interest the internal interaction rates are much larger than the external interaction rates. While this proposal has some attractive features, there are also some important unresolved issues. We shall refer to this scenario as case II.

A third scenario which is the most intriguing is also the most speculative. We now consider a system which is at canonical equilibrium but for which we cannot neglect the interaction potential. In this case the phase-space density function is given by (10) with $\Lambda_j = E_0$, but this exact expression is usually not tractable. However, it is sometimes the case that the small scale modes contribute much more strongly to the interaction potential than do the larger scale modes (this can be shown to be true for oceanic internal waves). In this case $\lambda V_I(p,q)$ in (10) provides a cutoff due to nonlinear interactions for modes which correspond to length scales smaller than μ_n. We can then approximate the phase-space density function by (9) if we choose Λ_j such that $\Lambda_j = E_0$ for modes which correspond to length scales larger than μ_n and then decreases rapidly to zero for modes which correspond to length scales smaller than μ_n. This approximation cannot be expected to provide correct detailed information about the exclusion of the small scale modes. While this is a serious shortcoming for the purpose of describing the Lagrangian spectra, we will show that the Eulerian spectra are not sensitive to such details. The ratio E_0/μ_n^3 must be limited but otherwise (9) can again be used for the calculation of statistical averages. We shall refer to this scenario as case III.

D. Summary

It is important to realize that the fundamental treatments which lead to (9) and (10) are in terms of Lagrangian variables. That is, they do not include contributions from the advective nonlinearity, since from a Lagrangian frame point of view the advective nonlinearity is irrelevant. In Sec. II we will develop expressions which enable us to write the Eulerian displacements and velocities in terms of the dynamical variables $p_j(t)$ and $q_j(t)$. We will find that while these expressions are not tractable for the purpose of finding the exact Eulerian fields, they lead to entirely tractable calculations for the Eulerian spectra. We will thus be able to study the consequence of strong nonlinear contributions from the advective nonlinearity in otherwise weakly interacting stochastic systems. We shall not attempt to establish that any particular system qualifies to be within that class. Rather, we will focus upon the important physical features exhibited by such systems and emphasize the differences between the Lagrangian and Eulerian spectra. We will develop the general relation between Lagrangian and Eulerian variables and define the statistical quantities in terms of which we describe the stochastic system. Of particular interest will be the four-dimensional frequency wave-number spectrum which is the average energy per unit frequency per unit three-dimensional wave vector as a function of frequency and wave vector. We will obtain an expression for the Eulerian frequency wave-number spectrum which, while tedious to evaluate, is clearly tractable.

In Sec. III we will consider an example with one spatial dimension and obtain an expression for the two-dimensional frequency wave-number spectrum (i.e., the average energy per unit frequency per unit one-dimensional wave number). The one-dimensional exam-

ple is rich in physical detail and it will be clear that in general the Eulerian spectra are significantly different from the corresponding Lagrangian spectra. In the final section we discuss the results. In particular, we will consider the application of these methods to the problem of oceanic internal waves. It is possible that the oceanic internal wave field is near canonical equilibrium, but that strong contributions from the advective nonlinearity result in Eulerian spectra which are significantly different from the corresponding Lagrangian spectra. These considerations may ultimately play an important role in obtaining a fundamental understanding of the observed distribution of oceanic internal waves. While much of our discussion is in terms of oceanic internal waves, it is clear that there is a potential applicability to a variety of other geophysical systems.

We now recapitulate our point of view and precisely what is done in this paper. We assume that the translational modes can be neglected, that for the relatively short times τ linear time evolution is adequate, and that the phase-space density function is given by (9). We make no attempt to extend the fundamental theories[3,9] which might be used to derive (9) or to justify the application of (9) to any particular physical system. Rather, we apply (9) to the practical problem of computing the Eulerian frequency wave-number spectrum and show that it is important to account for the difference between Eulerian and Lagrangian variables. We find that in general the Eulerian spectra are significantly different from the corresponding Lagrangian spectra. These considerations are clearly important if one is to be able to make meaningful comparisons between theoretical calculations and the results of experiments which involve the measurement of Eulerian variables.

II. FORMULATION

We shall illustrate our development by considering the Eulerian velocity; the development of the corresponding expressions for the Eulerian displacement will be obvious. The Eulerian velocity is defined in terms of an average over a collection of Lagrangian fluid parcels. We begin by dividing the fluid into a nunber of Eulerian cells which are macroscopically small, yet large enough to contain many Lagrangian fluid parcels. The Eulerian velocity is then given by the weighted average

$$\mathbf{v}_E(\mathbf{x},t)=\int \mathbf{v}_L(\boldsymbol{\chi},t)W(\mathbf{x},\boldsymbol{\chi},t)d^3\chi \ , \tag{11}$$

where it is to be understood that, unless otherwise noted, integrals are taken over the full range of the integration variable. The weighting function $W(\mathbf{x},\boldsymbol{\chi},t)$ in (11) is defined in terms of the coarse-grained delta function $\Delta(\mathbf{x}-\mathbf{u})$ such that

$$W(\mathbf{x},\boldsymbol{\chi},t)d^3\chi=\Delta(\mathbf{x}-\mathbf{u})d^3u$$
$$=\Delta(\mathbf{x}-\mathbf{u})J[\mathbf{u}]d^3\chi \ , \tag{12}$$

where $\mathbf{u}$ is the position of the Lagrangian fluid parcel given by

$$\mathbf{u}=\boldsymbol{\chi}+\boldsymbol{\eta}_L(\boldsymbol{\chi},t) \tag{13}$$

and $J[\mathbf{u}]$ is the Jacobian determinant associated with the transformation given by (13). While there are numerous possible choices, a convenient representation for the coarse-grained delta function is[12,13]

$$\Delta(\mathbf{x}-\mathbf{u})=(1/\sigma\sqrt{2\pi})^3\exp(-|\mathbf{x}-\mathbf{u}|^2/2\sigma^2) \ , \tag{14}$$

where σ is the linear dimension of the Eulerian cell.

