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Hyperfine splitting in muonium, positronium, and hydrogen, deduced from a solution of Dirac's equation in Kerr-Newman geometry

C. L. Pekeris
Department of Applied Mathematics, The Weizmann Institute, Rehovot, Israel

K. Frankowski
Computer Science Department, University of Minnesota, Minneapolis, Minnesota 55455
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The nucleus is represented as a Kerr-Newman source, under the assumption that the angular momentum of the source is equal to the intrinsic spin angular momentum of the nucleus. The theoretical values of the hyperfine splitting obtained from a solution of Dirac's equation for this model of the muon, the positron, and the proton agree with the observed values in muonium, positronium, and hydrogen, to within the uncertainty in the respective QED corrections—except for an unaccounted factor of 2. In hydrogen one also has to introduce the observed value of the magnetic moment of the proton. The singularity in Dirac's radial differential equations is shifted from the origin, where it is located in the case of flat space, to a pair of conjugate points on the imaginary axis of the radial coordinate. Consequently, the energy eigenvalues have a trace of an imaginary part, making the bound states of our model unstable.

I. INTRODUCTION

The Kerr[1]-Newman[2] metric is a solution of the coupled Einstein-Maxwell field equations for a source possessing mass, charge, and an intrinsic angular momentum. In this paper we wish to explore the consequences of treating the atomic nucleus as a Kerr-Newman source under the assumption[3] that the angular momentum of the source is to be identified with the intrinsic spin angular momentum $I_N\hbar$ of the nucleus. The hyperfine splitting (HFS) was chosen for study, because in Kerr-Newman geometry, the magnetic quantum number m appears explicitly in Dirac's radial differential equations, in contrast to the HFS degeneracy in flat space. Besides the mass and charge, there appears in the Kerr-Newman metric a third parameter a, having the dimensions of a length. a is a measure of the scale of the spatial distribution of the electromagnetic field inside the source. By studying the asymptotic form of the metric and of the electromagnetic field at large distances from source, one deduces that the angular momentum J and the magnetic dipole moment μ_D of the source are given by

$$J = am_N c \ , \tag{1}$$

$$\mu_D = eZa \ . \tag{2}$$

Here, we have already used the mass m_N of the nucleus and its charge eZ. Putting now, in accordance with our basic assumption,

$$J = I_N\hbar \ , \tag{3}$$

we obtain for the constant a the following expression:

$$a = \frac{I_N\hbar}{m_N c} \ . \tag{4}$$

For the proton, muon, and positron, with $I_N=\frac{1}{2}$, a, as given in (4), is equal to $\frac{1}{2}$ the Compton wavelength of the nucleus. Using (4) in (2), we get

$$\mu_D = \frac{eZ\hbar}{2m_N c} \ . \tag{5}$$

This relation fits the muon with a gyromagnetic ratio[4] equal to 2, except, of course, for the magnetic anomaly, for which we have to invoke QED. Now, our results on the HFS in muonium agree with the experimental values to within 0.94 ppm, except for a missing factor of 2. In the case of positronium, our theoretical value of the HFS agrees with the observed value to within 68 ppm, compared to the deviation of current theory of 66 ppm.

The proton's magnetic moment is, of course, not equal to 1 nuclear magneton, as would follow from Eq. (5), but is greater by a factor of 2.79. If we apply this factor of 2.79 to our theoretical values of the HFS in the ground state of hydrogen, in addition to the factor of 2 appearing in the case of muonium and positronium, we obtain agreement with the experimental value to within the uncertainty in the QED corrections which we had to apply, namely, to within 2.6 ppm. In the ground state of hydrogen, the experimental HFS value is known (1984) to the fantastic accuracy of 6×10^{-7} ppm, a target which theory is not likely to reach before the turn of the century.

II. DIRAC'S EQUATION IN KERR-NEWMAN GEOMETRY

Dirac's equation in Kerr geometry was separated by Chandrasekhar,[5] and the separation was then extended to Kerr-Newman geometry by Page.[6] Using Boyer-

Lindquist[7] coordinates, and on omitting a common external factor of $\exp[im\varphi-(iEt/\hbar)]$, Chandrasekhar showed that the four components of the Dirac spinor, which he denotes by F_1, F_2 and G_1, G_2, can be separated into the forms

$$F_1=\frac{R(r)S(\theta)}{(r-ia\cos\theta)},\quad F_2=R_{+1/2}(r)S(\pi-\theta)\ , \tag{6}$$

$$G_1=R_{+1/2}(r)S(\theta),\quad G_2=\frac{R(r)S(\pi-\theta)}{(r+ia\cos\theta)}\ , \tag{7}$$

where the functions $R(r)$ and $R_{+1/2}$ obey a pair of simultaneous first-order ordinary linear differential equations with complex coefficients. It was shown by Pekeris[3] that, on writing

$$\begin{aligned} &R(r)=[F(r)+iG(r)]\ ,\\ &R_{+1/2}(r)=\sqrt{[2/\Delta(r)]}[F(r)-iG(r)]\ , \end{aligned} \tag{8}$$

where Δ is given in Eq. (13) below, Chandrasekhar's complex equations can be transformed into simultaneous real equations for the functions F and G, which in the context of the nuclear source take on the form

$$\begin{aligned}(1+x^2)\frac{dF}{dx}=k(1+x^2)^{1/2}F&\\ +[-\omega\epsilon(1+x^2)-\alpha Zx+m&\\ +\omega x(1+x^2)^{1/2}]G\ ,&\end{aligned} \tag{9}$$

$$\begin{aligned}(1+x^2)\frac{dG}{dx}=-k(1+x^2)^{1/2}G&\\ +[\omega\epsilon(1+x^2)+\alpha Zx-m&\\ +\omega x(1+x^2)^{1/2}]F\ .&\end{aligned} \tag{10}$$

Here, m is the magnetic quantum number,

$$r=ax\ , \tag{11}$$

$$\epsilon=\frac{E}{m_ec^2},\quad \omega\equiv I_N\frac{m_e}{m_N}\ , \tag{12}$$

and, to a high degree of approximation,

$$\Delta=a^2(1+x^2)\ . \tag{13}$$

The constant k is an eigenvalue of $S(\theta)$, which obeys the (real) differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dS}{d\theta}\right]+\frac{\omega\sin\theta}{(k+\omega\cos\theta)}\frac{dS}{d\theta}+\left[(\tfrac{1}{2}+\omega\epsilon\cos\theta)^2-\left[\frac{m-\frac{1}{2}\cos\theta}{\sin\theta}\right]^2-\tfrac{3}{4}+2\omega\epsilon m-\omega^2\epsilon^2 +\frac{\omega}{(k+\omega\cos\theta)}(\tfrac{1}{2}\cos\theta+\omega\epsilon\sin^2\theta-m)-\omega^2\cos^2\theta+k^2\right]S=0\ . \tag{14}$$

The parameters ϵ and k are thus two coupled eigenvalues which are to be determined from the boundary conditions obeyed by the solutions of Eqs. (9), (10), and (14).

