# Nonlinear theory of a two-photon correlated-spontaneous-emission laser: A coherently pumped two-level–two-photon laser

Ning Lu, Fang-Xiao Zhao, and J. Bergou*
*Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131*
(Received 19 December 1988)

We develop a nonlinear theory of a two-photon correlated-spontaneous-emission laser (CEL) by using an effective interaction Hamiltonian for a two-level system coupled by a two-photon transition. Assuming that the active atoms are prepared initially in a coherent superposition of two atomic levels involved in the two-photon transition, we derive a master equation for the field-density operator by using our quantum theory for coherently pumped lasers. The steady-state properties of the two-photon CEL are studied by converting the field master equation into a Fokker-Planck equation for the antinormal-ordering $Q$ representation of the field-density operator. Because of the injected atomic coherence, the drift and diffusion coefficients become phase sensitive. This leads to laser phase locking and an extra two-photon CEL gain. The laser field can build up from a vacuum in the no-population-inversion region, in contrast to an ordinary two-photon laser for which triggering is needed. We find an approximate steady-state solution of the $Q$ representation for the laser field, which consists of two identical peaks of elliptical type. We calculate the phase variance and, for any given mean photon number, obtain the minimum variance in the phase quadrature as a function of the initial atomic variables. Squeezing of the quantum noise in the phase quadrature is found and it exhibits the following features: (1) it is possible only when the laser intensity is smaller than a certain value; (2) it becomes most significant for small mean photon number, which is achievable in the no-population-inversion region; and (3) a maximum of 50% squeezing can be asymptotically approached in the small laser intensity limit. As a by-product we also study the ordinary two-photon laser and find, e.g., photon-number variance and laser linewidth.

## I. INTRODUCTION

Laser devices based on two-photon transition processes have received considerable interest in the last two decades.[1–16] These devices were originally believed[2] to lead to the realization of a continuously tunable coherent-light source, whose output frequencies satisfy the equation $\nu_1+\nu_2=\omega_{ac}$. Here $\nu_1$ and $\nu_2$ are the frequencies of the two photons, respectively, generated by the laser, and $\hbar\omega_{ac}$ is the energy difference between the two atomic levels involved. More recently, however, it is the prospect of generating squeezed light[5,11,12,14] that brought increased attention to the two-photon laser. Because of the quadratic form of the interaction Hamiltonian in these two-photon devices, squeezing might be possible under certain circumstances.

The steady-state photon statistics of an incoherently pumped two-photon laser (and micromaser) has been investigated extensively.[4,8,10,13,15] The operation of a two-photon laser, however, is quite different from that of a conventional one-photon laser. The self-starting problem[13] is encountered in the two-photon laser, whereas a one-photon laser can build up in the laser cavity from a vacuum. This is mainly due to the fact that the probability for a two-photon transition is several orders of magnitude smaller than that for an ordinary one-photon transition. Besides, other nonlinear processes also compete with the two-photon lasing transition. Because of all these difficulties, up to now only a handful of experiments on two-photon lasers have been reported,[9,16] in sharp contrast to the wealth of published theoretical works. Recent development of Rydberg-atom two-photon micromasers[14,16] makes the verification of theoretical predictions possible. Following the early proposal of generating squeezed light[17] in a two-photon laser,[5] the possibility of squeezing in an ordinary two-photon laser was investigated and ruled out.[11,12]

We have recently shown,[18] in the linear theory of a two-photon correlated-spontaneous-emission laser (CEL) (consisting of coherently pumped, cascade three-level atoms interacting with a single mode of the radiation field), that the phase noise in such a laser can be below the shot-noise level, i.e., less than that of a field in a coherent state.[19] It was shown that the phase squeezing can be achieved simultaneously with net linear gain when the active atoms are prepared initially in a coherent superposition of the top and bottom levels. A maximum of 50% phase squeezing below the shot-noise level [i.e., total phase variance $\langle(\Delta\phi)^2\rangle=\frac{1}{2}(4n_0)^{-1}$, $n_0$ being the mean photon number] can be asymptotically approached when the intermediate level of the cascade three-level atoms is far off resonance with the one-photon transition, while the top and bottom levels are maintained at the two-photon resonance. Under these conditions the two-photon transition dominates one-photon transitions, and such a cascade three-level laser becomes equivalent to a two-level–two-photon laser. The squeezed-state generation mechanism of a two-photon transition[5,17] then leads to squeezing of the quantum noise in the phase quadrature.

It is the purpose of this paper to develop a nonlinear theory of the two-photon CEL (i.e., active atoms are prepared initially in a coherent superposition of two levels involved in the two-photon lasing transition). We adopt a model of an effective interaction Hamiltonian[3] commonly used by many authors[4,5,8] (in so doing, a dynamic Stark shift is neglected[20]). It can be shown that this model corresponds to the cascade three-level laser with the intermediate level far off resonance with the one-photon transition.[13,20,21] Following the works on the CEL,[22] we have developed a quantum theory of a coherently pumped laser with injected atomic coherence[23] by appropriately generalizing the Scully-Lamb laser theory[24] originally introduced for incoherently pumped lasers. In this paper we apply this quantum theory of coherently pumped lasers to the effective interaction Hamiltonian to obtain a nonlinear theory of the two-photon CEL. Compared to the linear theory of Ref. 18, the nonlinear theory presented in this work is capable of investigating squeezing in the laser-phase quadrature in terms of Hermitian-quadrature operators $a_1$ and $a_2$ directly. We find that the injected atomic coherence leads to laser-phase locking and provides an extra two-photon CEL gain. The laser field can thus build up from a vacuum without triggering in the no-population-inversion region. The amount of squeezing for the in-quadrature operator $a_2$ is the same as that for laser phase. Such squeezing is most significant in the no-population-inversion region and the minimum quadrature noise is $\langle(\Delta a_2)^2\rangle=\frac{1}{8}$, representing 50% squeezing. We point out that in order to achieve these results, a proper phase relation must be satisfied among all randomly injected active atoms.

In Sec. II we derive the master equation for the reduced-field-density operator. This field master equation is transformed into a Fokker-Planck equation for the antinormal-ordering distribution $Q$ function[5,25] of the field-density operator[26] in Sec. III and that for the Glauber-Sudarshan $P$ representation[19] in Appendix A. As a comparison, we study separately the steady-state properties of an incoherently pumped two-photon laser in Sec. IV A and those of a coherently pumped two-photon laser in Sec. IV B by using the $Q$-function approach. The mean photon number, laser frequency, natural linewidth, solution of the $Q$ function, and photon number variance of the ordinary two-photon laser and the mean photon number, stable laser phases, approximate solution of the $Q$ function, and phase variance of the two-photon CEL are found in Sec. IV. The present results based on the two-level–two-photon model are compared to the results of Ref. 18 and the effect of a dynamical Stark shift is discussed in Appendix B. In Sec. V the noise properties of the two-photon CEL are further studied in terms of Hermitian-quadrature operators $a_1$ and $a_2$, which correspond to laser amplitude and phase quadratures, respectively. For any given laser intensity, we find the minimum variance in the $a_2$ quadrature as a function of the initial atomic variables. Finally, a physical explanation to the proper form of the initial atomic coherence and stable laser phases is given in Sec. VI.

## II. MASTER EQUATION

We consider a two-photon single-mode laser (see Fig. 1) with active atoms prepared initially in a coherent superposition of two atomic levels $a$ and $c$, $\rho_{ac}^{j}(t_j)\neq 0$. The two levels $a$ and $c$ are of the same parity and intermediate levels $b_i$ are far off resonance with the one-photon transition, so that the atomic transition is a two-photon transition process. The frequencies of the two photons in one atomic transition from the level $a$ to level $c$ are degenerate in a single-mode laser cavity considered in this work. For simplicity, we use an effective interaction Hamiltonian to describe the atom-field interaction under the assumption of large atom-field detunings for the intermediate levels (i.e., a two-level–two-photon laser model). The quantum theory of a coherently pumped laser has been developed recently[23] by properly generalizing the Scully-Lamb theory of lasers.[24] We first summarize the newly developed theory of the coherently pumped laser here, and then apply it to a two-photon correlated-spontaneous-emission laser (i.e., to a two-photon laser with injected atomic coherence) to derive the master equation for the reduced density operator of the laser field.

We assume that the active atoms are injected randomly into the laser cavity to start the interaction with the laser field. To be specific we denote $t_j$ as the injection time of the $j$th atom. For a single-mode laser, the reduced density operator for the laser field in the interaction picture satisfies the following equation:[23]

$$\dot{\rho}=-i(\Omega-\nu)[a^\dagger a,\rho]-i\sum_j \Theta(t-t_j)\,\mathrm{Tr}_{A^j}[\tilde{V}_j,\tilde{\rho}_j^f] + \tfrac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\ . \qquad (2.1)$$

Here $\Omega$ is the cavity-mode frequency, $\nu$ is the actual laser frequency, $a$ ($a^\dagger$) is the field annihilation (creation) operator, $\tilde{\rho}_j^f$ is the reduced density operator for the $j$th atom and the field in the interaction picture, $\gamma$ is the cavity-loss rate, and

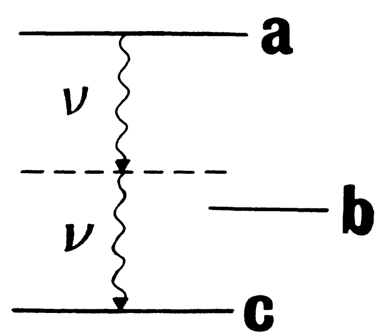


FIG. 1. Atomic levels of a two-level–two-photon laser.

$$\Theta(t-t_j)=\begin{cases}1, & t\geq t_j\\ 0, & t<t_j\end{cases} \tag{2.2}$$

is a step function used here to ensure that the $j$th atom begins its interaction with the laser field at time $t=t_j$. Note that $\hbar\tilde{V}_j$ is the interaction Hamiltonian of the $j$th atom with the laser field in the interaction picture, which is obtained from that in the Schrödinger picture $\hbar V_j$ through the unitary transformation

$$\tilde{V}_j=\exp[i\nu a^\dagger a t+iH_j^{\rm at}(t-t_j)]V_j \times\exp[-i\nu a^\dagger a t-iH_j^{\rm at}(t-t_j)] , \tag{2.3}$$

where $\hbar H_j^{\rm at}=\sum_A \hbar\omega_A|A^j\rangle\langle A^j|$ is the free Hamiltonian of the $j$th atom, and $\hbar\omega_A$ is the energy of the atomic state $|A^j\rangle$.

Corresponding to the unitary transformation prescribed in Eq. (2.3), the field-density operator $\rho$ in the interaction picture coincides with its counterpart $\rho^{\rm Sch}$ in the Schrödinger picture at time $t=0$,

$$\rho(0)=\rho^{\rm Sch}(0) , \tag{2.4}$$

and the reduced-density operator for the $j$th atom in the interaction picture, $\tilde{\rho}^j={\rm Tr}_f\tilde{\rho}_j^f$, coincides with that in the Schrödinger picture $\rho^j$ at the injection time $t_j$ of the $j$th atom,

$$\tilde{\rho}^j(t_j)=\rho^j(t_j) . \tag{2.5}$$

Treating active atoms in the field as independent of each other as in the Scully-Lamb laser theory,[24] we find that the reduced density operator $\tilde{\rho}_j^f$ obeys the following equation in the interaction picture:

$$\frac{d}{dt}\tilde{\rho}_j^f=-i\Theta(t-t_j)[\tilde{V}_j,\tilde{\rho}_j^f]-\tfrac{1}{2}(\Gamma^j\tilde{\rho}_j^f+\tilde{\rho}_j^f\Gamma^j) , \tag{2.6}$$

where

$$\Gamma^j=\sum_A\Gamma_A|A^j\rangle\langle A^j| \tag{2.7}$$

is the decay operator for the $j$th atom and $\Gamma_A$ is the decay rate of the atomic level $A$. Equations (2.1) and (2.6) are two basic equations for studying laser problems involving either coherent pumping or incoherent pumping in the good-cavity limit ($\Gamma_A\gg\gamma$).

We now derive the master equation for the reduced-field-density operator $\rho$ of the two-photon CEL (see Fig. 1). When the laser field is far off resonance with the intermediate levels $b_i$, the interaction of the radiation field with atomic levels $a$ and $c$ via the intermediate levels $b_i$ can be described by the following phenomenological effective interaction Hamiltonian[5,10,12,20] $\hbar V_j$ in the Schrödinger picture:

$$V_j=g|a^j\rangle\langle c^j|a^2+g^*(a^\dagger)^2|c^j\rangle\langle a^j| , \tag{2.8}$$

in which a dynamic Stark shift is neglected. Here $g$ is the effective coupling constant for the two-photon transition between levels $a$ and $c$. Using the transformation given in Eq. (2.3), the effective interaction Hamiltonian $\hbar\tilde{V}_j$ in the interaction picture can be found readily

$$\tilde{V}_j=g|a^j\rangle\langle c^j|a^2e^{i(\Delta t-\omega_{ac}t_j)} +g^*(a^\dagger)^2|c^j\rangle\langle a^j|e^{-i(\Delta t-\omega_{ac}t_j)} , \tag{2.9}$$

where

$$\Delta=\omega_{ac}-2\nu=\omega_a-\omega_c-2\nu \tag{2.10}$$

is the detuning for the two-photon transition.

Summation over the randomly injected atoms in Eq. (2.1) may be replaced by integration over the injection time $t_j$, i.e., $\sum_j\Theta(t-t_j)\rightarrow r_a\int_{-\infty}^t dt_j$, where $r_a$ is the atomic injection rate. The equations of motion for the field-density matrix elements are found after the substitution of Eq. (2.9) into Eq. (2.1),

$$\begin{aligned}\dot{\rho}_{nm}=&-i(\Omega-\nu)(n-m)\rho_{nm}-ir_a\int_{-\infty}^t dt_j\{ge^{i(\Delta t-\omega_{ac}t_j)}[\sqrt{(n+1)(n+2)}\tilde{\rho}^j_{cn+2,am}-\sqrt{m(m-1)}\tilde{\rho}^j_{cn,am-2}]\\&+g^*e^{-i(\Delta t-\omega_{ac}t_j)}[\sqrt{n(n-1)}\tilde{\rho}^j_{an-2,cm}-\sqrt{(m+1)(m+2)}\tilde{\rho}^j_{an,cm+2}]\}\\&+\gamma\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm} .\end{aligned} \tag{2.11}$$

Here $\tilde{\rho}^j_{An,A'm}$ ($A,A'=a,c$) are the matrix elements of the density operator $\tilde{\rho}_j^f$ as found from Eq. (2.6). To facilitate the calculation of $\tilde{\rho}^j_{An,A'm}$, we introduce $b^j_{An}$ such that

$$\tilde{\rho}^j_{An,Am}=b^j_{An}(b^j_{A'm})^*,\quad A,A'=a,c . \tag{2.12}$$

Upon substitution of Eqs. (2.7), (2.9), and (2.12) into Eq. (2.6) one obtains the equations of motion for $b^j_{An}$,

$$\dot{b}^j_{a,n}=-\Gamma_a b^j_{a,n}-ig\sqrt{(n+1)(n+2)} \times e^{i(\Delta t-\omega_{ac}t_j)}b^j_{c,n+2} , \tag{2.13a}$$

$$\dot{b}^j_{c,n+2}=-\Gamma_c b^j_{c,n+2}-ig^*\sqrt{(n+1)(n+2)} \times e^{-i(\Delta t-\omega_{ac}t_j)}b^j_{a,n} \tag{2.13b}$$

for $t\geq t_j$. From now on we consider equal atomic decay rates $\Gamma_a=\Gamma_b\equiv\Gamma$ only. The solution of Eqs. (2.13) in terms of initial conditions $b^j_{An}(t_j)$ can be found easily

$$\begin{aligned}b^j_{a,n}(t)=&e^{-(1/2)(\Gamma-i\Delta)(t-t_j)}\\&\times\{[\cos y_n-i(\Delta/\Omega_n)\sin y_n]b^j_{a,n}(t_j)\\&-i[2g\sqrt{(n+1)(n+2)}/\Omega_n]\\&\times\sin y_n e^{-i2\nu t_j}b^j_{c,n+2}(t_j)\} ,\end{aligned} \tag{2.14a}$$

$$b^{j}_{c,n+2}(t)=e^{-(1/2)(\Gamma+i\Delta)(t-t_j)}\times\{[\cos y_n+i(\Delta/\Omega_n)\sin y_n]b^{j}_{c,n+2}(t_j)-i[2g^*\sqrt{(n+1)(n+2)}/\Omega_n]\times\sin y_{n+1}e^{i2\nu t_j}c^{j}_{a,n}(t_j)\}\ , \tag{2.14b}$$

where

$$y_n=\tfrac{1}{2}\Omega_n(t-t_j)\ ,\qquad \Omega_n=[4|g|^2(n+1)(n+2)+\Delta^2]^{1/2}\ . \tag{2.15}$$

Substituting Eqs. (2.14) into Eqs. (2.12), one obtains $\tilde{\rho}^{j}_{An,A'm}(t)$ expressed in terms of the density-matrix elements of the operator $\tilde{\rho}^{f}_{j}(t_j)=\rho(t_j)\otimes\tilde{\rho}^{j}(t_j)$, since the $j$th atom is injected at time $t_j$. The coarse-grained time rate of change for the field operator $\rho$ is readily found from Eq. (2.11) under the condition $\gamma\ll\Gamma$, i.e., the laser field does not change appreciably on a time scale of atomic lifetime. In this case one can make the approximation $\rho(t_j)\approx\rho(t)$ for the field operator when $t_j>t-\Gamma^{-1}$. For atoms entering the cavity (much) earlier than $t-\Gamma^{-1}$, i.e., $t_j<t-\Gamma^{-1}$, one may still replace $\rho(t_j)$ with $\rho(t)$ by noting that $\tilde{\rho}^{f}_{j}$ decays with an overall factor $e^{-\Gamma(t-t_j)}\ll 1$. In other words, the coarse-grained time rate of change for the field operator $\rho$ can be obtained by substituting Eqs. (2.14), via Eqs. (2.12), into Eq. (2.11) and using Eqs. (2.15) and $\rho(t_j)\simeq\rho(t)$.