In many practical applications we may take the limit $\sigma\rightarrow 0$ in which case (14) becomes the Dirac delta function $\delta(\mathbf{x}-\mathbf{u})$ and by using (11) and (12) we may write

$$\mathbf{v}_E(\mathbf{x},t)=\int \mathbf{v}_L(\boldsymbol{\chi},t)\delta(\mathbf{x}-\mathbf{u})d^3u$$
$$=\mathbf{v}_L(\boldsymbol{\chi}(\mathbf{x},t),t) \ , \tag{15}$$

where $\boldsymbol{\chi}(\mathbf{x},t)$ is the solution obtained by inverting (13) with $\mathbf{u}=\mathbf{x}$. While (15) is a simple-looking expression, in order to implement an exact evaluation we must invert (13) which, in all but trivial cases, is intractable. If, however, we are interested in statistical quantities, then for the three cases discussed in Sec. I C the problem of inverting (13) can be avoided and tractable expressions for the Eulerian spectra can be obtained.

All of the second-order statistical quantities are obtained from the two space point two-time point Eulerian velocity correlation functions which are defined by

$$C_{Ev\alpha\beta}(\mathbf{X},\tau)=E[v_{E\alpha}(\mathbf{x}+\mathbf{X},t+\tau)v_{E\beta}(\mathbf{x},t)] \ . \tag{16}$$

The correlation functions defined by (16) are given in terms of Eulerian variables and correspond to the usual quantities which are obtained experimentally. There are analogous correlation functions which are defined in terms of Lagrangian variables (to be discussed later). We note that the left-hand side (lhs) of (16) assumes spatial homogeneity and temporal stationarity (i.e., independent of $\mathbf{x}$ and t) while the right-hand side (rhs) is not so restricted. The definition given by (16) is general, but in this paper we shall consider only the homogeneous stationary case. Of particular importance for our studies here are the four-dimensional Eulerian frequency wave-number spectra which are defined by

$$S_{Ev\alpha\beta}(\mathbf{k},\omega)=\int d^3X\int d\tau\, C_{Ev\alpha\beta}(\mathbf{X},\tau)\exp[-i(\mathbf{k}\cdot\mathbf{X}-\omega\tau)] \ . \tag{17}$$

The coarse-grained delta function given by (14) can be written in terms of its Fourier transform as

$$\Delta(\mathbf{x}-\mathbf{u})=(1/2\pi)^3\int \exp[i\mathbf{m}\cdot(\mathbf{x}-\mathbf{u})-m^2\sigma^2/2]d^3m \ . \tag{18}$$

By using (11), (12), and (18) the expression for the Eulerian velocity becomes

$$v_{E\alpha}(\mathbf{x},t)=(1/2\pi)^3\int d^3\chi\int d^3m\; v_{L\alpha}(\chi,t)J[\mathbf{u}(\chi,t)]\exp\{i\mathbf{m}\cdot[\mathbf{x}-\chi-\boldsymbol{\eta}_L(\chi,t)]-m^2\sigma^2/2\}\ . \tag{19}$$

Then by using (16) and (19) the correlation functions can be written as

$$C_{Ev\alpha\beta}(\mathbf{X},\tau)=(1/2\pi)^6\int d^3\chi\int d^3\chi'\int d^3m\int d^3m' M_{\alpha\beta}(\mathbf{y},\tau,\mathbf{m},\mathbf{m}')\exp[i\mathbf{m}\cdot(\mathbf{x}+\mathbf{X}-\chi)-\mathbf{m}'\cdot(\mathbf{x}-\chi')-(m^2+m'^2)\sigma^2/2]\ , \tag{20}$$

where

$$M_{\alpha\beta}(\mathbf{y},\tau,\mathbf{m},\mathbf{m}')=E(v_{L\alpha}(\chi,t+\tau)v_{L\beta}(\chi',t)J[\mathbf{u}(\chi,t+\tau)]J[\mathbf{u}(\chi',t)]\exp\{-i[\mathbf{m}\cdot\boldsymbol{\eta}_L(\chi,t+\tau)-\mathbf{m}'\cdot\boldsymbol{\eta}_L(\chi',t)]\}) \tag{21}$$

and

$$\mathbf{y}=\chi-\chi'\ . \tag{22}$$

We again note that we are considering the spatially homogeneous case. Next, by making the transformation (22) and $\mathbf{z}=(\chi+\chi')/2$, the integral over z results in $\delta(\mathbf{m}-\mathbf{m}')$ and (20) becomes

$$C_{Ev\alpha\beta}(\mathbf{X},\tau)=(1/2\pi)^3\int d^3y\int d^3m\; M_{\alpha\beta}(\mathbf{y},\tau,\mathbf{m})\exp[i\mathbf{m}\cdot(\mathbf{X}-\mathbf{y})-m^2\sigma^2]\ , \tag{23}$$

where since $\mathbf{m}'=\mathbf{m}$ we have suppressed the display of $\mathbf{m}'$. Finally, by using (17) and (23) we obtain

$$S_{Ev\alpha\beta}(\mathbf{k},\omega)=\int d^3y\int d\tau\, M_{\alpha\beta}(\mathbf{y},\tau,\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{y}-\omega\tau)-k^2\sigma^2]\ . \tag{24}$$

The frequency wave-number spectra given by (24) are of central importance for our development in this paper. However, in order to implement (24) we must first obtain a tractable expression for (21). In the case of a general statistical distribution the expectation value in (21) can be prohibitively difficult to calculate. However, if we consider the Gaussian form given by (9), then the expectation value in (21) can be computed exactly. It is straightforward to compute the partition function for (9) and we can then write

$$g(p,q)=\prod_{j=1}^{N}(\omega_j/2\pi\Lambda_j)\times\exp\{-(\omega_j/2\Lambda_j)[p_j^2(t)+q_j^2(t)]\}\ . \tag{25}$$

The expectation value of any function $F(p,q,t)$ of the dynamical variables is given by

$$E[F(p,q,t)]=\int F(p,q,t)g(p,q)\prod_{j=1}^{N}dp_j dq_j\ . \tag{26}$$

The phase-space density function give by (9) restricts the population of the modes so that the contributions from the interaction potential are small. Thus, the time dependence of the dynamical variables can be approximated by (3). By using (1) and (3) it can be shown

$$\eta_{L\alpha}(\chi,t+\tau)=\sum_{j=1}^{N}\phi_{j\alpha}(\chi)[q_j(t)\cos(\omega_j\tau)+p_j(t)\sin(\omega_j\tau)] \tag{27}$$

and

$$v_{L\alpha}(\chi,t+\tau)=\sum_{j=1}^{N}\phi_{j\alpha}(\chi)\omega_j[p_j(t)\cos(\omega_j\tau)-q_j(t)\sin(\omega_j\tau)] \tag{28}$$

so that we can express the dynamical variables in terms of their explicit dependence upon the time difference τ and upon the values $p_j(t)$ and $q_j(t)$. From this point on we shall drop explicit display of the time t and simply write p_j and q_j. We will call this the weak-interaction approximation and while its use is perhaps somewhat speculative for case III, it is clearly appropriate for cases I and II.