For large values of r ($r\gg a$, $x\gg 1$), Eqs. (9) and (10) reduce to Dirac's radial equations in flat space

$$\frac{dF}{dr}-\frac{k_0}{r}F+\frac{1}{\hbar c}\left[E+\frac{Ze^2}{r}-E_0\right]G=0\ , \tag{15}$$

$$\frac{dG}{dr}+\frac{k_0}{r}G-\frac{1}{\hbar c}\left[E+\frac{Ze^2}{r}+E_0\right]F=0\ , \tag{16}$$

where $E_0=m_ec^2$, and

$$k_0=-(j+\tfrac{1}{2}),\quad j=l+\tfrac{1}{2} \tag{17}$$

$$k_0=+(j+\tfrac{1}{2}),\quad j=l-\tfrac{1}{2}\ . \tag{18}$$

III. NORMALIZATION CONDITION

We take as the normalization condition

$$I=\int(|F_1|^2+|F_2|^2+|G_1|^2+|G_2|^2)dV=1\ , \tag{19}$$

where dV denotes the spatial volume element. In Boyer-Lindquist coordinates,

$$dV=(r^2+a^2\cos^2\theta)dr\,\sin\theta\,d\theta\,d\phi\ , \tag{20}$$

and the normalization integral becomes, in view of (6), (7), and (13),

$$I=2\pi\int_0^\infty dr\int_0^\pi \sin\theta\, d\theta |R(r)|^2[S^2(\theta)+S^2(\pi-\theta)]$$
$$+2\pi\int_0^\infty dr\int_0^\pi \sin\theta\, d\theta \left[\frac{2}{\Delta}\right]|R(r)|^2[S^2(\theta)+S^2(\pi-\theta)](r^2+a^2\cos\theta)$$
$$=4\pi a\int_0^\infty dx\int_0^\pi \sin\theta\, d\theta |R(x)|^2 S^2(\theta)+8\pi a\int_0^\infty dx\int_0^\pi \sin\theta\, d\theta \left[\frac{x^2+\cos^2\theta}{1+x^2}\right]|R(x)|^2 S^2(\theta)\ . \tag{21}$$

The normalization condition (21) requires that $S^2(\theta)$ be integrable in the range $0<\theta<\pi$, and that $|R|^2$ be integrable in the range $0<x<\infty$. The angular function $S(\theta)$ is not symmetrical with respect to the equatorial plane. The singular points of the second-order differential Eq. (14) are $\theta=0$ and $\theta=\pi$. At each of these singular points, one of the two independent solutions is not square-integrable, and is rejected on that account. Taking the square-integrable solutions at each end of the range, we integrate (by power-series expansions) until the midpoint at $\theta=\pi/2$, where we match the logarithmic derivatives of the solutions. This matching condition yields the eigenvalue k, for given values of ϵ and m.

The situation is radically more complicated in the case of the radial function $R(x)$ which, equivalently to Eqs. (9) and (10), obeys the second-order differential equation

$$(1+x^2)\frac{d^2R}{dx^2}+x\frac{dR}{dx}-\frac{i\omega(1+x^2)}{(k+i\omega x)}\frac{dR}{dx}+\left[\frac{\chi^2-ix\chi}{(1+x^2)}-2i\omega\epsilon x-i\alpha Z+\frac{\omega\chi}{(k+i\omega x)}-\omega^2x^2-k^2\right]R=0\ , \tag{22}$$

where

$$\chi=[-\omega\epsilon(1+x^2)+m-\alpha Z x]\ . \tag{23}$$

Now, in flat space, the solutions of Dirac's equations (15) and (16) near the point $r=0$ are of the form $r^\nu\sum_{n=0}^{\infty}F_n r^n$, $r^\nu\sum_{n=0}^{\infty}G_n r^n$, where

$$\nu=\pm(k_0^2-\alpha^2Z^2)^{1/2}\ . \tag{24}$$

The square-integrability condition can be satisfied only for the positive value of ν. In the case of Eq. (22), on the other hand, the roots of the indicial equation at $x=0$ are

$$\nu=0,1\ , \tag{25}$$

and both independent solutions are square-integrable in the range $0<x<\infty$. Indeed, $x=0$ is not a singular point of the differential equation (22). This equation has only one singularity on the positive real axis, namely, at $x=\infty$. The physical implication of this result is that we have no eigenvalue problem for ϵ.

One way out of the difficulty is to ask, where *does* a second singularity of the differential equation (22) occur? It is located at the points

$$1+x^2=0,\quad x=\pm i\ . \tag{26}$$

Indeed, the points $r=\pm ia$ are what the points r_+ and r_- of black-hole theory reduce to in the nuclear context. While it was welcome at first to find that in the nuclear context the function $\Delta(r)$ does not vanish in real space, we are now forced to look for the singularities of $\Delta(r)$, in order to regain our eigenvalue problem.

It follows that the range of integration of Eqs. (21) and (22) has to be extended by adding the segment $x=i\rightarrow 0$, or the segment $x=-i\rightarrow 0$. The choice between the two segments is made by the requirement that the path of integration must terminate at a singular point, so as to allow us to pick only one of the two independent solutions at that point. For example, in the case of $m=+\frac{1}{2}$ (parallel alignment of the spins of the electron and nucleus), the point $x=+i$ is singular, while at the point $x=-i$ both independent solutions are square-integrable over the singularity of $(1+x^2)^{-1}$ in the last integral of Eq. (21). Hence, we start with $x=+i$. For $m=-\frac{1}{2}$ (antiparallel alignment), the opposite is true, and we start with $x=-i$. As a result, the eigenvalues ϵ and k are slightly complex. The fact that the Dirac Hamiltonian in Kerr-Newman geometry is Hermitian when the independent variables are real does not guarantee Hermiticity when one of the independent variables becomes complex.

The procedure which we followed in solving for the radial function $R(x)$ was first to integrate backwards from $x=\infty$ to some value x_0 for which the asymptotic series given in Eqs. (A13) and (A14) converge to the preset numerical accuracy. As shown in Eqs. (A4) and (A8), the asymptotic solutions for F and G depend only on the single arbitrary constant A appearing in Eq. (A4). From $x=x_0$ to $x=0$, the solution of Eqs. (A2) and (A3) is obtained by numerical integration, using an algorithm which ensures uniform maintenance of the preset accuracy. Denoting the values of F and G at the point $x=0$ by F_0 and G_0, we can use Eqs. (9) and (10) to evaluate the derivatives $(dF/dx)_0$ and $(dG/dx)_0$ in terms of F_0 and G_0. With $R_0=(F_0+iG_0)$, the ratio $(1/R_0)(dR/dx)_0$ is then independent of the arbitrary constant A. Similarly, the value of $(1/R_0)(dR/dx)_0$ obtained from the solution originating on the imaginary axis of x is independent of the single arbitrary constant of that solution. The matching of the two logarithmic derivates of R at $x=0$ yields the eigenvalue ϵ, for given values of k and m.