For the two-photon CEL we are interested in the following initial condition of the injected atoms in the Schrödinger picture:

$$\rho^{j}(t_j)=\begin{pmatrix}\rho_{aa} & \bar{\rho}_{ac}e^{-i2\nu t_j}\\ \bar{\rho}_{ca}e^{i2\nu t_j} & \rho_{cc}\end{pmatrix},\quad j=1,2,\ldots \tag{2.16}$$

where $\rho_{aa}$, $\rho_{cc}$, and $\bar{\rho}_{ac}=\bar{\rho}^{*}_{ca}$ are the same for all atoms. Using Eq. (2.5), the master equation for the laser field is obtained after the integration [here $n'\equiv(n+1)(n+2)$, $n''\equiv n(n-1)$, etc.]

$$\begin{aligned}\dot{\rho}_{nm}=&\{-\tfrac{1}{2}\alpha\rho_{aa}[n'+m'+i(n'-m')\delta+(\beta/4\alpha)(1+\delta^2)(n'-m')^2]\rho_{nm}+\alpha\rho_{cc}\sqrt{n'm'}\rho_{n+2,m+2}\\&+iS\bar{\rho}_{ac}[1-(\beta/4\alpha)(1+i\delta)(n'-m')]\sqrt{m'}\rho_{n,m+2}\\&-iS^*\bar{\rho}_{ca}[1+(\beta/4\alpha)(1-i\delta)(n'-m')]\sqrt{n'}\rho_{n+2,m}\}/\xi_{nm}\\&+\{\alpha\rho_{aa}\sqrt{n''m''}\rho_{n-2,m-2}-\tfrac{1}{2}\alpha\rho_{cc}[n''+m''-i(n''-m'')\delta+(\beta/4\alpha)(1+\delta^2)(n''-m'')^2]\rho_{nm}\\&-iS\bar{\rho}_{ac}[1+(\beta/4\alpha)(1+\delta)(n''-m'')]\sqrt{n''}\rho_{n-2,m}\\&+iS^*\bar{\rho}_{ca}[1-(\beta/4\alpha)(1-i\delta)(n''-m'')]\sqrt{m''}\rho_{n,m-2}\}/\xi_{n-2,m-2}\\&-i(\Omega-\nu)(n-m)\rho_{nm}+\gamma\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm}\ ,\end{aligned} \tag{2.17}$$

with

$$\xi_{nm}=1+\frac{\beta}{2\alpha}[(n+1)(n+2)+(m+1)(m+2)]+\frac{\beta^2}{16\alpha^2}(1+\delta^2)[(n+1)(n+2)-(m+1)(m+2)]^2\ , \tag{2.18}$$

where

$$\alpha=\frac{2r_a|g|^2}{\Gamma^2+\Delta^2},\quad \beta=\frac{8r_a|g|^4}{(\Gamma^2+\Delta^2)^2},\quad S=\frac{r_a g^*}{\Gamma+i\Delta},\quad \delta=\Delta/\Gamma\ . \tag{2.19}$$

Here $\alpha$ and $\beta$ are the linear-gain coefficient and saturation parameter of a two-photon laser, respectively. The coefficient $S$ is associated with the atomic coherence[23] ($\bar{\rho}_{ac}$) terms.

Equations (2.17) and (2.18) are very similar to the field master equation in a two-level–one-photon laser with injected atomic coherence[23] in terms of the "primed" quantities $n'$, $n''$, etc. These quantities are, however, quadratic in the photon number, reflecting the difference between the one-photon and two-photon lasers. On the other hand, Eq. (2.17) reduces to the master equation for an ordinary (i.e., incoherently pumped) two-photon laser when $\bar{\rho}_{ac}=\bar{\rho}^{*}_{ca}=0$.

The equations of motion for the diagonal elements $\rho_{nn}$ of the field-density matrix are obtained immediately from Eqs. (2.17) and (2.18) by setting $m=n$,

$$\begin{aligned}\dot{\rho}_{nn}=&-[1+(n+1)(n+2)\beta/\alpha]^{-1}[\alpha(n+1)(n+2)(\rho_{aa}\rho_{nn}-\rho_{cc}\rho_{n+2,n+2})-\sqrt{(n+1)(n+2)}(iS\bar{\rho}_{ac}\rho_{n,n+2}+\text{c.c.})]\\&+[1+n(n-1)\beta/\alpha]^{-1}[\alpha n(n-1)(\rho_{aa}\rho_{n-2,n-2}-\rho_{cc}\rho_{nn})-\sqrt{n(n-1)}(iS\bar{\rho}_{ac}\rho_{n-2,n-2}+\text{c.c.})]\\&+\gamma(n+1)\rho_{n+1,n+1}-\gamma n\rho_{nn}\ .\end{aligned} \tag{2.20}$$

Again Eq. (2.20) reduces to the equation of photon probability for an ordinary (i.e., without injected atomic coherence) two-photon laser[8] when $\bar{\rho}_{ac}=\bar{\rho}^*_{ca}=0$. The steady-state photon statistics of the two-photon CEL can be studied from Eq. (2.20) by setting the time derivative equal to zero ($\dot{\rho}_{nn}=0$). Besides the usual diagonal coupling between $\rho_{nn}$ and $\rho_{n+2,n+2}$ due to the two-photon transition, however, there exists an additional coupling to off-diagonal density-matrix elements $\rho_{n,n+2}$ and $\rho_{n+2,n}$.

Overall, due to the presence of the initial atomic coherence represented by terms containing $\bar{\rho}_{ac}$ and $\bar{\rho}_{ca}$ in Eqs. (2.17) and (2.20), the study of the two-photon CEL (including steady-state laser operation, noise properties, etc.) can be easily accomplished by converting the field master equation into a Fokker-Planck equation for a quasiprobability distribution function of the field operator $\rho$. This is the subject of the following sections.

## III. FOKKER-PLANCK EQUATION FOR THE $Q$ REPRESENTATION

Anticipating the possible squeezing of the laser field in the two-photon CEL, we choose to use the $Q$ representation[5,25] for the field-density operator $\rho$ in this work. The $Q$ representation is an antinormal-ordering distribution function defined by

$$Q(\mathcal{E},\mathcal{E}^*)=\pi^{-1}\langle\mathcal{E}|\rho|\mathcal{E}\rangle\ , \tag{3.1}$$

where $|\mathcal{E}\rangle$ is a coherent state, $a|\mathcal{E}\rangle=\mathcal{E}|\mathcal{E}\rangle$, and $\mathcal{E}^*$ is the complex conjugate of $\mathcal{E}$. One of appealing properties of the $Q$ function is[25]

$$0\le Q(\mathcal{E},\mathcal{E}^*)\le\pi^{-1}\quad\text{for all } \mathcal{E}\ . \tag{3.2}$$

The expectation value of an antinormally ordered function $F_{\text{anti}}(a,a^\dagger)$ can be evaluated with the help of the $Q$ function as[25]

$$\langle F_{\text{anti}}(a,a^\dagger)\rangle=\int F_{\text{anti}}(\mathcal{E},\mathcal{E}^*)Q(\mathcal{E},\mathcal{E}^*)d^2\mathcal{E}\ . \tag{3.3}$$

In order to transform the field master equation (2.17) into a Fokker-Planck equation for the $Q$ representation of the field density operator $\rho$, we rewrite Eq. (3.1) in the following form by using the expression of the coherent state[19] in the photon-number states and $\rho_{nm}=\langle n|\rho|m\rangle$:

$$Q(\mathcal{E},\mathcal{E}^*)=\pi^{-1}\sum_{n,m=0}^{\infty}e^{-|\mathcal{E}|^2}\frac{(\mathcal{E}^*)^n\mathcal{E}^m}{\sqrt{n!m!}}\rho_{nm}\ . \tag{3.4}$$

To derive the Fokker-Planck equation for the $Q$ function, the following formulas are helpful:

$$\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+|\mathcal{E}|^2\right]e^{-|\mathcal{E}|^2}\mathcal{E}^m=(m+1)e^{-|\mathcal{E}|^2}\mathcal{E}^m\ , \tag{3.5a}$$

$$\left[\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2+2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+|\mathcal{E}|^4\right]e^{-|\mathcal{E}|^2}\mathcal{E}^m=(m+1)(m+2)e^{-|\mathcal{E}|^2}\mathcal{E}^m\ . \tag{3.5b}$$

Similar expressions for $\mathcal{E}^*$ and $n$ are given by the replacements $\mathcal{E}\to\mathcal{E}^*$, $m\to n$.

Taking the time derivative on both sides of Eq. (3.4), substituting the master equation (2.17) into Eq. (3.4), and using Eqs. (3.5) we obtain an equation of motion for the $Q$ function. Assuming that the mean photon number of the two-photon CEL is much larger than 1, the equation of motion for $Q(\mathcal{E},\mathcal{E}^*,t)$ takes the following simple form after neglecting 1 compared to $|\mathcal{E}|^2$:

$$\begin{aligned}\frac{\partial Q(\mathcal{E},\mathcal{E}^*,t)}{\partial t}=&\Bigg\{-\alpha(\rho_{aa}-\rho_{cc})\left[(1-i\delta)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2\right]+\text{c.c.}\right]\\&-\tfrac{1}{2}(1+\delta^2)\beta(\rho_{aa}+\rho_{cc})\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2-\text{c.c.}\right]^2\\&+\alpha\rho_{cc}\left[4\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}|\mathcal{E}|^2+2\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\text{c.c.}\right]+\frac{\partial^4}{\partial\mathcal{E}^2\partial(\mathcal{E}^*)^2}\right]\Bigg\}L(\mathcal{E},\mathcal{E}^*,t)\\&+\Bigg\{iS\bar{\rho}_{ac}\left[2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}^*+\frac{\partial^2}{\partial\mathcal{E}^2}+\frac{\beta(1+i\delta)}{2\alpha}\left[2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2-\text{c.c.}\right](\mathcal{E}^*)^2\right.\\&\left.+\frac{\beta(1+i\delta)}{2\alpha}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2-\text{c.c.}\right]\left[\frac{\partial^2}{\partial\mathcal{E}^2}+2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}^*\right]\right]+\text{c.c}\Bigg\}L(\mathcal{E},\mathcal{E}^*,t)\\&+\gamma\left[\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}+\frac{1}{2}\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{1}{2}\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]Q(\mathcal{E},\mathcal{E}^*,t)\\&+i(\Omega-\nu)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]Q(\mathcal{E},\mathcal{E}^*,t)\ ,\end{aligned} \tag{3.6}$$

with

$$L(\mathcal{E},\mathcal{E}^*,t)=\left[1+\frac{\beta}{\alpha}|\mathcal{E}|^4+\frac{\beta}{\alpha}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2+\text{c.c.}\right]\right.$$
$$\left.+\frac{\beta^2(1+\delta^2)}{16\alpha^2}\left[2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2+\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2-\text{c.c.}\right]^2\right]^{-1}Q(\mathcal{E},\mathcal{E}^*,t)\ . \tag{3.7}$$

Equation (3.6) contains derivatives of all orders in $\mathcal{E}$ and $\mathcal{E}^*$ due to the presence of the inverse operator in $L(\mathcal{E},\mathcal{E}^*,t)$. In order to obtain the Fokker-Planck equation for the $Q$ function, we can expand Eq. (3.6) in terms of the derivatives and keep terms up to second order in the derivatives. Notice that the third- and higher-order derivative terms do not contribute to the calculations of the first and second moments, which are of interest for our present purpose. Consequently, we can safely neglect the third- and higher-order derivative terms here.

Since no zeroth-order term is left except in $L$, we need only expand $L(\mathcal{E},\mathcal{E}^*,t)$ up to first order in the derivatives, yielding

$$L(\mathcal{E},\mathcal{E}^*,t)=\frac{Q(\mathcal{E},\mathcal{E}^*,t)}{1+|\mathcal{E}|^4\beta/\alpha}-\frac{\beta}{\alpha(1+|\mathcal{E}|^4\beta/\alpha)}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]\frac{|\mathcal{E}|^2Q(\mathcal{E},\mathcal{E}^*,t)}{1+|\mathcal{E}|^4\beta/\alpha}\ . \tag{3.8}$$

Dropping terms containing third- and higher-order derivatives in Eq. (3.6) and neglecting 1 compared to $|\mathcal{E}|^2$, one obtains the Fokker-Planck equation for the $Q$ representation as

$$\frac{\partial Q(\mathcal{E},\mathcal{E}^*,t)}{\partial t}=\left[-\frac{\partial}{\partial\mathcal{E}}d_{\mathcal{E}}+\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}D_{\mathcal{E}\mathcal{E}^*}+\frac{\partial^2}{\partial\mathcal{E}^2}D_{\mathcal{E}\mathcal{E}}+\text{c.c.}\right]Q(\mathcal{E},\mathcal{E}^*,t)\ , \tag{3.9}$$

where the drift and diffusion coefficients for the $Q$ function are

$$d_{\mathcal{E}}=\{\alpha\mathcal{E}|\mathcal{E}|^2[(\rho_{aa}-\rho_{cc})(1-i\delta)-2i(g\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma]-2iS\bar{\rho}_{ac}\mathcal{E}^*\}/(1+|\mathcal{E}|^4\beta/\alpha)-\tfrac{1}{2}\gamma\mathcal{E}+i(\nu-\Omega)\mathcal{E}\ , \tag{3.10}$$

$$D_{\mathcal{E}\mathcal{E}^*}=\frac{\gamma}{2}+\frac{\alpha|\mathcal{E}|^2}{1+|\mathcal{E}|^4\beta/\alpha}\left[2\rho_{cc}+(1+\delta^2)(\rho_{aa}+\rho_{cc})\frac{\beta|\mathcal{E}|^4}{2\alpha}+(ig\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma\right]$$
$$+\frac{\beta|\mathcal{E}|^6(\rho_{aa}-\rho_{cc})+(iS^*\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})|\mathcal{E}|^2\beta/\alpha}{(1+|\mathcal{E}|^4\beta/\alpha)^2}\ , \tag{3.11a}$$

$$D_{\mathcal{E}\mathcal{E}}=\frac{iS\bar{\rho}_{ac}}{1+|\mathcal{E}|^4\beta/\alpha}+\frac{\alpha\mathcal{E}^2}{2(1+|\mathcal{E}|^4\beta/\alpha)}[2i(3\bar{\rho}_{ac}g^*+\bar{\rho}_{ca}g)/\Gamma-(\rho_{aa}-\rho_{cc})(1-i\delta)-(1+\delta^2)(\rho_{aa}+\rho_{cc})|\mathcal{E}|^4\beta/\alpha]$$
$$+\frac{\beta|\mathcal{E}|^2}{\alpha(1+|\mathcal{E}|^4\beta/\alpha)^2}\{\alpha\mathcal{E}^2[(\rho_{aa}-\rho_{cc})(1-i\delta)-2i(g\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma]-2iS\bar{\rho}_{ac}\}\ . \tag{3.11b}$$

As a comparison we give the Fokker-Planck equation in the Glauber-Sudarshan $P$ representation in Appendix A. One can see from Eqs. (3.10) and (A6) that, to the leading order (i.e., after dropping 1 as compared to $|\mathcal{E}|^2$), the drift coefficient $d_{\mathcal{E}}$ for the $Q$ representation is the same as the drift coefficient $d_{\mathcal{E}}^P$ for the $P$ representation,

$$d_{\mathcal{E}}=d_{\mathcal{E}}^P\ , \tag{3.12}$$

whereas the diffusion coefficients in the $Q$ and $P$ representations are different, $D_{\mathcal{E}\mathcal{E}^*}\neq D^P_{\mathcal{E}^*\mathcal{E}}$, $D_{\mathcal{E}\mathcal{E}}\neq D^P_{\mathcal{E}\mathcal{E}}$. For example, the cavity-loss rate $\gamma$ contributes to the diffusion coefficient[26] $D_{\mathcal{E}\mathcal{E}^*}$ [see Eq. (3.11a)] for the $Q$ representation, in contrast to the Fokker-Planck equation for the $P$ representation. The reason for this is that the $Q$ function is an antinormal-ordering distribution function. Without the injected atomic coherence ($\bar{\rho}_{ac}=0$), Eqs. (3.9)–(3.11) reduce to the $Q$'s Fokker-Planck equation for an ordinary two-photon laser.