The complicating factor in (21) is the complex exponential. All of the other factors in (21) generate products of the form $(p_{j_1}\cdots q_{j_n})$ where we may encounter any combination of the p_j and q_j and n can be as large as eight (i.e., up to three from each of the two Jacobian determinants and one from each of the two velocities). By using (27) the complex exponential in (21) can be written

$$\exp\{-i[\mathbf{m}\cdot\boldsymbol{\eta}_L(\chi,t+\tau)-\mathbf{m}'\cdot\boldsymbol{\eta}_L(\chi',t)]\}=\exp\left[-i\sum_{j=1}^{N}(P_jp_j+Q_jq_j)\right]\ , \tag{29}$$

where

$$P_j=\sum_{\alpha=1}^{3}m_\alpha\phi_{j\alpha}(\chi)\sin(\omega_j\tau) \tag{30}$$

and

$$Q_j=\sum_{\alpha=1}^{3}[m_\alpha\phi_{j\alpha}(\chi)\cos(\omega_j\tau)-m'_\alpha\phi_{j\alpha}(\chi')]\ . \tag{31}$$

The function $M_{\alpha\beta}(\mathbf{y},\tau,\mathbf{m},\mathbf{m}')$ will, therefore, consist of a sum of expectation values which, by using (25), (26), and (29), can be written in the form

$$E\left[p_{j_1}\cdots q_{j_n}\exp\left[-i\sum_{j=1}^{N}(P_jp_j+Q_jq_j)\right]\right]=(i)^n\frac{\partial}{\partial P_{j_1}}\cdots\frac{\partial}{\partial Q_{j_n}}E\left[\exp\left[-i\sum_{j=1}^{N}(P_jp_j+Q_jq_j)\right]\right]$$
$$=(i)^n\frac{\partial}{\partial P_{j_1}}\cdots\frac{\partial}{\partial Q_{j_n}}\exp\left[-\sum_{j=1}^{N}(\Lambda_j/2\omega_j)(P_j^2+Q_j^2)\right]. \quad (32)$$

By using (32) the expression given by (21) can be evaluated exactly. The only complication is that in the full three-dimensional case there are a substantial number of terms generated by the product of the two Jacobian determinants. While this makes the calculation tedious, it is only tedium and there are no fundamental difficulties. Because of the complexity associated with the full three-dimensional problem it is advisable to first consider a simplified one-dimensional example. We will find that the simplified example is rich in physical detail and there will be little doubt that the important features will carry over to three dimensions.

III. ONE-DIMENSIONAL EXAMPLE

In this section we shall consider a one-dimensional example for which we can obtain a relatively simple expression for the frequency wave-number spectrum. The system is translationally invariant and satisfies periodic boundary conditions so that the eigenfunctions $\phi_j(\chi)$ are given by

$$\phi_j(\chi)=\sqrt{1/2L\rho\omega_j}\exp(il_j\chi)\,, \quad (33)$$

where

$$l_j=(2\pi j/L),\quad -N/2\le j\le N/2\,. \quad (34)$$

L is the length of the system and ρ is the density. By using (1), (33), (34), and introducing a slight change in labeling we can write

$$\eta_L(\chi,t)=\sum_{j=-N/2}^{N/2}\sqrt{1/2L\rho\omega_j}\exp(il_j\chi)q_j(t) \quad (35)$$

and

$$v_L(\chi,t)=\sum_{j=-N/2}^{N/2}\sqrt{1/2L\rho\omega_j}\exp(il_j\chi)\omega_jp_j(t)\,. \quad (36)$$

The expressions given by (35) and (36) are given in terms of complex eigenfunctions and the dynamical variables $p_j(t)$ and $q_j(t)$ are also complex. Because $\eta_L(\chi,t)$ and $v_L(\chi,t)$ must be real, the $p_j(t)$ and $q_j(t)$ must satisfy

$$p_{-j}(t)=p_j^*(t)\,, \quad (37a)$$

$$q_{-j}(t)=q_j^*(t)\,, \quad (37b)$$

and hence the complex variables are not completely independent. However, it is straightforward to rewrite (35) and (36) in terms of real independent variables. By writing the complex dynamical variables in terms of their real and imaginary parts such that

$$p_j(t)=p_{j1}(t)+ip_{j2}(t)\,, \quad (38a)$$

$$q_j(t)=q_{j1}(t)+iq_{j2}(t)\,, \quad (38b)$$

(35) and (36) can be written

$$\eta_L(\chi,t)=\sum_{j=1}^{N/2}\sum_{s=1}^{2}\phi_{js}(\chi)q_{js}(t) \quad (39)$$

and

$$v_L(\chi,t)=\sum_{j=1}^{N/2}\sum_{s=1}^{2}\phi_{js}(\chi)\omega_jp_{js}(t)\,, \quad (40)$$

where

$$\phi_{j1}(\chi)=\sqrt{2/L\rho\omega_j}\cos(l_j\chi) \quad (41)$$

and

$$\phi_{j2}(\chi)=-\sqrt{2/L\rho\omega_j}\sin(l_j\chi)\,. \quad (42)$$

We note that the term $j=0$ corresponds to a uniform translation of the system and has been excluded from our formulation. The canonically conjugate dynamical variables $p_{js}(t)$ and $q_{js}(t)$ are real and completely independent. The Hamiltonian is given by

$$H(p,q)=\sum_{j=1}^{N/2}\sum_{s=1}^{2}(\omega_j/2)[p_{js}^2(t)+q_{js}^2(t)]+\lambda V_I(p,q)\,. \quad (43)$$

As a familiar example, the dispersion relation for longitudinal sound is given by

$$\omega_j=c|l_j|\,, \quad (44)$$

where c is the speed of sound.