The solution of Eq. (14) for the angular function $S(\theta)$, and therewith for the eigenvalue k, is obtained either directly by the use of the expansion of k in powers of ω given in Appendix C, or by using the matching condition (D18).

IV. HYPERFINE SPLITTING

Current theory[8] of hyperfine splitting in hydrogenic atoms is based on the Fermi formula,[9] which was derived by a perturbation calculation. For the ground state, the Fermi formula gives

$$E_F=\tfrac{16}{3}(\alpha Z)^2\left[\frac{\mu_N}{\mu_0}\right]Ry\ , \tag{27}$$

and the theoretical HFS values are usually given in the form

$$\Delta_{\mathrm{HFS}}=E_F(1+\delta_N)M_{RN}\ . \tag{28}$$

Here, μ_N denotes the nuclear magnetic moment, and μ_0 is the Bohr magneton. M_{RN} is the reduced mass correction which, in the case of HFS, is given by[10]

$$M_{RN}=\left[\frac{m_N}{m_N+m_e}\right]^3 . \tag{29}$$

The term δ_N includes the Breit relativistic correction,[11] the electron anomaly a_e, and various QED radiative corrections R_{QED} which are specific to HFS,

$$(1+\delta_N)=(1+a_e+R_{\mathrm{QED}}+\tfrac{3}{2}\alpha^2Z^2)\ . \tag{30}$$

Breit showed that, if one uses the relativistic Dirac theory, instead of the Pauli approximation used by Fermi,then the Fermi formula has to be multiplied by a factor of $(1+\frac{3}{2}\alpha^2Z^2)$.

We determine the hyperfine splitting from the difference of the energy eigenvalues ϵ evaluated for $m=+\frac{1}{2}$ and $m=-\frac{1}{2}$,

$$\Delta E=A\ \Delta(\epsilon),\quad \Delta(\epsilon)=[\epsilon(m=+\tfrac{1}{2})-\epsilon(m=-\tfrac{1}{2})]\ . \tag{31}$$

Using the values of the fundamental constants given by Aguilar-Benitez *et al.*,[12] we determine the constant A from the relation

$$E=m_ec^2\epsilon=\frac{2}{\alpha^2}R_\infty\epsilon\equiv A\epsilon\ . \tag{32}$$

On multiplying R_∞ by c to obtain the Rydberg frequency, we get

$$A=1.235\,590\,72\times10^{14}\ \mathrm{MHz}\ . \tag{33}$$

A is, of course, equal to $(m_ec^2/2\pi\hbar)$. Our procedure was to treat the value of a fundamental constant as if it were accurate to the last figure, disregarding the stated uncertainty, which for α is 0.8 ppm, for R_∞ is 0.076 ppm, and for c is 0.0043 ppm.

Our theoretical values of the hyperfine splitting are obtained from the formulas

$$\Delta_{\mathrm{HFS}}=C_N A\ \Delta(\epsilon)_N\ ,\qquad C_N=(\mu_N/\mu_D)M_{RN}(1+\delta'_N)\ , \tag{34}$$

$$\delta'_N=(\delta_N-\tfrac{3}{2}\alpha^2Z^2)\ , \tag{35}$$

where the value of δ_N is taken from the paper of Kinoshita and Sapirstein.[8] This paper will be referred to in the sequel as KS. The factor (μ_N/μ_D) is required, because our model has a dipole magnetic moment of μ_D. We subtract the Breit correction of $\frac{3}{2}\alpha^2Z^2$ from δ_N, because our method does not suffer from relativistic insufficiency.

V. MUONIUM

In the differential equations (9) and (10) there appears, besides the fine-structure constant α, a second parameter ω defined in Eq. (12). We have used the values

$$\frac{1}{\alpha}=137.036\,04(11)\ , \tag{36}$$

$$\omega_\mu=\frac{1}{2}\frac{m_e}{m_\mu}=\frac{0.5}{206.768\,331}=2.418\,165\,29\times10^{-3}\ . \tag{37}$$

Put

$$\epsilon=\epsilon_0+\epsilon'\ , \tag{38}$$

$$\epsilon_0=\left[1+\frac{\alpha^2Z^2}{[(k_0^2-\alpha^2Z^2)^{1/2}+n-|k_0|]^2}\right]^{-1/2} . \tag{39}$$

ϵ_0 denotes the Dirac energy eigenvalue in flat space, which is independent of the magnetic quantum number m, and n is the principal quantum number. We tabulate below only the deviation ϵ', multiplied by the factor A given in (33). We have, for the two states of muonium with $m=+\frac{1}{2}$ and $m=-\frac{1}{2}$,

m	$A\epsilon'$	k
$+\frac{1}{2}$	$(1128.4118-i2.9\times10^{-2})$ MHz	$-0.997\,581\,877\,6-i3.8\times10^{-19}$
$-\frac{1}{2}$	$(-1131.1533-i3.0\times10^{-2})$ MHz	$-1.002\,418\,122\,4+i4.0\times10^{-19}$

Disregarding, in the first instance, the imaginary parts, we have

$$A\ \Delta(\epsilon)=2259.5651\ \mathrm{MHz}\ . \tag{40}$$

To obtain the HFS, we have to evaluate the constant C_μ defined in Eq. (34). (μ_μ^+/μ_D) is simply 1 plus the muon magnetic anomaly,

$$(\mu_\mu^+/\mu_D)=1.001\,165\,911\ . \tag{41}$$

From the value of (m_μ/m_e) given in (37), we obtain, by (29), the reduced-mass correction for the muon,

$$M_{R\mu}=0.985\,630\,226\ . \tag{42}$$

From the tabulated values of the various terms in δ_μ given in KS, and on using Eq. (35), we obtain

$$(1+\delta'_\mu)=1.000\,877\,73\ . \tag{43}$$

By (34), (41), and (42), we have

$$2C_\mu=1.975\,291\,02\ . \tag{44}$$

The factor of 2 is needed in order to make the theoretical value of HFS agree with the observed value. With this, as yet unexplained, factor of 2, the theoretical value of the hyperfine splitting in muonium comes out,

$$\Delta_{\rm HFS}=2C_\mu A\,\Delta(\epsilon)_\mu=4463.2987\ \text{MHz}\ , \tag{45}$$

compared with the experimental value of 4463.302 88(16) MHz.[13] The agreement between the two values is within 0.94 ppm. The uncertainty in α is 0.8 ppm, in m_e is 2.7 ppm, and in m_μ is 2.7 ppm. The theoretical value for HFS in muonium given in KS is 4463.3047(1.7)(1.0). But for the factor of 2, our theoretical value is of the same order of uncertainty.