To facilitate the analysis of the laser intensity and phase, we rewrite the Fokker-Planck equation (3.9) in terms of intensity and phase variables, $I$ and $\phi$, via the relation $\mathcal{E}=\sqrt{I}e^{i\phi}$,

$$\frac{\partial Q_2(I,\phi,t)}{\partial t}=\left[-\frac{\partial}{\partial I}d_I-\frac{\partial}{\partial\phi}d_\phi+\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}+2\frac{\partial^2}{\partial I\partial\phi}D_{I\phi}\right]Q_2(I,\phi,t)\ , \tag{3.13}$$

where

$$D_{II}=2I[D_{\mathcal{E}\mathcal{E}^*}+\text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})]\ , \tag{3.14a}$$

$$D_{\phi\phi}=\frac{1}{2I}[D_{\mathcal{E}\mathcal{E}^*}-\text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})]\ , \tag{3.14b}$$

$$D_{I\phi}=\text{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi}) \tag{3.14c}$$

are the intensity, phase, and cross diffusion coefficients for the $Q$ function, respectively, and

$$d_I=2\sqrt{I}\ \text{Re}(d_{\mathcal{E}}e^{-i\phi})+2D_{\mathcal{E}\mathcal{E}^*}\ , \tag{3.15a}$$

$$d_\phi = I^{-1/2}\,\mathrm{Im}(d_{\mathcal{E}}e^{-i\phi}) - I^{-1}\,\mathrm{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi}) \tag{3.15b}$$

are the intensity- and phase-drift coefficients for the $Q$ function, respectively. Also we have let

$$Q(\mathcal{E},\mathcal{E}^*,t) = 2Q_2(I,\phi,t) , \tag{3.16a}$$

so that

$$\int Q(\mathcal{E},\mathcal{E}^*,t)d^2\mathcal{E} = \int Q_2(I,\phi,t)dI\,d\phi = 1 . \tag{3.16b}$$

Note that the relations (3.14) and (3.15) are very general, valid for a Fokker-Planck equation in any representation. As can be seen from Eqs. (3.10) and (3.11), the terms $2D_{\mathcal{E}\mathcal{E}^*}$ in Eq. (3.15a) and $-I^{-1}\,\mathrm{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})$ in Eq. (3.15b) are smaller than their respective leading drift terms by a factor of $I^{-1} = |\mathcal{E}|^{-2}$ and, consequently, can be neglected under the previous assumption that the mean photon number is much larger than 1.

Substituting Eqs. (3.10) into Eqs. (3.15), one finds

$$d_I = (G-\gamma)I , \tag{3.17a}$$

$$\begin{aligned} d_\phi = \nu - \Omega - \{ &\alpha I(\rho_{aa}-\rho_{cc})\delta \\ &+ 2|S\bar{\rho}_{ac}|[\cos(\theta-2\phi-\arctan\delta) \\ &\quad + (1+\delta^2)^{1/2}(I^2\beta/\alpha) \\ &\quad \times\cos(\theta-2\phi)]\}/(1+I^2\beta/\alpha) , \end{aligned} \tag{3.17b}$$

with

$$G = \frac{2\alpha I(\rho_{aa}-\rho_{cc}) + 4|S\bar{\rho}_{ac}|\sin(\theta-2\phi-\arctan\delta)}{1+I^2\beta/\alpha} \tag{3.17c}$$

being a saturated gain. Here we have introduced the notation $g^*\bar{\rho}_{ac} = |g^*\bar{\rho}_{ac}|e^{i\theta}$. Compared to the drift coefficients for the $P$ representation [see Eqs. (A9)], we note that the intensity- and phase-drift coefficients for the $Q$ and $P$ representations are the same to the leading order,

$$d_I = d_I^P, \quad d_\phi = d_\phi^P . \tag{3.18}$$

An extra contribution to the gain (as compared to the incoherently pumped laser) due to the injected atomic coherence $\bar{\rho}_{ac}$ is evident from Eq. (3.17c). Moreover, this contribution does not vanish as $I \to 0$. As will be seen in Sec. IV, this leads to the observation that the two-photon CEL can build up from a vacuum without external triggering, in contrast to an ordinary two-photon laser.

Substituting Eqs. (3.11) into Eqs. (3.14), one finds that at two-photon resonance ($\Delta = 0$),

$$\begin{aligned} D_{II} = \gamma I &+ \frac{\alpha I^2[5\rho_{cc}-\rho_{aa}+(3\rho_{aa}+\rho_{cc})I^2\beta/\alpha]}{(1+I^2\beta/\alpha)^2} \\ &- \frac{2|S\bar{\rho}_{ac}|I(1-3I^2\beta/\alpha)}{(1+I^2\beta/\alpha)^2}\sin(\theta-2\phi) , \end{aligned} \tag{3.19a}$$

$$\begin{aligned} D_{\phi\phi} = \frac{\gamma}{4I} &+ \frac{\alpha(\rho_{aa}-\rho_{cc})}{2} + \frac{\alpha(\rho_{cc}-\rho_{aa})}{4(1+I^2\beta/\alpha)} \\ &+ \frac{|S\bar{\rho}_{ac}|(1+2I^2\beta/\alpha)}{2I(1+I^2\beta/\alpha)}\sin(\theta-2\phi) , \end{aligned} \tag{3.19b}$$

$$D_{I\phi} = \frac{|S\bar{\rho}_{ac}|}{1+I^2\beta/\alpha}\cos(\theta-2\phi) . \tag{3.19c}$$

Without the injected atomic coherence ($\bar{\rho}_{ac} = 0$), the drift and diffusion coefficients are phase insensitive and reduce to those for an ordinary two-photon laser. With injected atomic coherence ($\bar{\rho}_{ac} \neq 0$), however, all of the drift and diffusion coefficients become phase sensitive, i.e., depend on phase $\phi$. As we will see in the following sections, this leads to phase locking and subsequently squeezing in the two-photon CEL.

## IV. STEADY-STATE OPERATION AND PHASE SQUEEZING

We investigate the intensity and phase properties of the two-photon CEL by using the $Q$ function in this section. From Eq. (3.3) one gets

$$\langle \hat{n} \rangle = \langle \vdots \hat{n} \vdots \rangle - 1 = \langle I \rangle - 1 \tag{4.1}$$

for the mean photon number, and

$$\langle (\Delta\hat{n})^2 \rangle = \langle \vdots (\Delta\hat{n})^2 \vdots \rangle - \langle \vdots \hat{n} \vdots \rangle = \langle (\delta I)^2 \rangle - \langle I \rangle \tag{4.2}$$

for photon-number variance, where $\hat{n} = a^\dagger a$ is the photon number operator, $\vdots\ \vdots$ denotes antinormal ordering of the operators $a$ and $a^\dagger$, and $\delta I = I - \langle I \rangle$. In the case of laser-phase locking and large mean photon number, $\langle \hat{n} \rangle \gg 1$, the phase variance of the laser field can be regarded as $\langle (\Delta\phi)^2 \rangle = \langle (\Delta a_2)^2 \rangle / \langle \hat{n} \rangle$, where $a_2$ is the in-quadrature operator of the laser field [see Eq. (5.1b)]. Consequently, we have the following relation for laser-phase variance [see Eqs. (5.7)]:

$$\langle (\Delta\phi)^2 \rangle = \langle (\delta\phi)^2 \rangle - \frac{1}{4\langle \hat{n} \rangle} , \tag{4.3}$$

where $\delta\phi = \phi - \phi_0$, $\phi_0$ is one of possible steady-state laser phases, and $\langle (\delta\phi)^2 \rangle = \langle \vdots (\Delta a_2)^2 \vdots \rangle / \langle \hat{n} \rangle$ represents antinormally ordered phase variance.

The equations of motion for the intensity (photon number) and phase of the laser field are obtained by using Eq. (3.13),

$$\frac{d}{dt}\langle I \rangle = \langle d_I \rangle , \tag{4.4a}$$

$$\frac{d}{dt}\langle \phi \rangle = \langle d_\phi \rangle . \tag{4.4b}$$

The equations of motion for antinormally ordered (photon-) number and phase variances are also obtained from Eq. (3.13) as

$$\frac{d}{dt}\langle (\delta I)^2 \rangle = 2\langle d_I \delta I \rangle + 2\langle D_{II} \rangle , \tag{4.5a}$$

$$\frac{d}{dt}\langle (\delta\phi)^2 \rangle = 2\langle d_\phi \delta\phi \rangle + 2\langle D_{\phi\phi} \rangle , \tag{4.5b}$$

$$\frac{d}{dt}\langle \delta I \delta\phi \rangle = \langle d_I \delta\phi \rangle + \langle d_\phi \delta I \rangle + 2\langle D_{I\phi} \rangle \; . \qquad (4.5c)$$

We can assume that the quasiprobability distribution function $Q_2(I,\phi,t)$ is well peaked at the mean photon number $n_0 \simeq \langle I \rangle$ and, if they exist, locked phase values $\phi_0$ in steady state. Thus we can expand $d_I$ and $d_\phi$ in Eqs. (4.4) around $(I_0,\phi_0)$ up to first order in $\delta I$ and $\delta\phi$ and find that the steady-state mean photon number $n_0$ and locked phase values $\phi_0$ satisfy the following deterministic equations:

$$d_I(n_0,\phi_0)=0 \; , \qquad (4.6a)$$

$$d_\phi(n_0,\phi_0)=0 \; . \qquad (4.6b)$$

Consistent with our assumption $n_0 \gg 1$, Eq. (4.6a) is equivalent to [see Eq. (3.17a)]

$$G(n_0,\phi_0)=\gamma \; . \qquad (4.7)$$

In the following we investigate the incoherently pumped ($\bar{\rho}_{ac}=0$) two-photon laser and the coherently pumped ($\bar{\rho}_{ac}\neq 0$) one separately.

### A. Incoherently pumped two-photon laser, $\bar{\rho}_{ac}=0$

To appreciate the effects of injected atomic coherence on laser operation, we first derive some results for an ordinary two-photon laser using the $Q$-function approach. The ordinary two-photon lasers were studied theoretically in Refs. 4, 7, 8, 10–13, and 15 using other methods. Recently, the two-photon micromaser was investigated both theoretically and experimentally by Haroche *et al.*[14,16] Without the initial atomic coherence (i.e., $\bar{\rho}_{ac}=0$), Eq. (3.17c) yields the saturated gain for an incoherently pumped two-photon laser

$$G(I)=\frac{2\alpha I(\rho_{aa}-\rho_{cc})}{1+I^2\beta/\alpha} \; , \qquad (4.8a)$$

and Eqs. (3.17b) and (4.8a) give the corresponding phase-drift coefficient

$$d_\phi=\nu-\Omega-\tfrac{1}{2}G(I)\delta \; . \qquad (4.8b)$$

Note that both drift coefficients are phase insensitive, and population inversion $\rho_{aa}>\rho_{cc}$ is needed in this case to have a positive gain and overcome the cavity loss. The gain $G$ as a function of laser intensity $I$, Eq. (4.8a), is plotted in Fig. 2. One sees that, starting from zero, the gain $G$ increases with $I$ for $I<\sqrt{\alpha/\beta}$ and reaches its maximum value

$$G_{\max}=|S|(\rho_{aa}-\rho_{cc}) \qquad (4.9)$$

at $I=\sqrt{\alpha/\beta}$. [Note $|S|=\alpha\sqrt{\alpha/\beta}$ due to Eqs. (2.19).] For $I>\sqrt{\alpha/\beta}$, the gain $G$ decreases with $I$. In order to meet Eq. (4.7), $G_{\max}>\gamma$ must be satisfied. When this is

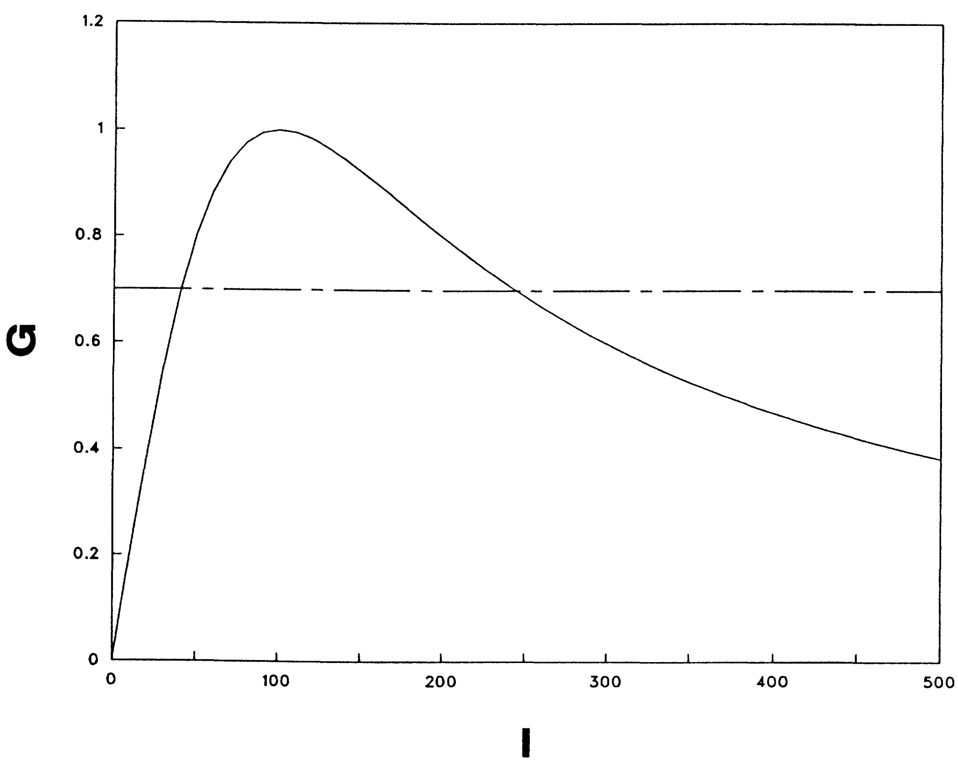


FIG. 2. Gain $G$ of an ordinary two-photon laser as a function of the laser intensity (photon number) $I$ for $\beta/\alpha=10^{-4}$, $\alpha(\rho_{aa}-\rho_{cc})=10^{-2}$. The long-dash–short-dash line represents the cavity loss rate $\gamma$. The two cross points are two intensity-locking points. The cross point with larger $I$ is a stable locking point while the one with smaller $I$ is an unstable locking point.

the case, Eq. (4.7) gives two positive solutions for $n_0$, and only the larger one

$$n_0=\sqrt{\alpha/\beta}[G_{\max}\gamma^{-1}+(G_{\max}^2\gamma^{-2}-1)^{1/2}] \tag{4.10}$$

satisfies $\partial G(n_0)/\partial I<0$, as is apparent in Fig. 2. Since a stable locking for the laser intensity requires

$$\frac{\partial d_I(n_0)}{\partial I}=n_0\frac{\partial G(n_0)}{\partial I}<0 , \tag{4.11}$$

where Eqs. (3.17a) and (4.7) have been used, one knows that only the larger $n_0$ ($>\sqrt{\alpha/\beta}$) given in Eq. (4.10) is stable.

Since $G$ vanishes as $I$ approaches zero and $\sqrt{\alpha/\beta}\geq\Gamma/2g\gg1$ in two-photon lasers, the field in an incoherently pumped two-photon laser cannot build up from a vacuum via spontaneous emission, in contrast to an incoherently pumped two-level–one-photon laser. Thus, in order to achieve stable operation in an ordinary two-photon laser, some kind of triggering is required.