Before proceeding to the calculation of the Eulerian spectra, we shall first compute the corresponding Lagrangian spectra. The two space point two-time point Lagrangian correlation function is defined by

$$C_{Lv}(X,\tau)=E[v_L(\chi+X,t+\tau)v_L(\chi,t)]\,. \quad (45)$$

(Recall that this is a one-dimensional example so that there is only one component of velocity and the label χ is one dimensional.) In the weak-interaction approximation the time dependence of $p_j(t)$ is given by (3a) so that by using (40) we can write

$$v_L(\chi,t+\tau)=\sum_{j=1}^{N/2}\sum_{s=1}^{2}\phi_{js}(\chi)\omega_j[p_{js}\cos(\omega_j\tau)-q_{js}\sin(\omega_j\tau)]\,, \quad (46)$$

where it is understood that the p_{js} and q_{js} are at the time t. By using (43) the phase-space density function corresponding to (25) can be written

$$g(p,q)=\prod_{j=1}^{N/2}\prod_{s=1}^{2}(\omega_j/2\pi\Lambda_j)\exp[-(\omega_j/2\Lambda_j)(p_{js}^2+q_{js}^2)]\,. \quad (47)$$

Then by using (45)–(47) it can be shown that

$$C_{Lv}(X,\tau)=\sum_{j=1}^{N/2}\sum_{s=1}^{2}\Lambda_j\omega_j\phi_{js}(\chi+X)\phi_{js}(\chi)\cos(\omega_j\tau) \quad (48)$$

and by using (41), (42), and (48) we find

$$C_{Lv}(X,\tau)=\sum_{j=1}^{N/2}(2\Lambda_j/L\rho)\cos(l_jX)\cos(\omega_j\tau)\ . \quad (49)$$

At this point it is convenient to pass to the limit that L becomes arbitrarily large so that the discrete variable l_j is replaced by the continuous variable l and (49) becomes

$$C_{Lv}(X,\tau)=(1/2\pi\rho)\int\Lambda(l)\cos(lX)\cos[\omega(l)\tau]dl\ , \quad (50)$$

where we have extended the integral to include both positive and negative values of l. The Lagrangian frequency wave-number spectrum is the two-dimensional Fourier transform of (50) given by

$$S_{Lv}(k,\omega)=\int dX\int d\tau\, C_{Lv}(X,\tau)\exp[-i(kX-\omega\tau)]$$

$$=(\pi/\rho)\Lambda(k)[\delta(\omega-\omega(k))+\delta(\omega+\omega(k))]\ , \quad (51)$$

where $\Lambda(k)$ symmetric about $k=0$. The delta functions in (51) confine the system to the dispersion surface, and we shall refer to this type of system as wavelike. The wave-number spectrum $\hat{S}_{Lv}(k)$ is given by

$$\hat{S}_{Lv}(k)=(1/2\pi)\int S_{Lv}(k,\omega)d\omega=(1/\rho)\Lambda(k) \quad (52)$$

which is simply proportional to the average energy per mode.

It is clear from (51) and (52) that the Lagrangian spectra are directly sensitive to the details of $\Lambda(k)$. For case II the heat bath determines the structure of $\Lambda(k)$ so that the wave-number spectrum given by (52) is a direct measure of the nature of the heat bath. In case III canonical equilibrium establishes the distribution such that $\Lambda(k)=E_0$ for small wave numbers and then decreases rapidly to zero for wave numbers which correspond to length scales smaller than μ_n. In case III the decaying region of $\Lambda(k)$ is an approximation to the effects of strong nonlinear interactions and the detailed structure in this region cannot be expected to be correct. If our interest is in these details of the Lagrangian spectra, then a more complete treatment must be given. However, if our interest is in the Eulerian spectra, then we will show that, over at least a part of the wave-number domain, they are not sensitive to these details. We shall find it convenient to introduce the convergence factor $h(k)$ such that

$$\Lambda(k)=E_0h(k)\ . \quad (53)$$

For case III, E_0 is the average energy per mode and $h(k)$ is unity if the wave number corresponds to length scales greater than μ_n and then decreases rapidly to zero for length scales smaller than μ_n. For case II, we shall consider E_0 to be the maximum average energy per mode and $h(k)$ is determined by the heat bath but must approach zero for length scales which are smaller than μ_h.

We shall now consider the case of Eulerian variables. In the one-dimensional case the Jacobian determinant is given by

$$J[u(\chi,t)]=1+\partial\eta_L(\chi,t)/\partial\chi\ . \quad (54)$$

By using (19) and (54) the Eulerian velocity can be written

$$v_E(x,t)=(1/2\pi)\int d\chi\int dm\, v_L(\chi,t)[1+\partial\eta_L(\chi,t)/\partial\chi]\exp\{im[x-\chi-\eta_L(\chi,t)]-m^2\sigma^2/2\}$$

$$=(1/2\pi)\int d\chi\int dm\, v_L(\chi,t)(i/m)\partial\exp\{im[x-\chi-\eta_L(\chi,t)]-m^2\sigma^2/2\}/\partial\chi$$

$$=-(i/2\pi)\int d\chi\int dm(1/m)\exp\{im[x-\chi-\eta_L(\chi,t)]-m^2\sigma^2/2\}\partial v_L(\chi,t)/\partial\chi\ , \quad (55)$$

where in the last step, which involves integration by parts, the integrated pieces vanish because of the periodic boundary conditions. Then by using (16) and (55) the Eulerian correlation function can be written

$$C_{Ev}(X,\tau)=(1/2\pi)^2\int d\chi\int d\chi'\int dm\int dm'M(y,\tau,m,m')\exp[im(x+X-\chi)-im'(x-\chi')-(m^2+m'^2)\sigma^2/2]\ , \quad (56)$$

where

$$M(y,\tau,m,m')=(1/mm')E([\partial v_L(\chi,t+\tau)/\partial\chi][\partial v_L(\chi',t)/\partial\chi']\exp\{-i[m\eta_L(\chi,t+\tau)-m'\eta_L(\chi',t)]\}) \quad (57)$$

and recall that y is the one-dimensional version of (22).