VI. HYDROGEN

In the case of hydrogen we have

$$\omega_P=\frac{1}{2}\frac{m_e}{m_P}=\frac{0.5}{1836.151\,52(70)}=2.723\,086\,818\times10^{-4}\ , \tag{46}$$

$$(\mu_P/\mu_D)=2.792\,845\,6(11),\quad (M_{RP})=0.998\,367\,926\ . \tag{47}$$

Using the value of δ_P given in KS, we obtain

$$(1+\delta'_P)=1.001\,021\,452\ . \tag{48}$$

Hence

$$2C_P=5.582\,267\,06\ . \tag{49}$$

In Table I we give results for the hyperfine splitting in the $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ states of hydrogen. Our theoretical value for the HFS in the ground state is 1420.4020 MHz, compared with the experimental value[14] of 1420.405 751 766 7(9) MHz. Except for the unexplained factor of 2, the agreement is within 2.6 ppm. The theoretical value given in KS deviates from the experimental value by 0.002 308(1278) MHz, i.e., by 1.6 ppm. The δ_P in KS contains also a term of -0.046 MHz stemming from the "static proton structure" effect on HFS.

We wish now to check whether our levels for the $2S_{1/2}$ and $2P_{1/2}$ states in hydrogen do not conflict with the measurements of the $(2S_{1/2}-2P_{1/2})$ split in the Lamb shift experiments. In Table I we have used for these

TABLE I. Hyperfine splitting in hydrogen. Here $A=1.235\,590\,72\times10^{14}$ MHz, $2C_P=5.582\,267\,06$, $\Delta(\epsilon)=[\epsilon(m=+\frac{1}{2})-\epsilon(m=-\frac{1}{2})]$, $\epsilon=\epsilon_0+\epsilon'$,

$$\epsilon_0=\left[1+\frac{\alpha^2Z^2}{[(k_0^2-\alpha^2Z^2)^{1/2}+n-|k_0|]^2}\right]^{-1/2},$$

and $E=m_ec^2\epsilon$.

State	m	$A\epsilon'$ (MHz)	$2C_PA\,\Delta(\epsilon)_P$ (MHz)	Mean (MHz)
$1S_{1/2}$	$\frac{1}{2}$	127.206 99	710.103 39	$-0.194\,21$
	$-\frac{1}{2}$	$-127.241\,78$	$-710.297\,60$	
	$A\,\Delta(\epsilon)$	254.448 77	HFS 1420.4010	
$2S_{1/2}$	$\frac{1}{2}$	15.901 40	88.7659	-0.0242
	$-\frac{1}{2}$	$-15.905\,75$	-88.7901	
	$A\,\Delta(\epsilon)$	31.807 15	HFS 177.5560	
$2P_{1/2}$	$\frac{1}{2}$	5.301 07	29.5920	-0.0007
	$-\frac{1}{2}$	$-5.301\,23$	-29.5929	
	$A\,\Delta(\epsilon)$	10.602 30	HFS 59.1849	

states the same value of C_P as for the ground state, even though there are indications of a slight state dependence. On the assumption that in the Lamb shift experiments, the two polarizations with $m=+\frac{1}{2}$ and $m=-\frac{1}{2}$ occurred in equal proportions, the $2S_{1/2}$ and $2P_{1/2}$ levels would be represented by the values labeled "mean" in Table I. The difference of the mean values is equal to 0.0235 MHz. This is only 2.6 times greater than the uncertainty of the experimental value[15] of 1057.845(9) quoted by Kinoshita and Sapirstein[8] in their comparison of theory with experiment. The theory is uncertain by 0.018 MHz because of a recently reported jump of 14% in the measured[16] value of the root-mean-square electromagnetic radius of the proton. Hence our results do not conflict with the Lamb shift measurements in hydrogen. Note that the HFS values for $2S_{1/2}$ and $2P_{1/2}$ states are close to $\frac{1}{8}$ and $\frac{1}{24}$, respectively, of the HFS value in the $1S_{1/2}$ state, as in the Fermi theory.

VII. POSITRONIUM

In positronium, the mass of the nucleus is equal to the mass of the electron, so that we have from (12) and (29)

$$\omega_{e^+}=\frac{1}{2}\frac{m_e^-}{m_e^+}=0.5\ ,$$
$$M_{R\,\text{Ps}}=\tfrac{1}{8}\ , \tag{50}$$
$$(\mu_{e^+}/\mu_D)=1.001\,159\,652\,2\ .$$

The results for the singlet and triplet states of positronium are

State	m	$A\epsilon'$ (MHz)	k
1^3S	$+\frac{1}{2}$	$175\,131.11-i832.91$	$-0.500\,008\,874\,9-i2.25\times10^{-12}$
1^2S	$-\frac{1}{2}$	$-292\,138.96-i1745.53$	$-1.499\,991\,123\,9+i4.71\times10^{-12}$

Here, we have used the previous notation of $m=+\frac{1}{2}$ for the triplet state and $m=-\frac{1}{2}$ for the singlet state, which refer to the value of m_z of the electron only, instead of the usual total spin designation of $m=1$ and $m=0$, respectively. Disregarding, again in the first instance, the imaginary part of ϵ', we have

$$A\ \Delta(\epsilon')=467\,270.07\ \text{MHz}\ . \tag{51}$$

In the case of positronium, there exists an annihilation channel which makes a contribution to the HFS amounting to some $\frac{3}{4}$ of the HFS stemming from nonannihilation origins.[17] Our results relate only to the latter part, for which we get the value[8,17]

$$(1+\delta'_{e^+})=0.997\,621\,500\ . \tag{52}$$

Substituting the values given in Eqs. (50), (51), and (52) into Eq. (34), we get

$$\begin{aligned} 2C_{e^+}&=0.249\,694\,598\,5\ ,\\ 2C_{e^+}A\ \Delta(\epsilon')&=116\,674.81\ \text{MHz}\ . \end{aligned} \tag{53}$$

Adding the value of the contribution to HFS arising from the annihilation channel,[17] namely, 86 728.19 MHz, we obtain a total of 203 403.00 MHz, compared with the observed value[18] of 203 389.10(74) MHz. Our theoretical value exceeds the experimental value by 13.9 MHz, compared to the excess of 13.4 MHz given in KS.