The actual frequency[13] $\nu$ of an ordinary two-photon laser can be found by substituting Eqs. (4.8b) and (4.7) into Eq. (4.6b),

$$\nu=\frac{\Gamma\Omega+\gamma(\omega_{ac}/2)}{\Gamma+\gamma} . \tag{4.12}$$

Since $\Gamma\gg\gamma$, Eq. (4.12) predicts a pulling of the actual laser frequency $\nu$ away from the cavity mode frequency $\Omega$ to the direction of half of the two-photon atomic transition frequency $\omega_{ac}/2$, when $\Omega$ and $\omega_{ac}/2$ do not coincide.

The diffusion coefficients are phase insensitive when $\bar{\rho}_{ac}=0$. The steady-state diffusion coefficients at two-photon resonance ($\Delta=0$) are obtained readily from Eqs. (3.19) by setting $\bar{\rho}_{ac}=0$ and using Eqs. (4.8a) and (4.7),

$$D_{II}(n_0)=\frac{\rho_{aa}+3\rho_{cc}+2\gamma n_0\beta/\alpha^2}{2(\rho_{aa}-\rho_{cc})}\gamma n_0 , \tag{4.13a}$$

$$D_{\phi\phi}(n_0)=\frac{\gamma+4\alpha n_0(\rho_{aa}+\rho_{cc})}{8n_0} , \tag{4.13b}$$

$$D_{I\phi}=0 . \tag{4.13c}$$

We note that $D_{\phi\phi}(n_0)$ is the same as that for the $P$ representation [see Eq. (A11)], $D_{\phi\phi}(n_0)=D^P_{\phi\phi}(n_0)$, and is half the natural linewidth[24,27] of the incoherently pumped two-photon laser. The coefficient $D_{II}(n_0)$ is proportional to the antinormally ordered photon-number variance $\langle(\delta I)^2\rangle$ in steady state or, approximately, proportional to the width of the steady-state quasiprobability distribution function $Q_2(I)$. (The drift and diffusion coefficients are independent of the phase $\phi$ and so is the steady-state $Q$ function.) According to the Fokker-Planck equation (3.13), $Q_2(I)$ satisfies the equation

$$\frac{\partial}{\partial I}\left[d_I-\frac{\partial}{\partial I}D_{II}\right]Q_2(I)=0 . \tag{4.14}$$

Equation (4.14) has the solution

$$Q_2(I)=\frac{N_I}{D_{II}}\exp\left[\int_0^I\frac{d_I(I')}{D_{II}(I')}dI'\right] , \tag{4.15}$$

where $N_I$ is a normalization constant to be determined from Eq. (3.16b). Substitution of Eqs. (3.17a), (4.8a), and (3.19a) (with $\bar{\rho}_{ac}=0$) into Eq. (4.15) will give an explicit expression for $Q_2(I)$. As an approximation, one can expand $d_I$ and $D_{II}$ around $I=n_0$ up to first and zeroth order in $\delta I$, respectively, and obtains from Eq. (4.15) by using Eqs. (4.11), (4.8a), (4.13a), and (4.10),

$$\begin{aligned}Q_2(I)&=N_I\exp\left[-\frac{|\partial d_I(n_0)/\partial I|}{2D_{II}(n_0)}(I-n_0)^2\right]\\&=N_I\exp\left[-\frac{\rho_{aa}-\rho_{cc}-\gamma/\alpha n_0}{5\rho_{aa}-\rho_{cc}-\gamma/\alpha n_0}\frac{(I-n_0)^2}{n_0}\right] ,\end{aligned} \tag{4.16}$$

which is a sharply peaked (at $I=n_0$) ring-type Gaussian distribution with radius $n_0$ and distribution width

$$D_{II}(n_0)/|\partial d_I(n_0)/\partial I|\sim O(n_0) .$$

In Fig. 3 we plot $Q_2(I)$ for the case $\rho_{aa}=1$, $\rho_{cc}=0$, $\alpha n_0=1.1\gamma$.

Since $Q_2(I)$ is well peaked at $I=n_0$, the variance $\langle(\delta I)^2\rangle$ can be obtained from Eq. (4.5a) by setting $d/dt=0$ and expanding $d_I$ and $D_{II}$ around $I=n_0$ up to first order in $\delta I=I-n_0$. Substituting the resulting expression for $\langle(\delta I)^2\rangle$ into Eq. (4.2), one finds that the total photon-number variance in steady state is

$$\begin{aligned}\langle(\Delta\hat{n})^2\rangle&=\frac{D_{II}(n_0)}{|\partial d_I(n_0)/\partial I|}-n_0\\&=n_0\frac{3\rho_{aa}+\rho_{cc}}{2(\rho_{aa}-\rho_{cc}-\gamma/\alpha n_0)} ,\end{aligned} \tag{4.17}$$

which is larger than that of a Poisson distribution with the same mean photon number $n_0$. Alternatively, one can find the photon-number variance by using the $Q_2(I)$ given in Eq. (4.16) directly. This leads to the same result for $\langle(\Delta\hat{n})^2\rangle$ as that given in Eq. (4.17).

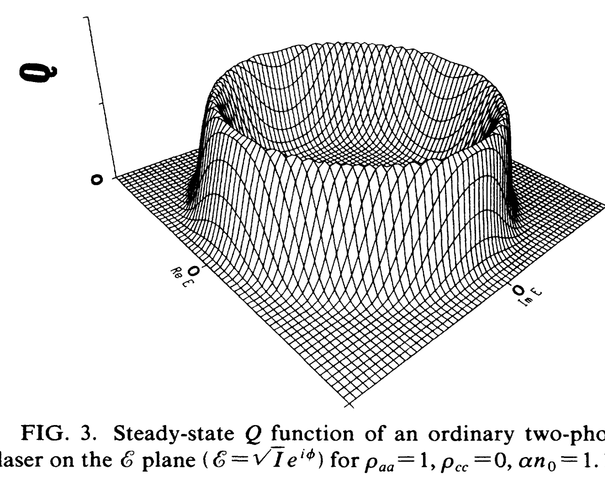


FIG. 3. Steady-state $Q$ function of an ordinary two-photon laser on the $\mathcal{E}$ plane ($\mathcal{E}=\sqrt{I}e^{i\phi}$) for $\rho_{aa}=1$, $\rho_{cc}=0$, $\alpha n_0=1.1\gamma$.

For the simple case $\rho_{aa}=1$, $\rho_{cc}=0$, $\Delta=0$, the photon-number variance and half the natural linewidth[28] are found from Eqs. (4.17) and (4.13b), respectively,

$$\langle(\Delta\hat{n})^2\rangle=\frac{3}{2}\frac{\alpha n_0}{\alpha n_0-\gamma}n_0\ , \tag{4.18a}$$

$$D_{\phi\phi}(n_0)=\frac{4\alpha n_0+\gamma}{8n_0}\ . \tag{4.18b}$$

One can compare these expressions with those of an ordinary one-photon laser[23,24] having the same mean photon number $n_0$ and cavity-loss rate $\gamma$,

$$\langle(\Delta\hat{n})^2\rangle_1=\frac{\alpha_1}{\alpha_1-\gamma}n_0\ , \tag{4.19a}$$

$$D^1_{\phi\phi}(n_0)=\frac{\alpha_1+\gamma}{8n_0}\ , \tag{4.19b}$$

where $\alpha_1$ is the linear gain (coefficient) for the one-photon laser. When $G_{\max}$ is slightly larger than $\gamma$, i.e., when the two-photon laser is just above its "threshold" $\gamma$, Eq. (4.10) gives $n_0\simeq\sqrt{\alpha/\beta}$ and, consequently, Eq. (4.8a) gives $\alpha n_0\simeq\gamma$. When $\alpha_1=\alpha n_0\simeq\gamma$, i.e., when both lasers are near threshold, the photon-number variance in the two-photon laser is 50% more than that in the one-photon laser, $\langle(\Delta\hat{n})^2\rangle=1.5\langle(\Delta\hat{n})^2\rangle_1$, and the linewidth of the two-photon laser is 150% more than that in the one-photon laser, $D_{\phi\phi}(n_0)=2.5D^1_{\phi\phi}(n_0)$. In another case $G_{\max}=\sqrt{\alpha^3/\beta}\gg\gamma$, Eq. (4.10) gives $n_0=2\alpha^2/\beta\gamma$, so that $\alpha n_0\gg\gamma$ (also, $n_0\sqrt{\beta/\alpha}\gg 1$). Consequently, we find from Eq. (4.18a) that[8] $\langle(\Delta\hat{n})^2\rangle=\frac{3}{2}n_0$. This is 50% larger than the photon-number variance of an ordinary one-photon laser operating far above threshold ($\alpha_1\gg\gamma$) with the same mean photon number $n_0$, $\langle(\Delta\hat{n})^2\rangle_1=n_0$ [see Eq. (4.19a)].

### B. Coherently pumped two-photon laser, $\bar{\rho}_{ac}\neq 0$

As compared to the incoherently pumped two-photon laser, two major differences are seen in Eqs. (3.17) when initial atomic coherence $\bar{\rho}_{ac}$ is present. Namely, there now exist phase locking and an extra two-photon gain. We will first examine the phase locking of the laser field and then find the laser intensity. For simplicity, we assume an actual resonant two-photon transition, $\Delta=\omega_{ac}-2\nu=0$, from now on. For the present effective-Hamiltonian model, in which a dynamic Stark shift is neglected, this could be achieved by setting the cavity-mode frequency $\Omega=\frac{1}{2}\omega_{ac}=\nu$. When the dynamic Stark shift is included, this could be obtained by offsetting the cavity-mode frequency, $\Omega=\frac{1}{2}\omega_{ac}-S$ with $g$ real, as discussed in Appendix B.

When $\Delta=0$, $\nu=\Omega$, the phase-drift coefficient given in Eq. (3.17b) reduces to

$$d_\phi=-2|S\bar{\rho}_{ac}|\cos(\theta-2\phi)\ . \tag{4.20}$$

Since $\partial d_\phi/\partial I=0$, the stable locking conditions for steady-state laser operation at $(n_0,\phi_0)$, which satisfies Eqs. (4.6), can be written as

$$\frac{\partial d_I(n_0,\phi_0)}{\partial I}=n_0\frac{\partial G(n_0,\phi_0)}{\partial I}<0\ , \tag{4.21a}$$

$$\frac{\partial d_\phi(n_0,\phi_0)}{\partial\phi}<0\ . \tag{4.21b}$$

Equation (4.21a) is similar to Eq. (4.11). The stable laser phases in steady state are thus found from Eqs. (4.6b), (4.20), and (4.21b) as satisfying $\sin(\theta-2\phi_0)=1$, i.e.,

$$\phi_0^k=\tfrac{1}{2}[\theta+\tfrac{1}{2}\pi+(-1)^k\pi],\quad k=1,2\ . \tag{4.22}$$

Depending on initial field fluctuations, the laser phase will be locked to *either* $\phi_0^1$ *or* $\phi_0^2$ in the steady state. Notice the difference from those of an incoherently pumped two-photon laser with injected signal[13] and a coherently pumped one-photon laser,[23] where only one phase is stable, and the similarity, on the other hand, to that of a squeezed-pump laser,[29] where there exist two stable phases.[18]

We now consider laser intensity at two-photon resonance ($\Delta=0$). As a matter of fact, $\phi$ is locked to $\phi_0$, given in Eq. (4.22), regardless of the laser intensity variable $I$. Thus, the laser gain $G(I,\phi)$ [see Eq. (3.17c)] as a function of $I$ can be replaced by that at stable phases $\phi_0$,

$$G(I,\phi_0)=2|S|\frac{I\sqrt{\beta/\alpha}(\rho_{aa}-\rho_{cc})+2|\bar{\rho}_{ac}|}{1+I^2\beta/\alpha}\ , \tag{4.23}$$

where Eqs. (2.19) have been used. Besides the usual two-photon gain [first term in the numerator on the right-hand side (rhs)], there appears an extra two-photon CEL gain induced by the atomic coherence. In contrast to the case of an ordinary two-photon laser, the gain $G(I,\phi_0)$ does not vanish as $I$ approaches zero due to the existence of the extra CEL gain. This means that triggering may not be necessary in the two-photon CEL. In Fig. 4 we plot the gain $G(I,\phi_0)$ as a function of $I$ for $\rho_{aa}>\rho_{cc}$, $\rho_{aa}=\rho_{cc}$, and $\rho_{aa}<\rho_{cc}$, respectively. One sees that population inversion $\rho_{aa}>\rho_{cc}$ is not necessary when $4|S\bar{\rho}_{ac}|>\gamma$ in order for the laser field to overcome the cavity loss and to be stable. In fact, as we will discuss in the following, large squeezing of the laser field occurs in the $\rho_{aa}<\rho_{cc}$ region. Also, as is evident in Fig. 4, the gain $G(I,\phi_0)$ assumes different behavior depending on whether or not there is a population inversion. When $\rho_{aa}>\rho_{cc}$ and

$$4|S\rho_{ac}|<\gamma<G_{\max}=|S|(\rho_{aa}-\rho_{cc})^2/(1-2|\bar{\rho}_{ac}|)\ ,$$

there are two positive solutions of $n_0$ to Eq. (4.7). Similar to the case of the incoherently pumped two-photon laser (see Sec. IV A), only the larger one satisfies Eq. (4.21a) and therefore is stable. When $\rho_{aa}>\rho_{cc}$ but $\gamma>G_{\max}$, or $\rho_{aa}\leq\rho_{cc}$ and $\gamma>4|S\bar{\rho}_{ac}|$, there is no positive solution of $n_0$ to Eq. (4.7); thus there is no stable operation point for the laser. In any other case, i.e., when $\gamma<4|S\bar{\rho}_{ac}|$, only one positive solution exists and it is stable. Within the last case, the laser field can build up from vacuum via spontaneous emission (i.e., without triggering), as can be understood from Fig. 4.

The stable $n_0$ in any of the above situations is

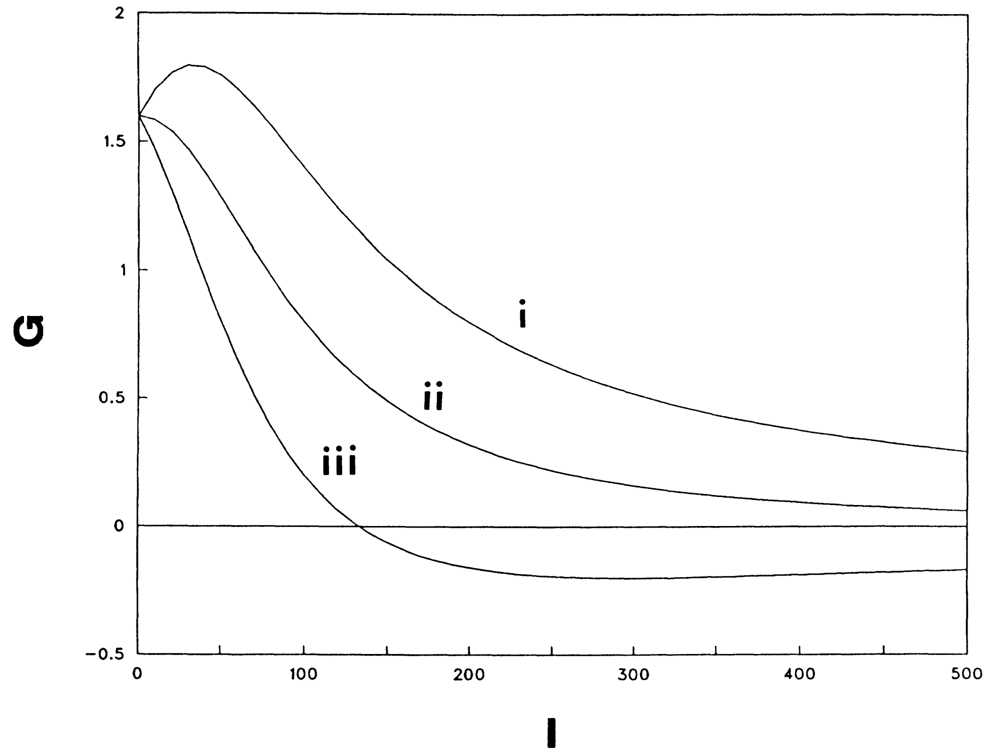


FIG. 4. Gain $G$ of the two-photon CEL at stable phase $\phi_0$ as a function of the laser intensity $I$ for $\alpha=10^{-2}$, $\beta=10^{-6}$, $|\bar{\rho}_{cc}|=(\rho_{aa}\rho_{cc})^{1/2}$, and curve $i$, $\rho_{aa}=0.8$, $\rho_{cc}=0.2$; curve $ii$, $\rho_{aa}=\rho_{cc}=0.5$; curve $iii$, $\rho_{aa}=0.2$, $\rho_{cc}=0.8$. A smaller $|s|$ value is used for curve $ii$ so that all three curves have the same linear gain $G|_{I=0}$.

$$n_0=\sqrt{\alpha/\beta}\{|S|(\rho_{aa}-\rho_{cc})\gamma^{-1}+[|S|^2(\rho_{aa}-\rho_{cc})^2\gamma^{-2}+4|S\bar{\rho}_{ac}|\gamma^{-1}-1]^{1/2}\}\ , \tag{4.24}$$

which reduces to Eq. (4.10) when $\bar{\rho}_{ac}=0$ as expected. As a result of the extra two-photon CEL gain, $n_0$ in Eq. (4.24) is larger than that in Eq. (4.10) for the same parameters $\alpha,\beta,\gamma$, and $\rho_{aa}-\rho_{cc}$ ($>0$). Note that the range of stable laser intensity $n_0$ changes dramatically according to the sign of $\rho_{aa}-\rho_{cc}$:

$$n_0>[(\sqrt{\eta^2+1}-\eta)\alpha/\beta]^{1/2},\quad \rho_{aa}>\rho_{cc} \tag{4.25a}$$

$$n_0<[(\sqrt{\eta^2+1}+\eta)\alpha/\beta]^{1/2},\quad \rho_{aa}<\rho_{cc} \tag{4.25b}$$

while $n_0$ can be any positive value when $\rho_{aa}=\rho_{cc}$, where $\eta=2|\bar{\rho}_{ac}/(\rho_{aa}-\rho_{cc})|$. The ranges of $n_0$ for $\rho_{aa}\le\rho_{cc}$ are in contrast to the usual two-photon laser studied in Sec. IV A.