By using (32), (39), (40), and (57) we find

$$M(y,\tau,m,m')=\sum_{j_1=1}^{\infty}\sum_{s_1=1}^{2}\sum_{j_2=1}^{\infty}\sum_{s_2=1}^{2}\left[\frac{\partial\phi_{j_1s_1}(\chi)}{\partial\chi}\right]\left[\frac{\partial\phi_{j_2s_2}(\chi')}{\partial\chi'}\right]$$
$$\times(\omega_{j_1}\omega_{j_2}/mm')E\left[[p_{j_1s_1}\cos(\omega_{j_1}\tau)-q_{j_1s_1}\sin(\omega_{j_1}\tau)]p_{j_2s_2}\right.$$
$$\left.\times\exp\left[-i\sum_{j=1}^{\infty}\sum_{s=1}^{2}(P_{js}p_{js}+Q_{js}q_{js})\right]\right]$$
$$=-\sum_{j_1=1}^{\infty}\sum_{s_1=1}^{2}\sum_{j_2=1}^{\infty}\sum_{s_2=1}^{2}\left[\frac{\partial\phi_{j_1s_1}(\chi)}{\partial\chi}\right]\left[\frac{\partial\phi_{j_2s_2}(\chi')}{\partial\chi'}\right](\omega_{j_1}\omega_{j_2}/mm')$$
$$\times\left[\cos(\omega_{j_1}\tau)\frac{\partial^2}{\partial P_{j_1s_1}\partial P_{j_2s_2}}-\sin(\omega_{j_1}\tau)\frac{\partial^2}{\partial Q_{j_1s_1}\partial P_{j_2s_2}}\right]$$
$$\times\exp\left[-\sum_{j=1}^{\infty}\sum_{s=1}^{2}(\Lambda_j/2\omega_j)(P_{js}^2+Q_{js}^2)\right], \tag{58}$$

where

$$P_{js}=m\phi_{js}(\chi)\sin(\omega_j\tau)\ , \tag{59}$$

$$Q_{js}=m\phi_{js}(\chi)\cos(\omega_j\tau)-m'\phi_{js}(\chi')\ , \tag{60}$$

and since we can control the occupation of the states by the convergence factor $h(k)$ we have extended the upper bound on the various sums over j to infinity. The derivatives in (58) can be evaluated to obtain

$$M(y,\tau,m,m')=-\sum_{j_1=1}^{\infty}\sum_{s_1=1}^{2}\sum_{j_2=1}^{\infty}\sum_{s_2=1}^{2}\left[\frac{\partial\phi_{j_1s_1}(\chi)}{\partial\chi}\right]\left[\frac{\partial\phi_{j_2s_2}(\chi')}{\partial\chi'}\right]$$
$$\times(1/mm')\Lambda_{j_1}\{\omega_{j_1}\delta_{j_1j_2}\delta_{s_1s_2}\cos(\omega_{j_1}\tau)$$
$$-[P_{j_1s_1}\cos(\omega_{j_1}\tau)+Q_{j_1s_1}\sin(\omega_{j_1}\tau)]$$
$$\times\Lambda_{j_2}P_{j_2s_2}\}\exp\left[-\sum_{j=1}^{\infty}\sum_{s=1}^{2}(\Lambda_j/2\omega_j)(P_{js}^2+Q_{js}^2)\right]. \tag{61}$$

Finally, by using (41), (42), (59), (60), and (61) it is tedious but straightforward to show that

$$M(y,\tau,m,m')=-(1/mm')\{[\partial^4D(y,\tau,m,m')/\partial y^2\partial\tau^2]-[\partial^2D(y,\tau,m,m')/\partial y\partial\tau]^2\}\exp[-mm'D(y,\tau,m,m')]\ , \tag{62}$$

where

$$D(y,\tau,m,m')=[(m^2+m'^2)/2mm']C_{L\eta}(0,0)-C_{L\eta}(y,\tau) \tag{63}$$

and the Lagrangian displacement correlation function $C_{L\eta}(y,\tau)$ is given by

$$C_{L\eta}(y,\tau)=(2/L\rho)\sum_{j=1}^{\infty}(\Lambda_j/\omega_j^2)\cos(l_jy)\cos(\omega_j\tau)=(1/2\pi\rho)\int[\Lambda(l)/\omega^2(l)]\cos(ly)\cos[\omega(l)\tau]dl\ . \tag{64}$$

We note that the last expression of (64) corresponds to the limit that L is arbitrarily large.

Since $M(y,\tau,m,m')$ depends upon $y=\chi-\chi'$ but not upon $z=(\chi+\chi')/2$, we can make the transformation from χ and χ' to y and z in (56) and by using (62) we obtain

$$C_{Ev}(X,\tau)=-(1/2\pi)\int dy\int dm(1/m^2)\left[\left[\frac{\partial^4D(y,\tau)}{\partial y^2\partial\tau^2}\right]-\left[\frac{\partial^2D(y,\tau)}{\partial y\partial\tau}\right]^2\right]\exp[im(X-y)-m^2D(y,\tau)-m^2\sigma^2]\ , \tag{65}$$

where, since $D(y,\tau,m,m'=m)$ is independent of m, we have suppressed the display of m and written $D(y,\tau)$. The frequency wave-number spectrum $S_{Ev}(k,\omega)$ is the two-dimensional Fourier transform of (65) which yields

$$S_{Ev}(k,\omega)=-\int dy\int d\tau(1/k^2)\left[\left[\frac{\partial^4D(y,\tau)}{\partial y^2\partial\tau^2}\right]-k^2\left[\frac{\partial^2D(y,\tau)}{\partial y\partial\tau}\right]^2\right]\exp[-i(ky-\omega\tau)-k^2D(y,\tau)-k^2\sigma^2]\ . \tag{66}$$

By using (53), (63), and (64) we can write

$$D(y,\tau)=(E_0/2\pi\rho)\int[h(l)/\omega^2(l)]\{1-\cos(ly)\cos[\omega(l)\tau]\}dl\ . \tag{67}$$

Equation (66) establishes the relationship between Lagrangian and Eulerian spectra, and is the vehicle by means of which we will explore the difference between the two.