VIII. STABILITY

Let

$$\epsilon=\epsilon_r-i\epsilon_i,\quad E=m_ec^2(\epsilon_r-i\epsilon_i)\ . \tag{54}$$

The imaginary term in Eq. (54) gives rise to a factor of $e^{-t/\tau}$ stemming from the expression $\exp[-i(Et/\hbar)]$, with

$$\tau=\frac{\hbar}{m_ec^2\epsilon_i}=\frac{1.288\times10^{-21}}{\epsilon_i}\ \text{sec}\ . \tag{55}$$

In muonium,

$$\epsilon_i=2.3\times10^{-16},\quad \tau=5.5\times10^{-6}\ \text{sec}\ . \tag{56}$$

Now, the mean life of the muon is $2.197\,09\times10^{-6}$ sec. Hence, in the muonium atom, the electron can stay bound with the muon for the life of the host. We conclude that, except for the missing factor of 2, our model of the muon as a Kerr-Newman source yields acceptable values for the hyperfine splitting and for the lifetime of muonium.

In the triplet state of positronium, $\epsilon_i=6.74\times10^{-12}$. Using Eq. (55) with the numerator doubled because of the reduced mass factor, we get a value of 3.8×10^{-10} sec for τ, which is much shorter than the observed value[19] of 1.418×10^{-7} sec. In the singlet state, $\epsilon_i=1.413\times10^{-11}$, giving a value of $\tau=1.82\times10^{-10}$ sec. This is of the same order of magnitude as the observed value[19] of 1.25×10^{-10} sec.

In hydrogen,

$$\epsilon_i=3.0\times10^{-18},\quad \tau=4.3\times10^{-4}\ \text{sec}\ ,$$

which is fatal.

IX. DISCUSSION

We set out to test the proposed model of the atomic nucleus as a Kerr-Newman source by evaluating the hyperfine splitting in hydrogenic atoms, and comparing the theoretical values with observations. Such a theoretical evaluation of HFS was made possible by Chandrasekhar's[5] separation of the Dirac equation in Kerr geometry. We found that, in order for this model of the nucleus to bind the electron in a discrete spectrum, the radial coordinate r in real space had to be extended by adding a segment of the imaginary axis of r. As a consequence of complexifying r, the resulting bound states proved to be unstable. Except perhaps for muonium, the instability is fatal for the model.

The instability speaks against extending the real domain into complex space. If we limit ourselves to real values of r, then the Hamiltonian is Hermitian, the instability disappears, but, since the differential equations are not singular at $r=0$, we lose the eigenvalue feature. A continuum of real eigenvalues is possible, depending on the arbitrary ratio of the amplitudes of the two independent solutions at $r=0$, each of which is square-integrable. In an effort to regain the boundary conditions at $r=0$, we converted Eqs. (9) and (10) into Eulerian equations of a variational problem, and applied *natural boundary conditions*[20] at $r=0$. The resulting HFS values were not in agreement with observation.

The conclusion—that the proposed model of the atomic nucleus as a Kerr-Newman source is unacceptable because its bound states are unstable, and because its HFS values are off by a factor of 2—would be logical, were it not for the fact that this missing factor is so close to 2, to within some parts in a million. The theoretical HFS values fit a source with a gyromagnetic ratio of 1 (as for a classical particle), rather than 2. In an effort to clarify this point, we have investigated[21] the electromagnetic field of the Kerr-Newman source, and found that the conclusion that this source has a gyromagnetic ratio of 2 is inescapable. Externally, i.e., for $r\gg a$, the electric field is close to that of a positive point charge, and the magnetic field approaches that of a positive magnetic dipole. However, the internal structure of the electromagnetic field of the Kerr-Newman source turns out to be extremely complicated.

The electric charge is smeared out over a circular disc as a surface charge. The disc is centered at the origin, and lies in a plane normal to the angular momentum vector. Its radius is equal to the parameter a. For a net positive charge e, the surface-charge density is *negative* throughout the interior of the disc, becoming infinitely negative as the rim of the disc is approached. On the rim, there is a positive line density of infinite intensity which more than compensates the negative charge distribution in the interior, leaving a net positive charge e.

The magnetic field is generated by current flowing in the *negative direction* in the interior of the disc, thereby generating a negative magnetic moment distribution there. On the rim flows a positive current of infinite intensity, which more than compensates the negative magnetic moments in the interior, and leaves a total integrated magnetic moment equal to *ea*, corresponding to a gyromagnetic ratio $g=2$. With the negative currents flowing throughout the interior of the disc, it could be that the theoretical HFS values reflect their presence.

Similarly, the instability may indicate that the electron senses the presence of negative surface charges in the interior of the disc. Indeed, inside a nearly spherical surface of radius a, centered at the origin, the force exerted by the nucleus on the electron ceases to be attractive, and becomes *repulsive*, due to proximity to the negative surface charges covering the equatorial phase.

APPENDIX A: ASYMPTOTIC EXPANSION OF THE RADIAL FUNCTION $R(r)$

The coefficients in the differential equations (9) and (10) can be rationalized by changing to the variable

$$y=(1+x^2)^{1/2}-x,\quad 0\le y\le 1,\quad 0\le x\le\infty \tag{A1}$$

whereby the equations become

$$(y^2+y^4)\frac{dG}{dy}-k(y+y^3)G+\left[\frac{\omega}{2}(1+\epsilon)+\alpha Z(y-y^3)-(2m-\omega\epsilon)y^2-\frac{\omega}{2}(1-\epsilon)y^4\right]F=0\ , \tag{A2}$$

$$(y^2+y^4)\frac{dF}{dy}+k(y+y^3)F+\left[\frac{\omega}{2}(1-\epsilon)-\alpha Z(y-y^3)+(2m-\omega\epsilon)y^2-\frac{\omega}{2}(1+\epsilon)y^4\right]G=0\ . \tag{A3}$$

In the vicinity of $y=0$ ($x=\infty$), the functions G and F behave like

$$G(y)=\frac{A}{r}e^{-\zeta/y}y^{-\tau}P(y),\quad F=Ae^{-\zeta/y}y^{-\tau}Q(y)\ , \tag{A4}$$

with $P(y)$ and $Q(y)$ regular. Here

$$r=\left[\frac{1-\epsilon}{1+\epsilon}\right]^{1/2},\quad \zeta=\frac{\omega}{2}(1-\epsilon^2)^{1/2},\quad \tau=\frac{\alpha Z\epsilon}{(1-\epsilon^2)^{1/2}}\ . \tag{A5}$$