Using Eqs. (4.7) and (4.23), the steady-state diffusion coefficients are found from Eqs. (3.19) to be

$$D_{II}(n_0,\phi_0)=\frac{n_0}{1+N_0^2}[|S|N_0(\rho_{aa}+3\rho_{cc})+2\gamma N_0^2+2|S\bar{\rho}_{ac}|(1+3N_0^4)(1+N_0^2)^{-1}]\ , \tag{4.26a}$$

$$\begin{aligned}D_{\phi\phi}(n_0,\phi_0)&=\frac{|S|}{2n_0(1+N_0^2)}\times[|\bar{\rho}_{ac}|+\rho_{aa}N_0+(\rho_{aa}+\rho_{cc})(\tfrac{1}{2}+N_0^2)N_0]+\frac{|S\bar{\rho}_{ac}|}{n_0}\\&=\frac{4\alpha n_0(\rho_{aa}+\rho_{cc})+\gamma}{8n_0}+\frac{|S\bar{\rho}_{ac}|}{n_0}\ ,\end{aligned} \tag{4.26b}$$

$$D_{I\phi}(\phi_0)=0\ , \tag{4.26c}$$

where

$$N_0=n_0\sqrt{\beta/\alpha} \tag{4.27}$$

is a normalized mean photon number and represents the degree of saturation.

The diffusion matrix

$$\begin{bmatrix}D_{II} & D_{I\phi}\\ D_{I\phi} & D_{\phi\phi}\end{bmatrix}$$

is positive definite at the locking points $(n_0,\phi_0)$. An approximate solution for the steady-state quasidistribution $Q_2(I,\phi)$ can be found by linearizing the Fokker-Planck equation (3.13). Expanding the drift coefficients $d_I$ and

$d_\phi$ and the diffusion coefficients $D_{II}$, $D_{\phi\phi}$, and $D_{I\phi}$ around the two locking points $(n_0,\phi_0^k)$ [$k=1,2$; see Eqs. (4.22) and (4.24)] up to first and zeroth order in $\delta I = I - n_0$ $\delta\phi_k = \phi - \phi_0^k$, respectively, and noting that $D_{I\phi}(\phi_0)=0$ and

$$\partial d_I(\phi_0)/\partial\phi = \partial d_\phi/\partial I = 0 \ , \tag{4.28}$$

one finds the linearized Fokker-Planck equation expressed in terms of intensity and phase variables, $I$ and $\phi$,

$$\begin{aligned}\frac{\partial}{\partial t}Q_2(I,\phi,t) = &-\frac{\partial}{\partial I}\left[\frac{\partial d_I(n_0,\phi_0)}{\partial I}\delta I - D_{II}(n_0,\phi)\frac{\partial}{\partial I}\right]Q_2(I,\phi,t)\\ &-\frac{\partial}{\partial\phi}\left[\frac{\partial d_\phi(\phi_0)}{\partial\phi}\delta\phi_k - D_{\phi\phi}(n_0,\phi_0)\frac{\partial}{\partial\phi}\right]Q_2(I,\phi,t) \quad \text{for } \phi \text{ near } \phi_k, \ \ k=1 \text{ or } 2\end{aligned} \tag{4.29}$$

where

$$\frac{\partial d_I(n_0,\phi_0)}{\partial I} = -\frac{4|S\bar{\rho}_{ac}| + \gamma(N_0^2-1)}{1+N_0^2} \ , \tag{4.30a}$$

$$\frac{\partial d_\phi(\phi_0)}{\partial\phi} = -4|S\bar{\rho}_{ac}| \ , \tag{4.30b}$$

and $D_{II}(n_0,\phi_0)$ and $D_{\phi\phi}(n_0,\phi_0)$ are given in Eqs. (4.26). The linearized Fokker-Planck equation (4.29) has the steady-state solution

$$\begin{aligned}Q_2(I,\phi) = N_{I\phi}\sum_{k=1}^{2}\exp\Bigg[&-\frac{|\partial d_I(n_0,\phi_0)/\partial I|}{2D_{II}(n_0,\phi_0)}(I-n_0)^2\\ &-\frac{|\partial d_\phi(\phi_0)/\partial\phi|}{2D_{\phi\phi}(n_0,\phi_0)}(\phi-\phi_0^k)^2\Bigg] \ ,\end{aligned} \tag{4.31}$$

where $N_{I\phi}$ is a normalization constant to be determined from Eq. (3.16b). This quasidistribution $Q_2(I,\phi)$ consists of two identical elliptical peaks at the two locking points $(n_0,\phi_0^k)$. The widths of the peaks for the intensity $I$ are

$$D_{II}(n_0,\phi_0)/|\partial d_I(n_0,\phi_0)/\partial I| \ ,$$

while the peak widths for the phase $\phi$ are

$$D_{\phi\phi}(n_0,\phi_0)/|\partial d_\phi(\phi_0)/\partial\phi| \ .$$

Since $Q_2(I,\phi)$ is sharply peaked at $(n_0,\phi_0^k)$ $(k=1,2)$, the antinormally ordered photon-number and phase variances in steady state, $\langle(\delta I)^2\rangle$ and $\langle(\delta\phi)^2\rangle$, can be obtained from Eqs. (4.5) by the same approach as we used in Sec. IV A, i.e., by setting $d/dt=0$ and expanding $d_I$, $d_\phi$, $D_{II}$, $D_{\phi\phi}$, and $D_{I\phi}$ around the two locking points $(n_0,\phi_0^k)$ up to first order in $\delta I$ and $\delta\phi_k$. In general one needs to solve three coupled first-order algebraic equations to obtain $\langle(\delta I)^2\rangle$ and $\langle(\delta\phi)^2\rangle$, since both $\langle(\delta I)^2\rangle$ and $\langle(\delta\phi)^2\rangle$ are coupled to $\langle\delta I\delta\phi\rangle$. For the present situation, however, this is not the case because of Eqs. (4.28).

From Eqs. (4.3) and (4.5b) the steady-state phase variance is found to be (note that $\phi_0$ will be *either* $\phi_0^1$ *or* $\phi_0^2$)

$$\begin{aligned}\langle(\Delta\phi)^2\rangle &= \frac{D_{\phi\phi}(n_0,\phi_0)}{|\partial d_\phi(\phi_0)/\partial\phi|} - \frac{1}{4n_0}\\ &= \frac{1+N_0|\bar{\rho}_{ac}|^{-1}(\rho_{aa}+\frac{1}{2}+N_0^2)}{8n_0(1+N_0^2)}\\ &= \frac{1}{4n_0}\frac{4\alpha n_0+\gamma}{8|S\bar{\rho}_{ac}|} \ ,\end{aligned} \tag{4.32}$$

where Eqs. (4.26b) and (4.30b) and the relation $\rho_{aa}+\rho_{cc}=1$ have been used. Another way to find the photon-number and phase variances is by using the $Q_2(I,\phi)$ in Eq. (4.31). Similar to the incoherently pumped laser discussed in Sec. IV A, this approach gives the same results for $\langle(\Delta\phi)^2\rangle$ as Eq. (4.32). The phase variance in Eq. (4.33) differs from that of a field in a coherent state with the same mean photon number $n_0$ by a factor of $(4\alpha n_0+\gamma)/8|S\bar{\rho}_{ac}|$. The squeezing in the laser-phase quadrature occurs whenever this factor becomes less than unity. For a significant squeezing to occur, we must then have a small normalized photon number $N_0 \ll |\bar{\rho}_{ac}| \le \frac{1}{2}$ (but still keeping $n_0 \gg 1$, i.e., $1 \ll n_0 \ll \sqrt{\alpha/\beta}|\bar{\rho}_{ac}|$). This can actually be achieved when (see Fig. 4),

$$\rho_{aa} < \rho_{cc}, \quad 4|S\bar{\rho}_{ac}| - \gamma \to 0^+, \quad \alpha/\beta > 10^5 \ . \tag{4.33}$$

(In this case $4\alpha n_0 \ll \gamma \simeq 4|S\bar{\rho}_{ac}|$.) Consequently, $\langle(\Delta\phi)^2\rangle = (8n_0)^{-1}$ can be approached asymptotically. That is, a maximum of 50% squeezing of the phase noise can be achieved in the two-photon CEL.

This result agrees with the conclusion reached in Ref. 18. For completeness we compare, in Appendix B, the results in the present paper with those in Ref. 18 in the region where both are valid.

## V. SQUEEZING OF QUADRATURE VARIANCE

To demonstrate the squeezing of the phase noise in the two-photon CEL more rigorously, we investigate the variance of the phase quadrature $a_2$. We define the Hermitian quadrature operators of the field as

$$a_1 = (ae^{-i\phi_0} + a^\dagger e^{i\phi_0})/2 \ , \tag{5.1a}$$

$$a_2 = (ae^{-i\phi_0} - a^\dagger e^{i\phi_0})/2i \ , \tag{5.1b}$$

where $\phi_0$ is chosen to be one of the steady-state locked-phase values [see Eq. (4.22)]. In this way, $a_1$ is associated with the amplitude of the laser field while $a_2$ with the laser phase, as we will see later.[30] The quadrature operators $a_1$ and $a_2$ satisfy the commutation relation

$$[a_1,a_2]=\tfrac{1}{2}i \ . \tag{5.2}$$

Corresponding to Eqs. (5.1), the $c$-number quadrature variables are

$$\mathcal{E}_1=(\mathcal{E}e^{-i\phi_0}+\mathcal{E}^*e^{i\phi_0})/2 \ , \tag{5.3a}$$

$$\mathcal{E}_2=(\mathcal{E}e^{-i\phi_0}-\mathcal{E}^*e^{i\phi_0})/2i \ . \tag{5.3b}$$

In terms of $\mathcal{E}_1$ and $\mathcal{E}_2$ the Fokker-Planck equation (3.9) takes the form

$$\frac{\partial Q(\mathcal{E}_1,\mathcal{E}_2,t)}{\partial t}=\left[-\frac{\partial}{\partial\mathcal{E}_1}d_1-\frac{\partial}{\partial\mathcal{E}_2}d_2+\frac{\partial^2}{\partial\mathcal{E}_1^2}D_{11}+\frac{\partial^2}{\partial\mathcal{E}_2^2}D_{22}+2\frac{\partial^2}{\partial\mathcal{E}_1\partial\mathcal{E}_2}D_{12}\right]\times Q(\mathcal{E}_1,\mathcal{E}_2,t) \ , \tag{5.4}$$

where the new drift and diffusion coefficients are related to the old ones by

$$d_1=\mathrm{Re}(d_{\mathcal{E}}e^{-i\phi_0}) \ , \tag{5.5a}$$

$$d_2=\mathrm{Im}(d_{\mathcal{E}}e^{-i\phi_0}) \ , \tag{5.5b}$$

$$D_{11}=\tfrac{1}{2}[D_{\mathcal{E}\mathcal{E}^*}+\mathrm{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi_0})] \ , \tag{5.6a}$$

$$D_{22}=\tfrac{1}{2}[D_{\mathcal{E}\mathcal{E}^*}-\mathrm{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi_0})] \ , \tag{5.6b}$$

$$D_{12}=\tfrac{1}{2}\,\mathrm{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi_0}) \ . \tag{5.6c}$$

Also in terms of $\mathcal{E}_1$ and $\mathcal{E}_2$ one finds from Eqs. (5.1), (5.2), and (3.3) that $\langle a_i\rangle=\langle\mathcal{E}_i\rangle$ $(i=1,2)$ and

$$\langle(\Delta a_i)^2\rangle=\langle:(\Delta a_i)^2:\rangle-\tfrac{1}{4}=\langle(\delta\mathcal{E}_i)^2\rangle-\tfrac{1}{4}, \quad i=1,2 \tag{5.7}$$

where $\Delta a_i=a_i-\langle a_i\rangle$, $\delta\mathcal{E}_i=\mathcal{E}_i-\langle\mathcal{E}_i\rangle$ $(i=1,2)$.

From the Fokker-Planck equation (5.4) one can derive the equations of motion for the expectation values $\langle\mathcal{E}_i\rangle$ of the quadrature operators $a_i$,

$$\frac{d}{dt}\langle\mathcal{E}_i\rangle=\langle d_i\rangle, \quad i=1,2 \tag{5.8}$$

and those for antinormally ordered quadrature variances $\langle(\delta\mathcal{E}_i)^2\rangle$ as well as for $\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle$

$$\frac{d}{dt}\langle(\delta\mathcal{E}_i)^2\rangle=2\langle d_i\delta\mathcal{E}_i\rangle+2\langle D_{ii}\rangle, \quad i=1,2 \tag{5.9a}$$

$$\frac{d}{dt}\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle=\langle d_1\delta\mathcal{E}_2\rangle+\langle d_2\delta\mathcal{E}_1\rangle+2\langle D_{12}\rangle \ . \tag{5.9b}$$

It is reasonable to assume that the quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2,t)$ is sharply peaked at the locking points, $(\mathcal{E}_1,\mathcal{E}_2)=(\mathcal{E}_{10},\mathcal{E}_{20})$, in steady state. Consequently one finds, by expanding $d_i$ in Eqs. (5.8) around $(\mathcal{E}_{10},\mathcal{E}_{20})$ up to first order in $\delta\mathcal{E}_1$ and $\delta\mathcal{E}_2$, that $\mathcal{E}_{10}$ and $\mathcal{E}_{20}$ satisfy the deterministic equations in steady state

$$d_i(\mathcal{E}_{10},\mathcal{E}_{20})=0, \quad i=1,2 \ . \tag{5.10}$$

Since $\phi_0$ in Eqs. (5.3) is one of the steady-state locked-phase values, one knows from $\mathcal{E}_0=\pm\sqrt{n_0}e^{i\phi_0}$ that

$$\begin{aligned}&\mathcal{E}_{10}^k=(-1)^k\sqrt{n_0}, \quad k=1,2\\&\mathcal{E}_{20}=0 \ .\end{aligned} \tag{5.11}$$

[Alternatively one can find $\mathcal{E}_{10}$ and $\phi_0$ from Eqs. (5.10) by requiring $\mathcal{E}_{20}=0$.] A comparison of Eqs. (5.5) with Eqs. (3.15) shows that, to leading order, at the steady-state locking points given in Eqs. (5.11)

$$d_1(\mathcal{E}_{10},\mathcal{E}_{20})=d_I(n_0,\phi_0)/2\sqrt{n_0} \ , \tag{5.12a}$$

$$d_2(\mathcal{E}_{10},\mathcal{E}_{20})=d_\phi(n_0,\phi_0)\sqrt{n_0} \ , \tag{5.12b}$$

which agree with Eqs. (4.6) and (5.10). Similarly, comparison of Eqs. (5.6) with Eqs. (3.14) gives the relations between the various diffusion coefficients at the steady-state locking points given in Eqs. (5.11) as

$$D_{11}(\mathcal{E}_{10},\mathcal{E}_{20})=D_{II}(n_0,\phi_0)/4n_0 \ , \tag{5.13a}$$

$$D_{22}(\mathcal{E}_{10},\mathcal{E}_{20})=D_{\phi\phi}(n_0,\phi_0)n_0 \ , \tag{5.13b}$$

$$D_{12}(\mathcal{E}_{10},\mathcal{E}_{20})=\tfrac{1}{2}D_{I\phi}(n_0,\phi_0) \ . \tag{5.13c}$$

The association of $\mathcal{E}_1$ with the intensity and $\mathcal{E}_2$ with the phase is obvious from Eqs. (5.11)–(5.13).