It is also useful to compute the Eulerian wave-number spectrum $\hat{S}_{Ev}(k)$ which by analogy with (52) is given by

$$\hat{S}_{Ev}(k)=(1/2\pi)\int S_{Ev}(k,\omega)d\omega \ . \tag{68}$$

By using (66) and (68) we find

$$\hat{S}_{Ev}(k)=-(1/k^2)\int \left[\frac{\partial^4 D(y,0)}{\partial y^2\partial\tau^2}\right]\exp[-iky-k^2D(y,0)-k^2\sigma^2]dy \ , \tag{69}$$

where we have used (67) to show that $\partial^2 D(y,0)/\partial y\partial\tau=0$. The important physical properties of (66) and (69) will be discussed in Sec. IV.

IV. DISCUSSION

The Eulerian spectra given by (66) and (69) are central results of this paper. We will show that in general they yield expressions which are different from the corresponding Lagrangian spectra given by (51) and (52). In this section we shall discuss these important differences, but before doing so it is useful to examine in more detail the conditions for the weak-interaction approximation to be valid. For fluid systems the expression for the Hamiltonian given by (2) is an expansion in powers of $[J(\mathbf{u})-1]$ which in the one-dimensional case is given by

$$\lambda=[J(\mathbf{u})-1]=\frac{\partial\eta_L}{\partial\chi} \ . \tag{70}$$

If that expansion is to be valid, then λ must be small relative to unity. We can estimate the strength of the nonlinear interactions by computing the expectation value of λ^2. By using (25), (39), (41), (42), and (53) it is straightforward to show that

$$E[\lambda^2]=(E_0/2\pi\rho)\int[h(l)l^2/\omega^2(l)]dl \ . \tag{71}$$

Thus, as a heuristic criterion we can require that $E[\lambda^2]$ given by (71) be somewhat less than unity for the expansion in (2) to be valid.

We shall find it convenient for the following discussion to define three useful parameters. The first is the previously defined Lagrangian length scale μ which we now define more precisely as

$$\mu=\left[(1/2)\int h(l)dl\right]^{-1} \ . \tag{72}$$

We note that μ is a measure of the length scale at which the convergence factor $h(l)$ limits the participation of the small scale modes. Secondly we define the effective wave speed $\bar{c}$ such that

$$\bar{c}=\left[(\mu/2)\int[h(l)l^2/\omega^2(l)]dl\right]^{-1/2} \ . \tag{73}$$

If, for example, we use the dispersion relation for sound given by (44), then we find that the effective wave speed is simply the speed of sound. Finally, we define the Eulerian length scale ν such that

$$(\nu/\mu)^2=E[\lambda^2] \ . \tag{74}$$

By using (71)–(74) it is easy to show that

$$\nu=(E_0\mu/\pi\rho\bar{c}^2)^{1/2} \ . \tag{75}$$

For most distributions ν is approximately equal to the root-mean-square Lagrangian displacement and we will find that it plays an important role in our discussion of the Eulerian spectra. We note that the ratio ν/μ must be somewhat smaller than unity for the expansion in (2) to be valid.

By using (67), (72), and (73) we define the dimensionless function $F(y,\tau)$ such that

$$D(y,\tau)=\nu^2F(y,\tau) \tag{76}$$

and

$$F(y,\tau)=(2\bar{c}^2/\mu)\int[h(l)/\omega^2(l)]\{1-\cos(ly)\cos[\omega(l)\tau]\}dl \ . \tag{77}$$

Then by using (66), (76), and (77) we find

$$S_{Ev}(k,\omega)=-(\nu/k)^2\int dy\int d\tau\left[\left[\frac{\partial^4F(y,\tau)}{\partial y^2\partial\tau^2}\right]-(k\nu)^2\left[\frac{\partial F(y,\tau)}{\partial y\partial\tau}\right]^2\right]\exp[-i(ky-\omega\tau)-(k\nu)^2F(y,\tau)-(k\sigma)^2] \ . \tag{78}$$

We note that the above discussion is specific to the one-dimensional system. For a three-dimensional system the convergence factor is a function of the three-dimensional vector $\boldsymbol{l}$. Further, for a stratified fluid it is likely that there is one length scale which corresponds to the cutoff of horizontal wave numbers and a different length scale which corresponds to the cutoff of vertical wave numbers. The treatment of the three-dimensional system is otherwise analogous to that for the one-dimensional system. We also point out that the above discussion concerns only the compressional potential energy. For stratified fluids there is also an expansion for the gravita-

tional potential energy which must be considered. In a typical case the gravitational nonlinearities tend to be smaller than the compressional nonlinearities, and in any case the treatment of the two is entirely similar. The specific discussion in the remainder of this section is for the one-dimensional compressional wave system. The extension to more general systems is obvious.

We shall now examine the general features exhibited by (78), first for small values of the product $k\nu$, and then for large values. The factor $\exp[-(k\sigma)^2]$ simply accounts for the small but nonzero size of the Eulerian cell and for our purposes here we may set this factor equal to unity. For small wave numbers such that $k\nu \ll 1$ it is useful to expand the factor

$$\exp[-(k\nu)^2 F(y,\tau)]$$

in (78) to obtain

$$S_{Ev}(k,\omega)=\sum_{j=0}^{\infty} S_{Evj}(k,\omega) , \tag{79a}$$

where

$$S_{Ev0}(k,\omega)=-(\nu/k)^2\int dy\int d\tau\frac{\partial^4 F(y,\tau)}{\partial y^2\partial\tau^2}\exp[-i(ky-\omega\tau)] \tag{79b}$$

and

$$S_{Evj}(k,\omega)=-\nu^4[(-1)^j/j!](k\nu)^{2(j-1)}\int dy\int d\tau[F(y,\tau)]^{j-1}\left[\left[\frac{\partial^4 F(y,\tau)}{\partial y^2\partial\tau^2}\right]F(y,\tau)+\left[\frac{\partial^2 F(y,\tau)}{\partial y\partial\tau}\right]^2\right]$$
$$\times\exp[-i(ky-\omega\tau)],\quad 1\le j<\infty . \tag{79c}$$