Substitution of (A4) in Eqs. (A2) and (A3) yields

$$(y^2+y^4)\frac{dP}{dy}+[\zeta(1+y^2)-(\tau+k)(y+y^3)]P+r\left[\frac{\omega}{2}(1+\epsilon)+\alpha Z(y-y^3)-(2m-\omega\epsilon)y^2-\frac{\omega}{2}(1-\epsilon)y^4\right]Q=0\ , \tag{A6}$$

$$(y^2+y^4)\frac{dQ}{dy}+[\zeta(1+y^2)+(-\tau+k)(y+y^3)]Q+\frac{1}{r}\left[\frac{\omega}{2}(1-\epsilon)-\alpha Z(y-y^3)+(2m-\omega\epsilon)y^2-\frac{\omega}{2}(1+\epsilon)y^4\right]P=0\ . \tag{A7}$$

Now, put

$$U=P+Q,\quad V=P-Q\ , \tag{A8}$$

$$(2m-\omega\epsilon)=\kappa,\quad s=\frac{\omega}{(1-\epsilon^2)^{1/2}},\quad \eta=\frac{1}{(1-\epsilon^2)^{1/2}}\ , \tag{A9}$$

and add Eq. (A6) to (A7), obtaining

$$(y^2+y^4)\frac{dU}{dy}+\left[2\zeta-2\tau y+(\zeta+\epsilon\eta\kappa)y^2-\frac{s}{2}(1+\epsilon^2)y^4\right]U+\left[-\left[k+\frac{\tau}{\epsilon}\right]y-\left[k-\frac{\tau}{\epsilon}\right]y^3+\eta\kappa y^2-\epsilon s y^4\right]V=0\ . \tag{A10}$$

Subtraction of (A7) from (A6) yields

$$(y^2+y^4)\frac{dV}{dy}+\left[-2\tau y^3+(\zeta-\epsilon\eta\kappa)y^2+\frac{s}{2}(1+\epsilon^2)y^4\right]V+\left[-\left[k-\frac{\tau}{\epsilon}\right]y-\eta\kappa y^2-\left[k+\frac{\tau}{\epsilon}\right]y^3+s\epsilon y^4\right]U=0\ . \tag{A11}$$

From Eqs. (A10) and (A11) we obtain the following recursion relations for the coefficients in the expansions:

$$U(y)=\sum_{n=0}^{\infty}U_n y^n,\quad V(y)=\sum_{n=0}^{\infty}V_n y^n\ , \tag{A12}$$

$$2\zeta U_n=[2\tau-(n-1)]U_{n-1}-(\zeta+\eta\epsilon\kappa)U_{n-2}-(n-3)U_{n-3}+\frac{s}{2}(1+\epsilon^2)U_{n-4}+\left[k+\frac{\tau}{\epsilon}\right]V_{n-1}-\eta\kappa V_{n-2}$$
$$+\left[k-\frac{\tau}{\epsilon}\right]V_{n-3}+s\epsilon V_{n-4}\ , \tag{A13}$$

$$nV_n=(-\zeta+\epsilon\eta\kappa)V_{n-1}+[2\tau-(n-2)]V_{n-2}-\frac{s}{2}(1+\epsilon^2)V_{n-3}+\left[k-\frac{\tau}{\epsilon}\right]U_n+\eta\kappa U_{n-1}+\left[k+\frac{\tau}{\epsilon}\right]U_{n-2}-s\epsilon U_{n-3}\ . \tag{A14}$$

Here, V_0 is arbitrary. On setting $V_0=1$, U_0, U_n, and V_n for $n=1,2,3,\ldots$ are uniquely determined.

APPENDIX B: SOLUTION OF THE DIFFERENTIAL EQUATION FOR THE RADIAL FUNCTION $R(r)$

1. Power-series expansion of $R(x)$ near $x=+i$

In Eq. (22), put

$$x=i(1-2y)\ , \tag{B1}$$

so that the points $x=(i,0,-i)$ correspond to the points $y=(0,\frac{1}{2},1)$. With

$$D=(k-\omega+2\omega y)=k+i\omega x\ , \tag{B2}$$

Eq. (22) now reads

$$(y-y^2)\frac{d^2R}{dy^2}+\tfrac{1}{2}(1-2y)\frac{dR}{dy}+\frac{\omega}{D}\left[-(2y-2y^2)\frac{dR}{dy}+[4\omega\epsilon(y-y^2)-m+i\alpha Z(1-2y)]R\right]$$
$$+\{k^2-\omega^2-\alpha^2Z^2+\omega\epsilon[2m-1-2i\alpha Z+(2+4i\alpha Z)y]+4\omega^2(1-\epsilon^2)(y-y^2)\}R$$
$$-\frac{1}{(4y-4y^2)}[(m-i\alpha Z)(m+1-i\alpha Z)-2m(1-2i\alpha Z)y]R=0\ . \tag{B3}$$

Let

$$R(y)=K(y)F(y),\quad K(y)=y^{q/2}(1-y)^{p/2}\ , \tag{B4}$$

where the exponents q and p, chosen so that $R(y)$ does not become infinite, at $y=0$ and at $y=1$, respectively, are given by

	q	p
$m>0$	$m+1-i\alpha Z$	$m+i\alpha Z$
$m<0$	$-m+i\alpha Z$	$-m+1-i\alpha Z$

Writing

$$u=|m+\tfrac{1}{2}|,\quad v=|m-\tfrac{1}{2}|,\quad s=\frac{m}{|m|}\ , \tag{B5}$$

we have

$$q=(u+\tfrac{1}{2}-i\alpha Zs),\quad p=(v+\tfrac{1}{2}+i\alpha Zs)\ . \tag{B6}$$

Equation (B3) can now be written in the form

$$DLF+\omega MF=0\ , \tag{B7}$$

where the differential operators L and M are defined by

$$L=\left[(y-y^2)\frac{d^2}{dy^2}+[u+1-i\alpha Zs-(2+2|m|)y]\frac{d}{dy}+(C-2\omega\epsilon-\alpha^2Z^2-2i\alpha Z\omega\epsilon)\right.$$
$$\left.+[\omega\epsilon(2+4i\alpha Z)+4\omega^2(1-\epsilon^2)]y-4\omega^2(1-\epsilon^2)y^2\right]\ , \tag{B8}$$

$$M=\left[-2(y-y^2)\frac{d}{dy}-u-\tfrac{1}{2}+i\alpha Zs-m+i\alpha Z+(1+2|m|+4\omega\epsilon-2i\alpha Z)y-4\omega\epsilon y^2\right], \tag{B9}$$

and

$$C=[k^2-\omega^2-(|m|+\tfrac{1}{2})^2+\omega\epsilon(1+2m)] . \tag{B10}$$

The recursion relation for the coefficients F_n in the expansion $F=\sum_{n=0}^{\infty}F_n y^n$ is