Just as in Sec. IV B, we consider actual two-photon resonant transition $(\omega_{ac}=2\nu)$ in the following. Substitution of Eqs. (3.10a), (5.3), and (4.22) into Eqs. (5.5) leads to

$$d_1=\mathcal{E}_1A_1-\mathcal{E}_2A_2 \ , \tag{5.14a}$$

$$d_2=\mathcal{E}_1A_2+\mathcal{E}_2A_1 \ , \tag{5.14b}$$

where

$$\begin{aligned}A_1=&[1+(\mathcal{E}_1^2+\mathcal{E}_2^2)^2\beta/\alpha]^{-1}\\&\times[\alpha(\rho_{aa}-\rho_{cc})(\mathcal{E}_1^2+\mathcal{E}_2^2)\\&+2|S\bar{\rho}_{ac}|(\mathcal{E}_1^2-\mathcal{E}_2^2)/(\mathcal{E}_1^2+\mathcal{E}_2^2)]-\tfrac{1}{2}\gamma \ ,\end{aligned} \tag{5.15a}$$

$$A_2=-4|S\bar{\rho}_{ac}|\mathcal{E}_1\mathcal{E}_2/(\mathcal{E}_1^2+\mathcal{E}_2^2) \ . \tag{5.15b}$$

Because of Eqs. (5.10) and (5.11) we note that

$$A_i(\mathcal{E}_{10},\mathcal{E}_{20})=0, \quad i=1,2 \ . \tag{5.16}$$

As we have done in Sec. IV, one can obtain an approximate steady-state quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ by linearizing the Fokker-Planck equation (5.4). Expanding the drift coefficients $d_1$ and $d_2$ around the two locking points $(\mathcal{E}_{10}^k,\mathcal{E}_{20})$ up to first order in $\delta\mathcal{E}_1^k=\mathcal{E}_1-\mathcal{E}_{10}^k$ and $\delta\mathcal{E}_2=\mathcal{E}_2-\mathcal{E}_{20}=\mathcal{E}_2$ (since $\mathcal{E}_{20}=0$), replacing the diffusion coefficients by their respective values at the locking points, and noting that $D_{12}(\mathcal{E}_{10},\mathcal{E}_{20})=0$ and

$$\frac{\partial d_i(\mathcal{E}_1,\mathcal{E}_{20})}{\partial\mathcal{E}_j}=\mathcal{E}_1\frac{\partial A_i(\mathcal{E}_1,\mathcal{E}_{20})}{\partial\mathcal{E}_j}=0\ ,\qquad i,j=1,2,\quad i\neq j\ , \tag{5.17}$$

we arrive at the linearized Fokker-Planck equation expressed in terms of quadrature variables $\mathcal{E}_1$ and $\mathcal{E}_2$,

$$\begin{aligned}\frac{\partial}{\partial t}Q(\mathcal{E}_1,\mathcal{E}_2,t)=&-\frac{\partial}{\partial\mathcal{E}_1}\left[\frac{\partial d_1(\mathcal{E}_{10}^k,\mathcal{E}_{20})}{\partial\mathcal{E}_1}\delta\mathcal{E}_1^k-D_{11}(\mathcal{E}_{10},\mathcal{E}_{20})\frac{\partial}{\partial\mathcal{E}_1}\right]Q(\mathcal{E}_1,\mathcal{E}_2,t)\\&-\frac{\partial}{\partial\mathcal{E}_2}\left[\frac{\partial d_2(\mathcal{E}_{10}^k,\mathcal{E}_{20})}{\partial\mathcal{E}_2}\mathcal{E}_2-D_{22}(\mathcal{E}_{10},\mathcal{E}_{20})\frac{\partial}{\partial\mathcal{E}_2}\right]Q(\mathcal{E}_1,\mathcal{E}_2,t),\quad k=\begin{cases}1 & \text{for } \mathcal{E}_1<0\\ 2 & \text{for } \mathcal{E}_2\geq 0\ .\end{cases}\end{aligned} \tag{5.18}$$

Here for both $k=1$ and $k=2$,

$$\begin{aligned}\frac{\partial d_1(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial\mathcal{E}_1}&=\mathcal{E}_{10}^k\frac{\partial A_1(\mathcal{E}_{10}^k,\mathcal{E}_{20})}{\partial\mathcal{E}_1}\\&=-\frac{4|S\bar{\rho}_{ac}|+\gamma(N_0^2-1)}{1+N_0^2}\\&=\frac{\partial d_I(n_0,\phi_0)}{\partial I}\ ,\end{aligned} \tag{5.19a}$$

$$\begin{aligned}\frac{\partial d_2(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial\mathcal{E}_2}&=\mathcal{E}_{10}^k\frac{\partial A_2(\mathcal{E}_{10}^k,\mathcal{E}_{20})}{\partial\mathcal{E}_2}=-4|S\bar{\rho}_{ac}|\\&=\partial d_\phi(\phi_0)/\partial\phi\ .\end{aligned} \tag{5.19b}$$

The steady-state solution of the linearized Fokker-Planck equation (5.18) is

$$\begin{aligned}Q(\mathcal{E}_1,\mathcal{E}_2)=N_{12}\sum_{k=1}^{2}\exp\Bigg[&-\frac{|\partial d_1(\mathcal{E}_{10}^k,\mathcal{E}_{20})/\partial\mathcal{E}_1|}{2D_{11}(\mathcal{E}_{10},\mathcal{E}_{20})}\\&\times(\mathcal{E}_1-\mathcal{E}_{10}^k)^2\\&-\frac{|\partial d_2(\mathcal{E}_{10}^k,\mathcal{E}_{20})/\partial\mathcal{E}_2|}{2D_{22}(\mathcal{E}_{10},\mathcal{E}_{20})}\mathcal{E}_2^2\Bigg]\ .\end{aligned} \tag{5.20}$$

Again, $N_{12}$ is a normalization constant. The quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ is made up of two identical elliptical peaks located at the two locking points $(\mathcal{E}_{10}^k,\mathcal{E}_{20})=(\pm\sqrt{n_0},0)$. The widths of the peaks in the $\mathcal{E}_i$ direction are

$$D_{ii}(\mathcal{E}_{10},\mathcal{E}_{20})/|\partial d_i(\mathcal{E}_{10},\mathcal{E}_{20})/\partial\mathcal{E}_i|\quad(i=1,2)\ .$$

We plot the quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ in Fig. 5 for the case $\rho_{aa}<\rho_{cc}$.

Because $Q(\mathcal{E}_1,\mathcal{E}_2)$ is well peaked at $\mathcal{E}_2=0$, the antinormally ordered $a_2$'s variance in the steady state, $\langle(\delta\mathcal{E}_2)^2\rangle$, is readily obtained from Eqs. (5.9a) by putting $d/dt=0$ and expanding $d_2$ and $D_{22}$ around the two locking points $(\pm\sqrt{n_0},0)$ up to first order in $\delta\mathcal{E}_1^k=\mathcal{E}_1-\mathcal{E}_{10}^k$ and $\delta\mathcal{E}_2=\mathcal{E}_2$. Owing to Eqs. (5.17), which means that the fluctuations in the amplitude $(a_1)$ and phase $(a_2)$ quadratures are decoupled, the resulting expression for $\langle(\delta\mathcal{E}_2)^2\rangle$ is simple. Substituting this expression into Eqs. (5.7) and using Eqs. (5.19b), (5.13b), and (4.26b) and the relation $\rho_{aa}+\rho_{cc}=1$, one finds the total steady-state variance of $a_2$ as

$$\begin{aligned}\langle(\Delta a_2)^2\rangle&=\frac{D_{22}(\mathcal{E}_{10},\mathcal{E}_{20})}{|\partial d_2(\mathcal{E}_{10},\mathcal{E}_{20})/\partial\mathcal{E}_2|}-\frac{1}{4}\\&=\frac{1+N_0|\bar{\rho}_{ac}|^{-1}(\rho_{aa}+\frac{1}{2}+N_0^2)}{8(1+N_0^2)}\ ,\end{aligned} \tag{5.21}$$

which is just $n_0\langle(\Delta\phi)^2\rangle$ given in Eq. (4.33). The total steady-state variance $\langle(\Delta a_1)^2\rangle$ can be obtained similarly which is large in general. Once again the same results for the variance $\langle(\Delta a_2)^2\rangle$ can also be obtained from the quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ in Eq. (5.20). The same discussion as that given at the end of Sec. IV B can be repeated here. The main conclusion is that a 50% squeezing in the $a_2$ quadrature [i.e., $\langle(\Delta a_2)^2\rangle=\frac{1}{8}$, which is only half the vacuum noise level] can be asymptotically reached when $1\ll n_0\ll\sqrt{\alpha/\beta}|\bar{\rho}_{ac}|$, and this is experimentally feasible [see Eq. (4.33)].

It is also interesting to find the minimum value of the quadrature (or phase) noise level at a given laser intensity. For a given $N_0$, $\langle(\Delta a_2)^2\rangle$ in Eq. (5.21) is a function of the initial atomic variables $\rho_{aa}$ and $|\bar{\rho}_{ac}|$. Under the maximum atomic coherence condition

$$|\bar{\rho}_{ac}|=(\rho_{aa}\rho_{cc})^{1/2}=[\rho_{aa}(1-\rho_{aa})]^{1/2}\ ,$$

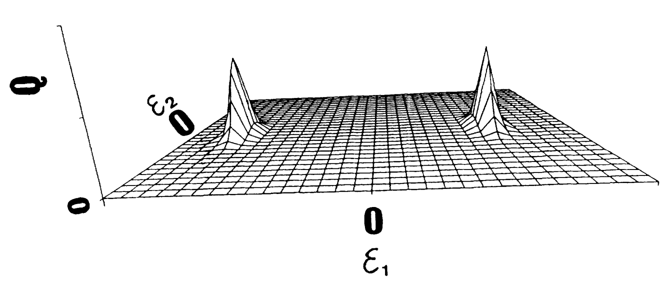

FIG. 5. Steady-state quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ of the two-photon CEL on the $\mathcal{E}$ plane $[\mathcal{E}=e^{i\phi_0}(\mathcal{E}_1+i\mathcal{E}_2)]$ for $\rho_{aa}<\rho_{cc}$.

the minimum value of $\langle(\Delta a_2)^2\rangle$ with respect to $\rho_{aa}$ can then be found to be

$$\langle(\Delta a_2)^2\rangle_{\min}=\frac{1+N_0(1+2N_0^2)^{1/2}(3+2N_0^2)^{1/2}}{8(1+N_0^2)}, \quad (5.22a)$$

when

$$\rho_{aa}=1-\rho_{cc}=\frac{1+2N_0^2}{4(1+N_0^2)}. \quad (5.22b)$$

[Since Eq. (4.7) must be satisfied, this requires that the cavity-loss rate $\gamma$ be adjusted according to Eq. (4.7).] This is in the $\rho_{aa}<\rho_{cc}$ region, in which $N_0 \ll 1$ is possible. Using Eqs. (5.22a) one can show easily that there exists squeezing of the $a_2$-quadrature (also phase) variance only when $N_0<1/\sqrt{2}$. Equation (5.22a) is plotted in Fig. 6. For $N_0 \ll 1$ (but keeping $n_0 \gg 1$), the $a_2$-quadrature variance increases linearly with $N_0$: $\langle(\Delta a_2)^2\rangle_{\min}=(1+\sqrt{3}N_0)/8$.

## VI. DISCUSSION AND SUMMARY

The two main effects of the two-photon CEL, stable phase locking and squeezing, both stem from the injected atomic coherence given by Eq. (2.16). That indeed this is the proper form of atomic coherence, necessary to achieve quantum noise quenching and squeezing, can be understood in the following way. Suppose that the initial atomic density matrix takes the following form:

$$\rho^j(t_j)=\begin{bmatrix}\rho_{aa} & \bar\rho_{ac}e^{-i2\omega_0 t_j}\\ \bar\rho_{ca}e^{i2\omega_0 t_j} & \rho_{cc}\end{bmatrix}, \quad j=1,2,\ldots \quad (6.1)$$

where $\rho_{aa},\rho_{cc},\bar\rho_{ac}=\bar\rho_{ca}^*$ are the same for all atoms but $\omega_0\neq\nu$. If we use this, instead of Eq. (2.16), as an initial condition we arrive at a master equation for the reduced field-density matrix, which is similar to Eq. (2.17) but with the following replacements in the $\bar\rho_{ac}$ and $\bar\rho_{ca}$ terms (including their denominators $\xi_{nm}$ and $\xi_{n-2,m-2}$):

$$\begin{aligned}&\bar\rho_{ac}=\bar\rho_{ca}^*\rightarrow\bar\rho_{ac}e^{i2(\nu-\omega_0)t},\\ &\Gamma\rightarrow\Gamma+i2(\nu-\omega_0)\ \text{ in terms containing } \bar\rho_{ac},\\ &\Gamma\rightarrow\Gamma-i2(\nu-\omega_0)\ \text{ in terms containing } \bar\rho_{ca},\\ &S\rightarrow\frac{r_a g^*}{\Gamma+i(\omega_{ac}-2\omega_0)}.\end{aligned} \quad (6.2)$$

When all injected atoms have the same initial coherence ($\omega_0=0$) then $S\rightarrow 0$ since $\omega_{ac}\gg\Gamma$. When $\omega_0$ is close but not equal to $\nu$, the phase $\theta$ of the atomic coherence [see its definition after Eqs. (3.17)] is replaced by $\theta+2(\nu-\omega_0)t$ in all the equations of Secs. III–V, as indicated by the first relation of (6.2). Consequently, no stationary phase locking is possible. Furthermore, the phase-diffusion coefficients $D_{\phi\phi}$ would vary periodically with time and any noise quenching and squeezing would be averaged out on a time scale $|\omega_0-\nu|^{-1}$.

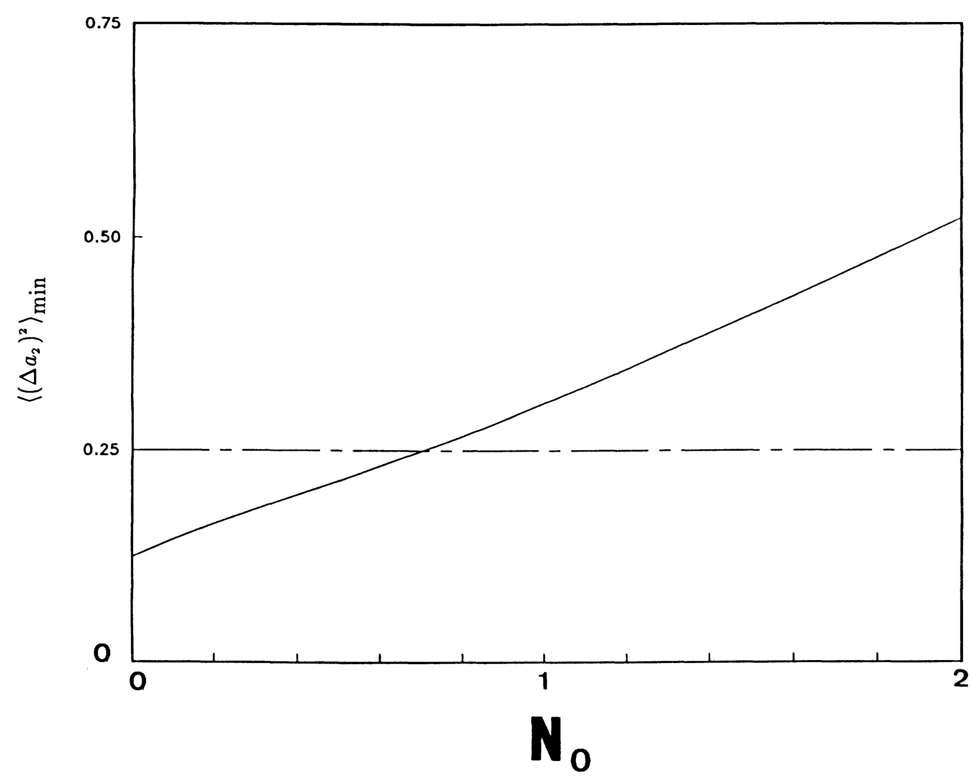


FIG. 6. Minimum variance in the $a_2$ quadrature (corresponding to the laser phase) as a function of the normalized laser intensity $N_0=\sqrt{\beta/\alpha}n_0$. The long-dash–short-dash line represents the vacuum noise level $\langle(\Delta a_2)^2\rangle=\frac{1}{4}$.