By using (77) it is straightforward to show that (79b) yields

$$S_{Ev0}(k,\omega)=(4\pi^2\nu^2\bar{c}^{\,2}/\mu)h(k)$$
$$\times[\delta(\omega-\omega(k))+\delta(\omega+\omega(k))] \tag{80}$$

which by using (75) is easily seen to be the same as the Lagrangian frequency wave-number spectrum given by (51). The term from (79c) denoted by $S_{Ev1}(k,\omega)$ is also of order $(k\nu)^0$. However, as we will demonstrate later, it is also smaller by the factor $(\nu/\mu)^2$ and can be viewed as a correction which in general tends to smear the delta functions in (80). The remaining terms from (79c), for which $2\le j<\infty$, are proportional to $(\nu/\mu)^2$ and are of order $(k\nu)^{2(j-1)}$ so that if $k\nu\ll 1$, then they can be neglected. Thus, if $\nu\ll\mu$ and if $k\nu\ll 1$ for all values of k which are of interest, then we have obtained the well-known result that in this limit the Lagrangian and Eulerian spectra are equivalent.

If $1\ll k\nu$, then because of the factor

$$\exp[-(k\nu)^2 F(y,\tau)]$$

only small values of y and τ will contribute to the integrals and we can generate an expansion in powers of $1/k\nu$ by expanding $F(y,\tau)$ in powers of y and τ to obtain

$$S_{Ev}(k,\omega)=(\nu/k)^2\int dy\int d\tau[(c^2/\mu^4)M+\cdots]\exp\{-\tfrac{1}{2}(k\nu)^2[(y/\mu)^2+(\bar{c}\tau/\mu)^2]-i(ky-\omega\tau)\}$$
$$=(4\pi^2\nu^2\bar{c}^{\,2}/\mu)(\nu/\mu)^2[M(1/k\nu)^4+\cdots](\mu/2\pi\bar{c})\exp\{-\tfrac{1}{2}[(\mu\omega/k\nu\bar{c})^2+(\mu/\nu)^2]\} , \tag{81}$$

where

$$M=2\mu^3\int h(l)l^2 dl . \tag{82}$$

For typical choices for the convergence factor in (82) the dimensionless parameter M is of order unity. While the specific arrangement of the parameters here is somewhat arbitrary, the above choice has been made for ease of comparison and manipulation between these and subsequent expressions. We note first that the frequency wave-number spectrum given by (81) is not proportional to the delta functions which confine the system to the dispersion surface, and hence from an Eulerian frame point of view the system is not wavelike. For large values of $k\nu$ the dominant behavior of (81) is the power-law decay which is proportional to $(1/k\nu)^4$ and this is true independent of how the convergence factor $h(k)$ is chosen. Hence, if $1\ll k\nu$, then the Eulerian frequency wave-number spectrum is in general significantly different from the corresponding Lagrangian spectrum.

We can also obtain an expansion in powers of $k\nu$ for the wave-number spectrum $\hat{S}_{Ev}(k)$ given by (69). This expansion is useful if $k\nu\ll 1$ and is most easily obtained by noting that its terms are given by

$$\hat{S}_{Evj}(k)=(1/2\pi)\int S_{Evj}(k,\omega)d\omega . \tag{83}$$

By using (80) and (83) it is straightforward to show that

$$\hat{S}_{Ev0}(k)=(2\pi\nu^2\bar{c}^{\,2}/\mu)h(k)2 \tag{84}$$

which by using (75) for ν is easily shown to be the same as the Lagrangian wave-number spectrum given by (52). By using (77), (79c), and (83) it can be shown that

$$\hat{S}_{Ev1}(k)=(2\pi\nu^2\bar{c}^{\,2}/\mu)(\nu/\mu)^2N(k)2\ , \tag{85}$$

where

$$N(k)=2\mu\bar{c}^{\,2}\int[h(k-l)(k-l)^2 -h(k)k^2][h(l)/\omega^2(l)]dl \tag{86}$$

which is dimensionless and for typical choices for the convergence factor is of order unity. Thus, we find that for small $k\nu$ the Eulerian wave-number spectrum is equal to the Lagrangian wave-number spectrum plus the correction term given by (85). The remaining terms, for which $2\le j<\infty$, are of order $(k\nu)^{2(j-1)}$ so that if $k\nu\ll 1$, then they can be neglected.

If $1\ll k\nu$, then by using (81) and (83) it can be shown that

$$\hat{S}_{Ev}(k)=(2\pi\nu^2\bar{c}^{\,2}/\mu)(\nu/\mu)^2[M(1/k\nu)^3+\cdots] \times(1/\sqrt{2\pi})\exp[-(\tfrac{1}{2})(\mu/\nu)^2]\ . \tag{87}$$

The Eulerian wave-number spectrum given by (87) is in general significantly different from the Lagrangian wave-number spectrum given by (52). We note that, independent of the specific choice for the convergence factor $h(l)$, the power-law decay proportional to $(1/k\nu)^3$ is obtained. In fact, the only detailed dependence upon the convergence factor is through the second moment integral M given by (82). Hence, we find that in the decaying region of the Eulerian wave-number spectrum all detailed information concerning the Lagrangian frame decay is lost and we simply find the $(1/k\nu)^3$ decay. This means that, independent of the detailed dynamics in the Lagrangian frame, we will always find the same large wave-number decay in the Eulerian frame. This also means, however, that experiments which focus upon the large wave-number decay in the Eulerian frame are not capable of obtaining detailed dynamical information.