$$F_{n+1}=\frac{1}{(n+1)(n+1+u-i\alpha Zs)}\Big\{[n(n+1+2|m|)-C+\alpha^2Z^2+2\omega\epsilon+2i\alpha Z\omega\epsilon]F_n$$
$$-[2\omega\epsilon+4\omega^2(1-\epsilon^2)+4i\alpha Z\omega\epsilon]F_{n-1}+4\omega^2(1-\epsilon^2)F_{n-2}$$
$$-\frac{2\omega}{(k-\omega)}L_{n-1}-\frac{\omega}{(k-\omega)}M_n\Big\}, \tag{B11}$$

with

$$L_n=(n+1)(n+1+u-i\alpha Zs)F_{n+1}+[C-2\omega\epsilon-\alpha^2Z^2-2i\alpha Z\omega\epsilon-n(n+1+2|m|)]F_n$$
$$+[2\omega\epsilon+4\omega^2(1-\epsilon^2)+4i\alpha Z\omega\epsilon]F_{n-1}-4\omega^2(1-\epsilon^2)F_{n-2}, \tag{B12}$$

$$M_n=(-2n-u-\tfrac{1}{2}-m+i\alpha Zs+i\alpha Z)F_n+(2n-1+2|m|+4\omega\epsilon-2i\alpha Z)F_{n-1}-4\omega\epsilon F_{n-2} . \tag{B13}$$

2. Power-series expansion of $R(x)$ near the point $x=-i$

We take as the independent variable z, defined by

$$z=1-y,\quad x=-i(1-2z) , \tag{B14}$$

so that the points $x=(-i,0,i)$ correspond to the points $z=(0,\frac{1}{2},1)$. With

$$D=(k+\omega-2\omega z) , \tag{B15}$$

Eq. (22) reads

$$(z-z^2)\frac{d^2R}{dz^2}+\tfrac{1}{2}(1-2z)\frac{dR}{dz}+\{k^2-\omega^2-\alpha^2Z^2+\omega\epsilon[(2m+1+2i\alpha Z-(2+4i\alpha Z)z]+4\omega^2(1-\epsilon^2)z-4\omega^2(1-\epsilon^2)z^2\}R$$
$$-\frac{1}{(4z-4z^2)}[p^2-p+2m(1-2i\alpha Z)z]R+\frac{\omega}{D}\left[(2z-2z^2)\frac{dR}{dz}+[(-m-i\alpha Z)+(4\omega\epsilon+2i\alpha Z)z-4\omega\epsilon z^2]R\right]=0 . \tag{B16}$$

Now, let

$$R(z)=K(z)F(z),\quad K(z)=z^{p/2}(1-z)^{q/2} , \tag{B17}$$

so that, for a given point x, $K(z)$ is equal to $K(y)$ defined in Eq. (B4). Equation (B16) then becomes

$$(z-z^2)\frac{d^2F}{dz^2}+[(v+1+i\alpha Zs)-(2+2|m|)z]\frac{dF}{dz}$$
$$+\{C-\alpha^2Z^2+2i\alpha Z\omega\epsilon+[-\omega\epsilon(2+4i\alpha Z)+4\omega^2(1-\epsilon^2)]z-4\omega^2(1-\epsilon^2)z^2\}F$$
$$+\frac{\omega}{D}\{(2z-2z^2)\frac{dF}{dz}+[(v+\tfrac{1}{2}-m+i\alpha Zs-i\alpha Z)-(1+2|m|-4\omega\epsilon-2i\alpha Z)z-4\omega\epsilon z^2]\}F=0 . \tag{B18}$$

Putting

$$F=\sum_{n=0}^{\infty}F_n z^n , \tag{B19}$$

the recursion relation for the coefficients F_n becomes

$$F_{n+1}=\frac{1}{(n+1)(n+1+v+i\alpha Zs)}\left[[n(n+1+2|m|)-C+\alpha^2Z^2-2i\alpha Z\omega\epsilon]F_n +[\omega\epsilon(2+4i\alpha Z)-4\omega^2(1-\epsilon^2)]F_{n-1}+4\omega^2(1-\epsilon^2)F_{n-2} -\frac{2\omega}{(k+\omega)}L_{n-1}-\frac{\omega}{(k+\omega)}M_n\right], \tag{B20}$$

where

$$L_n=(n+1)(n+1+v+i\alpha Zs)F_{n+1}+[C-\alpha^2Z^2+2i\alpha Z\omega\epsilon-n(n+1+2|m|)]F_n +[-\omega\epsilon(2+4i\alpha Z)+4\omega^2(1-\epsilon^2)]F_{n-1}-4\omega^2(1-\epsilon^2)F_{n-2}, \tag{B21}$$

$$M_n=(2n+v+\tfrac{1}{2}-m+i\alpha Zs-i\alpha Z)F_n+(-2n+1-2|m|+4\omega\epsilon+2i\alpha Z)F_{n-1}-4\omega\epsilon F_{n-2}. \tag{B22}$$

The radius of convergence of the series of $F(y)$ and $F(z)$ is 1. Hence, at the matching point, where the argument is $\frac{1}{2}$, the convergence is good.

APPENDIX C: EXPANSION OF k IN POWERS OF ω

Using standard perturbation theory in Eq. (14), k can be expanded into a power series in ω as follows.

1. $m=+\frac{1}{2}$. We have

$$k_+=1-\tfrac{1}{3}(2\epsilon-1)\omega+\tfrac{2}{27}(\epsilon+1)^2\omega^2 +\tfrac{4}{1215}(\epsilon+1)^2(7\epsilon-2)\omega^3 +\tfrac{2}{10\,935}(\epsilon+1)^3(13\epsilon-23)\omega^4+\cdots, \tag{C1}$$

$$k_-=-1+\tfrac{1}{3}(2\epsilon+1)\omega-\tfrac{2}{27}(\epsilon-1)^2\omega^2 -\tfrac{4}{1215}(\epsilon-1)^2(7\epsilon+2)\omega^3 -\tfrac{2}{10\,935}(\epsilon-1)^3(13\epsilon+23)\omega^4+\cdots. \tag{C2}$$

2. $m=-\frac{1}{2}$. We have

$$k_+=1+\tfrac{1}{3}(2\epsilon-1)\omega+\tfrac{2}{27}(\epsilon+1)^2\omega^2 -\tfrac{4}{1215}(\epsilon+1)^2(7\epsilon-2)\omega^3 +\frac{2}{10\,935}(\epsilon+1)^3(13\epsilon-23)\omega^4+\cdots, \tag{C3}$$

$$k_-=-1-\tfrac{1}{3}(2\epsilon+1)\omega-\tfrac{2}{27}(\epsilon-1)^2\omega^2 +\tfrac{4}{1215}(\epsilon-1)^2(7\epsilon+2)\omega^3 -\tfrac{2}{10\,935}(\epsilon-1)^3(13\epsilon+23)\omega^4+\cdots. \tag{C4}$$

Here, k_- applies to the $1S_{1/2}$ and $2S_{1/2}$ states and k_+ to the $2P_{1/2}$ state. For hydrogen, with $\omega=2.723\,086\,8\times10^{-4}$, the power series in ω were found to agree with the value obtained from the matching condition (D18) below to within one part in 10^{18}. Note that the expansions for $m=-\frac{1}{2}$ can be obtained from the expansions for $m=+\frac{1}{2}$ by reversing the sign of ω, a result that is to be expected from physical considerations.