Introducing the vector model described by the two-photon optical Bloch equations[31,32] it is possible to give a simple intuitive physical picture of the two-photon CEL in a semiclassical framework, same as for a one-photon laser with injected atomic coherence.[23]

We focus on steady-state laser operation in which the phase $\phi$ is locked to one of stable phase values $\phi_0^k$. The semiclassical version of the interaction Hamiltonian (2.9) is given by

$$V_j^{cl}=gn_0 e^{-i2(\nu t-\phi_0)}|a^j\rangle\langle c^j|+\text{H.c.}\ , \tag{6.3}$$

where $gn_0$ is half the Rabi frequency of the two-photon transition. The Bloch vector of the $j$th atom satisfies the following equations of motion:

$$\frac{d\mathbf{B}^j}{dt}=\mathbf{\Omega}_B\times\mathbf{B}^j-\Gamma\mathbf{B}^j\ , \tag{6.4}$$

where

$$\mathbf{B}^j=\begin{bmatrix}u^j\\ v^j\\ w^j\end{bmatrix}=\begin{bmatrix}\rho_{ac}^j e^{i2\nu t}+\rho_{ca}^j e^{-i2\nu t}\\ i(\rho_{ac}^j e^{i2\nu t}-\rho_{ca}^j e^{-i2\nu t})\\ \rho_{aa}^j-\rho_{cc}^j\end{bmatrix} \tag{6.5a}$$

is the Bloch vector of the $j$th atom for the two-photon transition and

$$\mathbf{\Omega}_B=\begin{bmatrix}2gn_0\cos(2\phi_0)\\ -2gn_0\sin(2\phi_0)\\ \Delta\end{bmatrix} \tag{6.5b}$$

is the two-photon driving-field vector (with $g>0$ now). Substitution of the initial condition (2.16) into Eq. (6.5a) leads to

$$\mathbf{B}^j(t_j)=\begin{bmatrix}2|\bar{\rho}_{ac}|\cos\theta\\ -2|\bar{\rho}_{ac}|\sin\theta\\ \rho_{aa}-\rho_{cc}\end{bmatrix}\ , \tag{6.6}$$

independent of $j$ (see Fig. 7). Thus, the initial condition (2.16) ensures that in the frame rotating at the frequency of the two-photon transition (i.e., in the Bloch vector space), all the injected atoms have the *same* initial condition and follow the same trajectories in the Bloch vector space. On the other hand, for the initial condition (6.1) different atoms have different initial oscillations and follow different trajectories in the Bloch vector space and the effects sensitive to the phase of the initial coherence (i.e., phase locking and noise reduction) average out to zero.

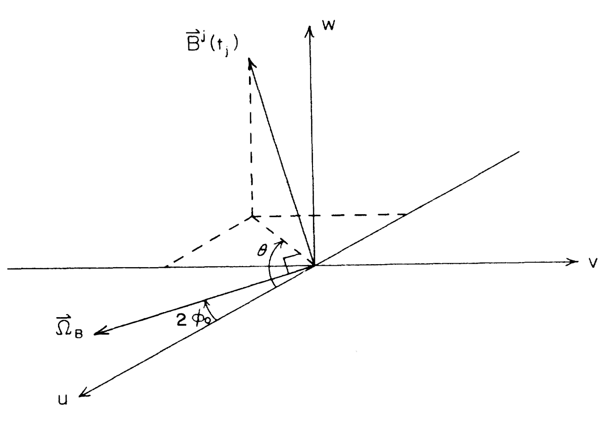


FIG. 7. Initial Bloch vector $\mathbf{B}^j(t_j)$ for the $j$th atom and steady-state driving-field vector $\mathbf{\Omega}_B$ for the laser field at one of stable-phase values $\phi_0^k$, $\theta-2\phi_0^k=\frac{1}{2}\pi$, in the Bloch vector space.

Besides providing an intuitive physical picture for understanding the role of the phase of the initial atomic coherence, the Bloch equations (6.4) also yield a similar simple picture for the time evolution of the atomic-density-matrix elements and for the relation between the phase $\theta$ of the atomic coherence and laser phase $\phi_0$. First, notice that, owing to atomic decay, a physical process happening earlier has a larger probability than that happening later and thus dominates. We consider the two-photon resonance case, $\Delta=0$, in which the driving vector $\mathbf{\Omega}_B$ is in the $uv$ plane (see Fig. 7). Since

$$\mathbf{\Omega}_B\cdot\mathbf{B}^j(t_j)=4g|\bar{\rho}_{ac}|\cos(\theta-2\phi_0)\ , \tag{6.7}$$

the condition of orthogonality is $\cos(\theta-2\phi_0)=0$, the same as the steady-state condition from Eq. (4.20), i.e., $\phi_0=\frac{1}{2}\theta+\frac{3}{4}\pi\pm l\pi/2$ ($l=0,1,\dots$). The driving vector rotates the Bloch vectors $\mathbf{B}^j$ ($j=1,2,\dots$) downward to $u=v=0$, $w<0$ (i.e., to $\rho_{aa}^j=0$) for even $l$ ($0,2,\dots$) and the laser intensity is increased, due to stimulated emission of the active atoms. Thus the $\phi_0$ values for even $l$ represent stable steady states for the field; stimulated emission gain compensates for the losses. On the other hand for odd $l$ ($1,3,\dots$) the Bloch vectors $\mathbf{B}^j$ ($j=1,2,\dots$) are first rotated upward to $u=v=0$, $w>0$ (i.e., to $\rho_{cc}^j=0$) and the laser intensity is decreased due to absorption. No steady state other than zero for the field is possible in this case. The stable steady-state values (even $l$) coincide with the values found in the fully quantized treatment in Eq. (4.22). Since for these values the Bloch vectors $\mathbf{B}^j$ are rotated downwards, independent of the relationship between $\rho_{aa}$ and $\rho_{cc}$, no inversion is necessary to have positive gain in the two-photon CEL. This again supports the conclusion reached in conjunction with Eq. (4.23). Also classically, if two harmonic oscillators with phases $\phi_1$ and $\phi_2$ are coupled, then the first oscillator gains energy from the second one if $\pi<\phi_1-\phi_2<2\pi$ and the second oscillator gains energy if $0<\phi_1-\phi_2<\pi$. In our case the atomic oscillator has phase $\theta$ and the laser oscillator has plane $2\phi_0$, and indeed, there is gain for the laser if $\theta-2\phi_0=\frac{1}{2}\pi$ and for the atoms if $\theta-2\phi_0=\frac{3}{2}\pi$.

It is interesting to compare our two-level–two-photon CEL model with a very similar model of the degenerate parametric oscillator investigated by Milburn and Walls.[33] In our case the two-photon transition is driven by atoms injected into the cavity in a coherent superposition of the levels involved in the lasing transition. In Ref. 33, however, the transition is driven by a coherent external field with amplitude ε (but the noise is treated quantum mechanically). In their paper an effective phenomenological Hamiltonian based on a nonlinear second-order susceptibility $\chi$ ($>0$) is used and the resulting Fokker-Planck equation for the field generated in the degenerate parametric oscillator is very similar to ours. Since their model is, strictly speaking, valid only below threshold the net gain is negative and also depletion of the pump field is neglected. Treating $\mathcal{E}$ and $\mathcal{E}^*$ no longer as complex conjugate to each other and keeping terms up to $g^2$ (i.e., dropping saturation terms), our Fokker-Planck equation (A5) is reduced to a linear Fokker-Planck equation in the generalized (complex) $P$ representation.[34] Comparing this equation with the Fokker-Planck equation (3) of Ref. 33, we find that the two Fokker-Planck equations are exactly the same if we identify the following correspondences (assuming $\omega_{ac}=2\nu=2\Omega$):

$$\begin{aligned}
&\rho_{aa}=1-\rho_{cc}\leftrightarrow 0\ ,\\
&r_a|\bar{\rho}_{ac}|\leftrightarrow \varepsilon\ ,\\
&2r_a|g|/\Gamma\leftrightarrow\chi\ ,\\
&r_a\leftrightarrow\gamma_2\ ,\\
&\gamma\leftrightarrow 2\gamma_1\ ,
\end{aligned} \tag{6.8}$$

where $\gamma_i$ are the cavity damping rates for the signal ($i=1$) and pump ($i=2$) modes used in Ref. 33. On the other hand, it is precisely those saturation terms that render our equation valid above threshold and lead to the possibility of squeezing in the above threshold region.

Before summarizing our results in this paper, we point out that in the deviation of our master equation for the reduced-field-density operator, Eq. (2.17), we introduced the coarse-grained-time-rate approximation as discussed after Eq. (2.15). If, instead, we carry out a more elaborate analysis (e.g., using the projection operator technique[35] to eliminate atomic variables) we find extra terms in Eq. (2.17) which contain $\bar{\rho}_{ac}^2$. While they reduce the fluctuations in the amplitude quadrature of the laser field (e.g., in a linear theory[36] of a coherently pumped one-photon laser[23]), these coherence-square terms do not affect the variance in the phase quadrature nor the amplitude- and phase-drift coefficients. Our main conclusions (concerning the phase noise) in this paper remain unchanged when these additional terms are included.

In summary, we have developed a quantum theory of the two-photon correlated-spontaneous-emission laser. As compared to standard treatments of the two-photon laser,[4,8,11] the new ingredient in our approach is the inclusion of the possibility that active atoms enter in the interaction region in a coherent superposition of the levels involved in the lasing transition. The standard treatments deal only with incoherent pumping, with no atomic coherence between the lasing levels. As a consequence, the possibility of generating squeezed states of light in an incoherently pumped two-photon laser has been ruled out.[11,12] We show here that it is precisely the injected atomic coherence that leads to phase locking, quantum-noise quieting, and even squeezing of the quantum noise in the phase quadrature in our treatment.

Starting from a frequently used effective Hamiltonian, the so-called two-level–two-photon laser model, we first derive (Sec. II) the master equation for the reduced-field-density operator of the two-photon CEL using the appropriate generalization[23] of the Scully-Lamb theory[24] to include the effect of atomic coherence into the laser operation. In the next step we convert this nonlinear master equation into a Fokker-Planck equation using both the antinormal-ordering $Q$ representation (Sec. III) and the Glauber-Sudarshan (normal-ordering) $P$ representation (Appendix A) for the density operator. Both the $Q$ and $P$ representations give the same drift coefficients (to leading order) but different diffusion coefficients. Steady-state laser operation is studied by using the drift coefficients of the laser intensity and phase in Sec. IV. As compared to the ordinary two-photon laser (i.e., the one without the injected atomic coherence) the difference are the following: (1) the phase locks to one of the stable values given by Eq. (4.22); (2) due to an extra two-photon CEL gain at the locking points, there is no need for triggering when linear gain $4|S\bar{\rho}_{ac}|>\gamma$. The noise properties of the two-photon CEL are investigated by using the $Q$ function. By linearizing the Fokker-Planck equation (3.13) around the steady-state locking points, we obtain the steady-state quasidistribution $Q_2(I,\phi)$ and find the phase variance in the steady state. Under the conditions of Eq. (4.33) the noise in the phase variable is squeezed and asymptotically approaches one half of the corresponding value for a coherent state with the same mean photon number (50% squeezing). In Sec. V we investigate the squeezing properties of the system further, in a more rigorous manner. Using Hermitian-quadrature operators $a_1$ and $a_2$ of the field we obtain an approximate steady-state quasidistribution $Q(\mathcal{E}_1,\mathcal{E}_2)$, which consists of two elliptically shaped Gaussian peaks located at $\mathcal{E}_1=\pm\sqrt{n_0}$, $\mathcal{E}_2=0$, and find that the percentage of squeezing in the $a_2$ quadrature is the same as that in the laser phase. In the case of stable phase locking, the $a_2$ quadrature plays a role analogous to that of the phase. We also calculate the minimum noise in the $a_2$ quadrature at any given mean photon number $n_0$ and find that there is no squeezing in the $a_2$ quadrature when $n_0>\sqrt{\alpha/2\beta}$. A maximum of 50% squeezing is found again when $1\ll n_0\ll\sqrt{\alpha/\beta}|\bar{\rho}_{ac}|$. We stress that a proper form of the initial atomic coherence [see Eq. (2.16)] is vital to such laser-phase locking and the squeezing of laser-phase noise. To achieve this goal all injected atoms should satisfy the relation $\rho_{ac}^j(t_j)=\bar{\rho}_{ac}e^{-i2\nu t_j}$, where $\bar{\rho}_{ac}$ is the same for all atoms ($\nu$ is the laser frequency and $t_j$ is the injection time of the $j$th atom), so that all injected atoms have the same phase with respect to the instantaneous total laser phase (i.e., including the frequency part $2\nu t$).

As a by-product of the study on the two-photon CEL, we also investigate an ordinary two-photon laser and obtain a steady-state mean photon number, laser frequency pulling relation, laser natural linewidth, and photon-number variance.

## ACKNOWLEDGMENTS

We gratefully acknowledge the support of the U.S. Office of Naval Research. We also gratefully thank M. O. Scully and C. Benkert for useful discussions.

## APPENDIX A: FOKKER-PLANCK EQUATION IN THE GLAUBER-SUDARSHAN P REPRESENTATION

We convert the master equation (2.17) for the reduced-field-density operator $\rho$ into a Fokker-Planck equation for the Glauber-Sudarshan $P$ representation[19] of the density operator $\rho$ in this appendix. Expanding the field operator $\rho$ in terms of the diagonal $P$ representation, one finds the expansion for the field-density-matrix elements,

$$\rho_{nm}=\int d^2\mathcal{E}P(\mathcal{E},\mathcal{E}^*,t)e^{-|\mathcal{E}|^2}\frac{\mathcal{E}^n(\mathcal{E}^*)^m}{\sqrt{n!m!}} . \tag{A1}$$

Assuming that the mean photon number $n_0$ of the two-photon CEL is much larger than 1, we can neglect 1 compared to $|\mathcal{E}|^2$ and obtain the following equation of motion for $P(\mathcal{E},\mathcal{E}^*,t)$:

$$\begin{aligned}\frac{\partial P(\mathcal{E},\mathcal{E}^*,t)}{\partial t}=&\Bigg\{-\alpha(\rho_{aa}-\rho_{cc})\left[(1-i\delta)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2-\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^2\right]+\text{c.c.}\right]\\&-\tfrac{1}{2}(1+\delta^2)\beta(\rho_{aa}+\rho_{cc})\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2-\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^*}\mathcal{E}^2-\text{c.c.}\right]^2+4\alpha\rho_{aa}\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}|\mathcal{E}|^2\Bigg\}M(\mathcal{E},\mathcal{E}^*,t)\\&+\Bigg\{iS\bar{\rho}_{ac}\left[2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}^*-\frac{\partial^2}{\partial\mathcal{E}^2}\right]+iS\bar{\rho}_{ac}\frac{\beta(1+i\delta)}{4\alpha}\left[(\mathcal{E}^*)^2+\left[\mathcal{E}^*-\frac{\partial}{\partial\mathcal{E}}\right]^2\right]\\&\qquad\times\left[\left[\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\right]^2(\mathcal{E}^*)^2-\mathcal{E}^2\left[\mathcal{E}^*-\frac{\partial}{\partial\mathcal{E}}\right]^2\right]+\text{c.c.}\Bigg\}M(\mathcal{E},\mathcal{E}^*,t)\\&+\frac{\gamma}{2}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]P(\mathcal{E},\mathcal{E}^*,t)+i(\Omega-\nu)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]P(\mathcal{E},\mathcal{E}^*,t) ,\end{aligned} \tag{A2}$$

with

$$\begin{aligned}M(\mathcal{E},\mathcal{E}^*,t)=&\Bigg[1+\frac{\beta}{\alpha}|\mathcal{E}|^4-\frac{\beta}{\alpha}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2-\frac{1}{2}\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}^*+\text{c.c.}\right]\\&+\frac{\beta^2(1+\delta^2)}{16\alpha^2}\left[2\frac{\partial}{\partial\mathcal{E}}\mathcal{E}|\mathcal{E}|^2-\frac{\partial^2}{\partial\mathcal{E}^2}\mathcal{E}-\text{c.c.}\right]^2\Bigg]^{-1}P(\mathcal{E},\mathcal{E}^*,t) .\end{aligned} \tag{A3}$$