We shall now consider the potential application of these methods to the problem of the statistical distribution of oceanic internal waves. There is, of course, a sense in which this discussion might be considered premature since internal waves are noncompressional waves which require at least a two-dimensional system to exist and we have examined the details of our methods for only the one-dimensional compressional wave example. On the other hand, the mathematical structure of linear internal waves is well understood (for a formulation in terms of Lagrangian variables see, e.g., Tolstoy[14]), and except for the imposition of specific relations between the vector components of the wave field, the mathematical structure is otherwise identical to that of our one-dimensional example. While we will not present the proof here, we have also shown that the large wave-number decay of the Eulerian wave-number spectrum in two dimensions is k^{-4} and in three dimensions is k^{-5}. We note that this is precisely what would be anticipated on the basis of dimensional arguments. Hence, there can be little doubt that the important general features exhibited by the one-dimensional example will carry over to three dimensions.

The model variance spectrum proposed by Garrett and Munk[1] has synthesized a variety of observations of oceanic temperature and velocity fields into a single unifying structure. The model assumes that these fields are due to a random superposition of linear internal waves and empirically adjusts the wave amplitudes in order to obtain agreement between their model and the various marginal spectra associated with the above collection of observed fields. While the GM model provides a useful and surprisingly reliable catalogue of existing experiments, it is empirical and does not presume to provide a first-principles explanation for the observed spectra. It is our ultimate goal to use the methods described in this paper to obtain such a first-principles explanation.

There have been other attempts to do this (for a recent review see Pomphrey[15]). By making the weak-interaction approximation, the action rate equation given by

$$\frac{\partial A_j}{\partial t}=\sum_{j'}\sum_{j''}\{T^{+}_{jj'j''}[A_{j'}A_{j''}-(A_jA_{j'}+A_jA_{j''})] +2T^{-}_{jj'j''}[A_{j'}A_{j''}-(A_jA_{j'}-A_jA_{j''})]\} \tag{88}$$

has been obtained by Hasselmann[5] and by Benney and Saffman.[5] In (88) A_j is the average wave action associated with the jth mode and the $T^{\pm}_{jj'j''}$ are coupling coefficients which depend upon the nonlinear interactions and conserve wave energy and momentum. Equation (88) has been used for numerous oceanographic calculations.[15] We note that (88) is also well known in statistical mechanics where it can be obtained by using the master equation for the phase-space density function derived by Prigogine[3] in the weak-interaction approximation.

It is well known that stationary solutions to (88) are given by $A_j=E_0/\omega_j$ which is the average wave action obtained from the canonical distribution. [We note that the more general form $A_j=E_0/(\omega_j+\boldsymbol{\beta}\cdot\mathbf{k}_j)$, where $\boldsymbol{\beta}$ is a constant vector, is sometimes quoted in the oceanographic literature. It is well known that this more general form includes a conserved component of net wave momentum which is possible in translationally invariant systems. It is straightforward to extend our treatment to include this case, but there is no need to complicate the discussion for our purposes in this paper.] For linear internal waves the dispersion relation ω_j approaches a constant for large $\mathbf{k}_j$ (actually large horizontal component of $\mathbf{k}_j$) and, thus, the above action spectrum is white for large $\mathbf{k}_j$. In contrast the action spectrum proposed by GM exhibits a large wave-number decay of between k_j^{-4} and k_j^{-5}. Because of this it is often assumed that the oceanic internal wave field is not near canonical equilibrium. As pointed out earlier, McComas and Müller[6] propose that if generation and dissipation terms are added to (88), then the GM spectrum will be a stationary solution to the altered equation. They present a heuristic construct and some numerical studies in support of this idea. While their results perhaps tend to support the conjecture, they are too incomplete to be taken as more than suggestive.

Another possibility is that the ocean is at (or near) canonical equilibrium. However, we must be careful to distinguish between Lagrangian and Eulerian variables.

The canonical distribution yields the action spectrum $A_j = E_0/\omega_j$ (or perhaps the more general form), but this is in terms of Lagrangian variables only. The empirical GM action spectrum is given in terms of Eulerian variables and, as we have shown, in the decaying region the two types of spectra are significantly different. Thus, it is the kinematic distortion caused by the transformation from Lagrangian to Eulerian variables which masks the true Lagrangian wave-number spectrum and produces an Eulerian wave-number spectrum which exhibits a power-law decay.

It is our proposal that the observed distribution of oceanic internal waves corresponds to either case II or III with the parameters chosen such that the decaying portion of the wave-number spectrum is within the region $1 \ll k\nu$. In either case the large wave-number decay of the Eulerian wave-number spectrum is the same and tends to be similar to that which is observed experimentally. The scenario described by case III is attractive because it provides an explanation for the quasiuniversal character of the GM spectrum. In this case the overall level E_0 is set by external considerations, but otherwise the system is self-regulating so that the spectral shape is determined by internal interactions and is universal. In both cases II and III the large wave-number decay is universal and perhaps this is all that is really needed. It is important to realize that our empirical knowledge of the oceanic spectra is somewhat sketchy. In a typical oceanographic experiment it is not the full four-dimensional frequency wave-number spectrum or even the three-dimensional wave-number spectrum which is actually measured. Instead, various one-dimensional marginal spectra (i.e., weighted integrals) obtained from the four-dimensional frequency wave-number spectrum are measured. Because of this the comparison between theory and experiment is not entirely straightforward. All we can really seek at this time, is a reasonable theory that exhibits general features which are similar to those which are experimentally observed. In order to establish that our proposal does indeed provide a first-principles explanation for the observed distribution of oceanic internal waves we must examine the above issues by detailed calculations for the full three-dimensional oceanic internal wave system. It is also important to consider the effects due to the translational modes. The methods we have introduced here can also be used to include the translational modes, but that task has yet to be accomplished. Nevertheless, the implications of the results from our one-dimensional example are clear cut and provide a compelling motivation to pursue the oceanic problem. It is also clear that these methods are potentially applicable to any stochastic fluid system and are, therefore, of broad general interest.

ACKNOWLEDGMENTS

This research was supported in part by the Office of Naval Research Contract No. N00024-85-C-5301 and in part by the J. H. Fitzgerald Dunning Foundation (K.R.A.) at the Department of Electrical and Computer Engineering of The Johns Hopkins University awarded by The Johns Hopkins University/Applied Physics Laboratory. The authors would like to thank Dr. L. W. Hart for many helpful discussions.

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