APPENDIX D: SOLUTION OF THE DIFFERENTIAL EQUATION FOR THE ANGULAR FUNCTION $S(\theta)$

1. Power-series expansion of $S(\theta)$ near $\theta=0$

Let

$$u=|m+\tfrac{1}{2}|,\quad v=|m-\tfrac{1}{2}|. \tag{D1}$$

In our case,

$$m=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\pm\tfrac{5}{2},\ldots, \qquad u+v=2|m|,\quad u-v=s,\quad s=\frac{m}{|m|}. \tag{D2}$$

Put

$$z=\tfrac{1}{2}(1-\cos\theta)=\sin^2\left[\frac{\theta}{2}\right]. \tag{D3}$$

$$D\equiv k+\omega\cos\theta=(k+\omega)-2\omega z, \tag{D4}$$

$$S(z)=K(z)F(z),\quad K(z)=z^{v/2}(1-z)^{u/2}, \tag{D5}$$

then Eq. (14) for $S(\theta)$ becomes

$$(z-z^2)\frac{d^2F}{dz^2}+[(1+v)-(2+2|m|)z]\frac{dF}{dz}+\frac{2\omega}{D}(z-z^2)\frac{dF}{dz}+\frac{\omega}{D}(v-2|m|z)F +\{C+[-2\omega\epsilon+4\omega^2(1-\epsilon^2)]z-4\omega^2(1-\epsilon^2)z^2\}F+\frac{\omega}{D}[(\tfrac{1}{2}-m)+(-1+4\omega\epsilon)z-4\omega\epsilon z^2]F=0, \tag{D6}$$

where

$$C=[k^2-\omega^2-(|m|+\tfrac{1}{2})^2+\omega\epsilon(1+2m)]\ . \tag{D7}$$

The recursion relation for the coefficients F_n in the expansion

$$F(z)=\sum_{n=0}^{\infty} F_n z^n \tag{D8}$$

is

$$F_{n+1}=\frac{1}{(n+1)(n+1+v)}\left[[n(n+1+2|m|)-C]F_n+[2\omega\epsilon-4\omega^2(1-\epsilon^2)]F_{n-1}\right.$$
$$\left.+4\omega^2(1-\epsilon^2)F_{n-2}+\frac{2\omega}{(k+\omega)}L_{n-1}-\frac{\omega}{(k+\omega)}M_n\right], \tag{D9}$$

where

$$L_{n-1}=n(n+v)F_n+[C-(n-1)(n+2|m|)]F_{n-1}+[-2\omega\epsilon+4\omega^2(1-\epsilon^2)]F_{n-2}-4\omega^2(1-\epsilon^2)F_{n-3}\ , \tag{D10}$$

$$M_n=(2n+v+\tfrac{1}{2}-m)F_n-(2n-1+2|m|-4\omega\epsilon)F_{n-1}-4\omega\epsilon F_{n-2}\ . \tag{D11}$$

Equation (D9) yields

$$F_1=-\frac{1}{(1+v)}\left[C+\frac{\omega}{(k+\omega)}(v+\tfrac{1}{2}-m)\right]F_0\ . \tag{D12}$$

Note that in flat space ($\omega=0$) the recursion relation becomes

$$F_{n+1}=\frac{1}{(n+1)(n+1+v)}[(n+|m|+\tfrac{1}{2})^2-k_0^2]F_n,\quad n=0,1,2,\ldots\ . \tag{D13}$$

The boundary condition for the eigenvalue k is that $(1-z)^{u/2}F$ should be finite at $z=1$, or, equivalently, that $F(1)$ should be finite.

2. Power-series expansion of $S(\theta)$ near $\theta=\pi$

In terms of the variable y defined by

$$y=1-z=\cos^2\left[\frac{\theta}{2}\right],\quad F(y)=\sum_{n=0}^{\infty} F_n y^n\ , \tag{D14}$$

the recursion relation (D9) is changed to

$$F_{n+1}=\frac{1}{(n+1)(n+1+u)}\left[[n(n+1+2|m|)-C+2\omega\epsilon]F_n-[2\omega\epsilon+4\omega^2(1-\epsilon^2)]F_{n-1}\right.$$
$$\left.+4\omega^2(1-\epsilon^2)F_{n-2}-\frac{2\omega}{(k-\omega)}L_{n-1}-\frac{\omega}{(k-\omega)}M_n\right], \tag{D15}$$

where

$$L_{n-1}=n(n+u)F_n+[C-2\omega\epsilon-(n-1)(n+2|m|)]F_{n-1}$$
$$+[2\omega\epsilon+4\omega^2(1-\epsilon^2)]F_{n-2}-4\omega^2(1-\epsilon^2)F_{n-3}\ , \tag{D16}$$

$$M_n=-(2n+u+\tfrac{1}{2}+m)F_n$$
$$+(2n-1+2|m|+4\omega\epsilon)F_n-4\omega\epsilon F_{n-2}\ . \tag{D17}$$

Given ϵ, m, and ω, one computes $F(z)$ up to $z=\frac{1}{2}$, say, and also $F(y)$ up to the same matching point $y=\frac{1}{2}$. The eigenvalue k is then determined from the matching condition

$$\left[\frac{1}{F(z)}\frac{dF}{dz}\right]_{z=1/2}+\left[\frac{1}{F(y)}\frac{dF}{dy}\right]_{y=1/2}=0\ . \tag{D18}$$

Since the radius of convergence of the series expansions for $F(y)$ and $F(z)$ is 1, these series converge well at the matching point of $\frac{1}{2}$.

In the case of positronium, with $\omega=\frac{1}{2}$, the denominator $(k+\omega)$ in Eq. (D9) becomes very small for $m=+\frac{1}{2}$, and the expansion $F(z)$ in Eq. (D8) fails. Instead of the matching condition (D18), we have used in this case $(m=+\frac{1}{2})$ the series $F(y)$ in (D14)–(D17), imposing the condition that at $y=1$, $F(y)$ does not become infinite. Starting with an initial value of k as given by the power series in Eq. (C2), one finds that the coefficients F_n at first decrease rapidly, reaching a flat plateau where they are of the same sign. The condition that the height of this plateau vanish yields a rapidly converging value for k, which turns out to differ from the initial value only in the 18th decimal.

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