Equation (A2) contains derivatives of all orders in $\mathcal{E}$ and $\mathcal{E}^*$ due to the presence of the inverse operator in $M(\mathcal{E},\mathcal{E}^*,t)$. Upon the expansion of $M(\mathcal{E},\mathcal{E}^*,t)$ up to first order in the derivaties we obtain

$$M(\mathcal{E},\mathcal{E}^*,t)=\frac{P(\mathcal{E},\mathcal{E}^*,t)}{1+|\mathcal{E}|^4\beta/\alpha}+\frac{\beta}{\alpha(1+|\mathcal{E}|^4\beta/\alpha)}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]\frac{|\mathcal{E}|^2P(\mathcal{E},\mathcal{E}^*,t)}{1+|\mathcal{E}|^4\beta/\alpha} . \tag{A4}$$

Substituting Eq. (A4) into Eq. (A2), keeping terms up to second order in the derivatives, and neglecting 1 compared to $|\mathcal{E}|^2$, we arrive at the Fokker-Planck equation in the $P$ representation

$$\frac{\partial P(\mathcal{E},\mathcal{E}^*,t)}{\partial t}=\left[-\frac{\partial}{\partial\mathcal{E}}d_{\mathcal{E}}^P+\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}D_{\mathcal{E}^*\mathcal{E}}^P+\frac{\partial^2}{\partial\mathcal{E}^2}D_{\mathcal{E}\mathcal{E}}^P+\text{c.c.}\right]P(\mathcal{E},\mathcal{E}^*,t) , \tag{A5}$$

where the drift and diffusion coefficients in the $P$ representation are

$$d_{\mathcal{E}}^P=\{\alpha\mathcal{E}|\mathcal{E}|^2[(\rho_{aa}-\rho_{cc})(1-i\delta)-2i(g\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma]-2iS\bar{\rho}_{ac}\mathcal{E}^*\}/(1+|\mathcal{E}|^4\beta/\alpha)-\tfrac{1}{2}\gamma\mathcal{E}+i(\nu-\Omega)\mathcal{E} , \tag{A6}$$

$$\begin{aligned}D_{\mathcal{E}^*\mathcal{E}}^P=&\frac{\alpha|\mathcal{E}|^2}{1+|\mathcal{E}|^4\beta/\alpha}\left[2\rho_{aa}+(1+\delta^2)(\rho_{aa}+\rho_{cc})\frac{\beta|\mathcal{E}|^4}{2\alpha}-(ig\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma\right]\\&-\frac{\beta|\mathcal{E}|^6(\rho_{aa}-\rho_{cc})+(iS^*\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})|\mathcal{E}|^2\beta/\alpha}{(1+|\mathcal{E}|^4\beta/\alpha)^2} ,\end{aligned} \tag{A7a}$$

$$\begin{aligned}D_{\mathcal{E}\mathcal{E}}^P=&\frac{1-|\mathcal{E}|^4\beta/\alpha}{(1+|\mathcal{E}|^4\beta/\alpha)^2}\{\tfrac{1}{2}\alpha\mathcal{E}^2[(\rho_{aa}-\rho_{cc})(1-i\delta)-(2ig\bar{\rho}_{ca}\mathcal{E}^2+\text{c.c.})/\Gamma]-iS\bar{\rho}_{ac}\}\\&-\frac{\beta(1+\delta^2)(\rho_{aa}+\rho_{cc})|\mathcal{E}|^4\mathcal{E}^2}{2(1+|\mathcal{E}|^4\beta/\alpha)}-\frac{4i\alpha g^*\bar{\rho}_{ac}|\mathcal{E}|^4}{\Gamma(1+|\mathcal{E}|^4\beta\alpha)^2} .\end{aligned} \tag{A7b}$$

Rewriting the Fokker-Planck equation (A5) in terms of intensity and phase variables, $I$ and $\phi$, through the relation $\mathcal{E}=\sqrt{I}e^{i\phi}$, one has

$$\frac{\partial P(I,\phi,t)}{\partial t}=\left[-\frac{\partial}{\partial I}d_I^P-\frac{\partial}{\partial\phi}d_\phi^P+\frac{\partial^2}{\partial I^2}D_{II}^P+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}^P+2\frac{\partial^2}{\partial I\partial\phi}D_{I\phi}^P\right]P(I,\phi,t)\ , \tag{A8}$$

where the drift and diffusion coefficients for the intensity and phase are readily found by using Eqs. (3.14), (3.15), (A6), and (A7) and $g^*\bar{\rho}_{ac}=|g^*\bar{\rho}_{ac}|e^{i\theta}$,

$$d_I^P=(G^P-\gamma)I\ , \tag{A9a}$$

$$d_\phi^P=\nu-\Omega-\{\alpha I(\rho_{aa}-\rho_{cc})\delta+2|S\bar{\rho}_{ac}|[\cos(\theta-2\phi-\arctan\delta)+(1+\delta^2)^{1/2}(I^2\beta/\alpha)\cos(\theta-2\phi)]\}/(1+I^2\beta/\alpha)\ , \tag{A9b}$$

$$G^P=\frac{2\alpha I(\rho_{aa}-\rho_{cc})+4|S\bar{\rho}_{ac}|\sin(\theta-2\phi-\arctan\delta)}{1+I^2\beta/\alpha}\ , \tag{A9c}$$

$$D_{II}^P=\frac{\alpha I^2}{1+I^2\beta/\alpha}\left[\frac{4(\rho_{aa}-\rho_{cc})}{1+I^2\beta/\alpha}+\rho_{aa}+3\rho_{cc}\right]+\frac{2|S\bar{\rho}_{ac}|I(1-3I^2\beta/\alpha)}{(1+I^2\beta/\alpha)^2}\sin(\theta-2\phi-\arctan\delta)\ , \tag{A10a}$$

$$\begin{aligned}D_{\phi\phi}^P=&\frac{\alpha}{4(1+I^2\beta/\alpha)}[\rho_{aa}-\rho_{cc}-2(\rho_{aa}+\rho_{cc})\delta^2]+\tfrac{1}{2}\alpha(1+\delta^2)(\rho_{aa}+\rho_{cc})\\&-\frac{|S\bar{\rho}_{ac}|}{2I(1+I^2\beta/\alpha)}[\sin(\theta-2\phi-\arctan\delta)+2\sqrt{1+\delta^2}(I^2\beta/\alpha)\sin(\theta-2\phi)]\ ,\end{aligned} \tag{A10b}$$

$$\begin{aligned}D_{I\phi}^P=&-\frac{\alpha I(\rho_{aa}-\rho_{cc})(1-I^2\beta/\alpha)\delta}{2(1+I^2\beta/\alpha)}-\frac{|S\bar{\rho}_{ac}|}{(1+I^2\beta/\alpha)^2}[(1-I^2\beta/\alpha)\cos(\theta-2\phi-\arctan\delta)\\&+2\sqrt{1+\delta^2}(I^2\beta/\alpha)\cos(\theta-2\phi)]\ .\end{aligned} \tag{A10c}$$

When $\Delta=0$, comparison of Eqs. (A10) with Eqs. (3.19) shows that $D_{II}+D_{II}^P$ and $D_{\phi\phi}+D_{\phi\phi}^P$ are independent of $|\bar{\rho}_{ac}|$, $\phi$, and $D_{I\phi}+D_{I\phi}^P=0$.

The corresponding equations for an incoherently pumped two-photon laser can be obtained from the above equations by setting $\bar{\rho}_{ac}=\bar{\rho}_{ca}^*=0$ in Eqs. (A2), (A6a), (A7), (A9), and (A10). At the stable locking point $I=n_0$ given by Eq. (4.10), one finds from Eq. (A10b),

$$D_{\phi\phi}^P(n_0)=\frac{4\alpha n_0(\rho_{aa}+\rho_{cc})+\gamma}{8n_0}\ . \tag{A11}$$

Using the Fokker-Planck equation (A8), one can calculate the steady-state photon-number variance[23] once again. The result is the same as that in Eq. (4.17).

## APPENDIX B: COMPARISON WITH REF. 18 AND DYNAMIC STARK SHIFT

We make comparison of the present work with Ref. 18 in the region where both are valid. The present nonlinear theory of the two-photon CEL is developed by using the effective atom-field interaction Hamiltonian (2.8) for a two-photon transition, which is valid when the intermediate level $b$ is far off resonance with one-photon transition. On the other hand, the linear theory of Ref. 18 was derived from an exact interaction Hamiltonian of a cascade three-level atom interacting with a single-mode laser field. Consequently the comparison should be made in the regime of linear theory and large one-photon detunings. We consider the actual two-photon resonant transition, $\Delta=\omega_{ac}-2\nu=0$, in the following comparison.

The linear theory of the two-level–two-photon CEL may be obtained by expanding the drift and diffusion coefficients in the present nonlinear theory in terms of the coupling constant $g$ and dropping terms containing $g^n$ ($n\geq 2$). From Eqs. (3.17) and (3.19) one finds that the drift and diffusion coefficients for the intensity and phase reduce to the following expression in the linear theory of the two-level–two-photon CEL:

$$G=4S|\bar{\rho}_{ac}|\sin(\theta-2\phi)\ , \tag{B1a}$$

$$d_\phi=\nu-\Omega-2S|\bar{\rho}_{ac}|\cos(\theta-2\phi)\ , \tag{B1b}$$

$$D_{II}=[\gamma-2S|\bar{\rho}_{ac}|\sin(\theta-2\phi)]I\ , \tag{B2a}$$

$$D_{\phi\phi}=[\gamma+2S|\bar{\rho}_{ac}|\sin(\theta-2\phi)]/4I\ , \tag{B2b}$$

$$D_{I\phi}=S|\bar{\rho}_{ac}|\cos(\theta-2\phi)\ . \tag{B2c}$$

Here $S=r_ag/\Gamma$, $g$ real, and $\theta=\arg\bar{\rho}_{ac}$. From Eq. (4.33) one finds the steady-state phase variance by using Eqs. (B1b) and (B2b)

$$\langle(\Delta\phi)^2\rangle=\frac{1}{4n_0}\left[\frac{\gamma}{4|S\bar{\rho}_{ac}|}-\frac{1}{2}\right]\simeq\frac{1}{8n_0}\ , \tag{B3}$$

since $G(\phi_0)=4|S\bar{\rho}_{ac}|\approx\gamma$.

The above expressions are valid when Eq. (2.8) is valid and $|\Delta_1|\gg\Gamma$, i.e.,

$$|\Delta_1|\gg g_1\sqrt{n_0},\Gamma\ , \tag{B4}$$

where $\Delta_1=\omega_{ab}-\nu=\nu-\omega_{bc}$ is the one-photon detuning of the intermediate level $b$, and $g_1$ is the atom-field coupling constants for the $a$-$b$ and $b$-$c$ transitions (taken to be the same).

The drift and diffusion coefficients given by the linear theory of Ref. 18 take the following form in the limit $|\Delta_1| \gg \Gamma$ (in the notation of this paper):

$$G^P = 4(r_a g_1^2/\Delta_1\Gamma)|\bar{\rho}_{ac}|\sin(\theta-2\phi)\ , \tag{B5a}$$

$$d_\phi^P = \nu - \Omega - (r_a g_1^2/\Delta_1\Gamma) - 2(r_a g_1^2/\Delta_1\Gamma)|\bar{\rho}_{ac}|\cos(\theta-2\phi)\ , \tag{B5b}$$

$$D_{\phi\phi}^P = -(2I)^{-1}(r_a g_1^2/\Delta_1\Gamma)|\bar{\rho}_{ac}|\sin(\theta-2\phi)\ . \tag{B6}$$

These expression lead to $\langle(\Delta\phi)^2\rangle = (8n_0)^{-1}$ if one insists on calculating the phase variance in the Glauber-Sudarshan $P$ representation, in agreement with Eq. (B3).

Comparison of Eqs. (B1) with (B5) shows that (1) the two sets of drift coefficients agree with each other if we identify the effective coupling constant $g$ by

$$g = g_1^2/\Delta_1\ , \tag{B7}$$

and (2) the effective interaction Hamiltonian $V_j$ in Eq. (2.8) does not include a dynamic Stark shift. When the dynamic Stark shift is included, one should set the cavity-mode frequency $\Omega = \frac{1}{2}\omega_{ac} - S$ in order to achieve an actual resonant two-photon transition (i.e., $\omega_{ac} = 2\nu$) in a two-photon laser device.

*Permanent address: Central Research Institute for Physics, H-1525 Budapest, P.O. Box 49, Hungary.

[1]P. P. Sorokin and N. Braslau, IBM J. Res. Dev. **8**, 177 (1964).
[2]A. M. Prokhorov, Science **149**, 828 (1965).
[3]Y. R. Shen, Phys. Rev. **155**, 921 (1967).
[4]K. J. McNeil and D. F. Walls, J. Phys. A **7**, 617 (1974); **8**, 104 (1975); **8**, 111 (1975).
[5]H. P. Yuen, Phys. Rev. A **13**, 226 (1976).
[6]L. M. Narducci, W. W. Eidson, P. Furcinitti, and D. C. Eteson, Phys. Rev. A **16**, 1665 (1977).
[7]N. Nayak and B. K. Mohanty, Phys. Rev. A **19**, 1204 (1979).
[8]M. S. Zubairy, Phys. Lett. **80A**, 225 (1980).
[9]B. Nikolaus, D. Z. Zhang, and P. E. Toschek, Phys. Rev. Lett. **47**, 171 (1981).
[10]M. Reid, K. J. McNeil, and D. F. Walls, Phys. Rev. A **24**, 2029 (1981).
[11]L. A. Lugiato and G. Strini, Opt. Commun. **41**, 374 (1982).
[12]M. D. Reid and D. F. Walls, Phys. Rev. A **28**, 332 (1983).
[13]Z. C. Wang and H. Haken, Z. Phys. B **55**, 361 (1984); **56**, 77 (1984); **56**, 83 (1984).
[14]M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. A **35**, 154 (1987); L. Davidovich, J. M. Raimond, M. Brune, and S. Haroche, *ibid.* **36**, 3771 (1987).
[15]S. Y. Zhu and X. S. Li, Phys. Rev. A **36**, 3889 (1987).
[16]M. Brune, J. M. Raimond, P. Goy, L. Davidovich, and S. Haroche, Phys. Rev. Lett. **59**, 1899 (1987).
[17]For reviews see D. F. Walls, Nature (London) **306**, 141 (1983); R. Loudon and P. Knight, J. Mod. Opt. **34**, 709 (1987).
[18]M. O. Scully, K. Wódkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Vehn, Phys. Rev. Lett. **60**, 1832 (1988).
[19]R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. M. DeWitt *et al.* (Gordon and Breach, New York, 1965); Phys. Rev. **131**, 2766 (1963).
[20]P. Alsing and M. S. Zubairy, J. Opt. Soc. Am. B **4**, 177 (1987).
[21]H. I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).
[22]M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).
[23]N. Lu and J. A. Bergou (unpublished).
[24]M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967); M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).
[25]Y. Kano, J. Math. Phys. **6**, 1913 (1965); C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[26]M. S. Zubairy, Phys. Rev. A **20**, 2464 (1979).
[27]M. Lax and W. H. Louisell, Phys. Rev. **185**, 568 (1969); W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[28]I. Ashraf and M. S. Zubairy (unpublished).
[29]M. A. M. Marte and D. F. Walls, Phys. Rev. A **37**, 1235 (1988).
[30]J. Bergou, M. Orszag, M. O. Scully, and K. Wódkiewicz (unpublished).
[31]For two-photon transitions in a three-level atomic system, see R. G. Brewer and E. L. Hahn, Phys. Rev. A **11**, 1641 (1975); D. Grischkowsky, M. M. T. Loy, and P. F. Liao, *ibid.* **12**, 2514 (1975); D. Grischkowsky and R. G. Brewer, *ibid.* **15**, 1789 (1977); F. T. Hioe and J. H. Eberly, *ibid.* **25**, 2168 (1982); F. T. Hioe, *ibid.* **28**, 879 (1983).
[32]Corresponding to the effective two-photon Hamiltonian in Eq. (2.8), the two-photon Bloch equations are in effect two-level–one-photon Bloch equations. For the two-level Bloch equations, see R. P. Feynman, F. L. Vernon, and R. W. Hellwarth, J. Appl. Phys. **28**, 49 (1957).
[33]G. Milburn and D. F. Walls, Opt. Commun. **39**, 401 (1981).
[34]P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980); P. D. Drummond, C. W. Gardiner, and D. F. Walls, Phys. Rev. A **24**, 914 (1981).
[35]G. S. Agarwal, Phys. Rev. A **34**, 4055 (1986).
[36]C. Benkert, M. O. Scully, J. Bergou, and N. Lu (unpublished).