Intensity fluctuations and cross correlations in coupled-mode optical systems

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In nonlinear multimode optical systems, stochastic fluctuations of the intensities of the competing modes have signatures (characteristic moments, cumulants, and cross correlations) that depend on the number of modes and on any constraints on the total intensity. For the cases of maximum entropy fluctuations within a constant total intensity and within a fluctuating total intensity, we derive the intensity probability distribution functions and calculate the moments, cumulants, and cross correlation. The latter two are shown to be sensitive indicators of the number of interacting modes. When the modes are statistically equivalent, particular relations between the variance of one mode and the cross correlation of any two modes can indicate the type of mode coupling independent of their number. The predictions of these models are compared to recent experimental measurements on a multimode source of amplified spontaneous emission. Statistical results for a system that randomly jumps among single-mode states while maintaining constant total intensity are also calculated and are compared with the statistics of a (deterministic) two-mode bidirectional ring-laser simulation. The results are widely applicable to other coupled-mode systems including multimode lasers.

I. INTRODUCTION

Intensity fluctuations in optical systems are often characterized by statistical quantities such as moments, cumulants, and correlation functions.[1-25] More recently, with the recognition that such fluctuations may be the result of dynamical rather than stochastic processes, time series of single-mode and total intensity fluctuations have been used to calculate entropies and dimensions to test for and distinguish between dynamical and stochastic origins.[24,26] However, entropies and dimensions are global averages which often have the same values for fluctuating signals that differ in many other characteristics. One way to distinguish among systems with similar dimensions and entropies is to calculate generalized entropies and dimensions.[27,28] Another way to specify more accurately a fluctuating system is to measure its statistical properties (moments, cumulants, cross correlations, multitime correlation functions). However, in general, not all dynamical and stochastic measures can be accurately determined from experimental signals of limited duration and limited precision. Since a signal is fully specified only by a complete set of higher-order measures, the limited information from both dynamical and stochastic measures may be necessary to infer the underlying properties or constraints of the system. In addition, it is useful to measure the statistical properties of dynamically fluctuating systems in order to discern similarities to or deviations from the behavior of purely stochastic systems.

In this paper we show that certain statistical measures can be used effectively in the analysis of multimode fluctuating signals. For example, we calculate moments and cumulants for three types of multimode stochastic processes and show that they can be used to distinguish among the constraints governing those processes. Measurements of moments or cumulants of single-mode fluctuations or of two-mode cross correlations can be used to infer the total number of interacting modes if particular constraints on the total intensity are either known or assumed.

Our particular interest in multimode systems includes amplified spontaneous emission[1-4,29-31] (where the modes are driven by stochastic noise) and multimode lasers[5-26] (where the modes may be driven stochastically by spontaneous emission, deterministically by coupled-mode dynamics or by a combination of both). In a wide variety of these systems the total output is limited by the rate at which energy is supplied by the pumping mechanism. When the modal intensities are strong and heavily saturate the medium, it has often been reported that the total intensity is nearly constant while the intensities of the individual modes fluctuate more widely.[7-9] This was most recently observed for multimode dye laser data which were analyzed for deterministic behavior.[24,26] This motivates studying stochastic modal intensity fluctuations within the constraint of a constant total intensity.

In selecting a model for coupled-mode interactions, one must first select between describing the modes by their intensities or by their amplitudes. Despite the neglect of phase fluctuations, stochastic models for coupled modal intensities[1-3,6,12,15] lead to the same intensity fluctuations as models using coupled modal field amplitudes[4,5,13] when the gain of the medium (and hence the total intensity) is heavily saturated. This indicates that phase fluctuations in these cases do not couple strongly with intensity fluctuations under heavily saturated conditions. Thus the results from studies of models for coupled intensities are broadly applicable and will be used in the present analysis.

In this paper we will compare the intensity fluctuations in four different models: in Sec. II, stochastic, maximum entropy fluctuations of the individual modal intensities within the constraint of a constant total intensity; in Sec. III, stochastic, maximum entropy fluctuations of individual modal intensities with a fluctuating total intensity; in Sec. IV, stochastic fluctuations of individual modal intensities within the constraint of a constant total intensity and under the condition that at any one time only one mode is on (the others are off); and in Sec. V, chaotic (deterministic) switching in a two-mode laser in which the total intensity is nearly constant. To distinguish the different cases or select the appropriate model for a given experimental system, several normalized statistics and cross-correlation functions must be used. Measurements of two modes simultaneously and calculation of the cross correlation permit inferences to be drawn regarding the fluctuations of the total intensity as well as determination of the number of modes participating in the interaction.

Stochastic models are clearly appropriate for sources (both cosmic and laboratory) of amplified spontaneous emission (ASE) where no evidence for low-dimensional deterministic chaos has been found in the random fluctuations.[31] Some multimode laser phenomena also seem to be best described by stochastic models (while others are clearly dynamical). Even for deterministic systems, stochastic models can provide benchmarks, thereby facilitating a clear determination of those properties arising from dynamics.

One application of these results would be in analyzing the rates of multiphoton processes generated by a multimode beam. Often the rates of an m-order process exceed the predictions based on the mth moment of the total intensity fluctuations. The excess is attributed to the larger relative fluctuations of the constituent modal intensities. For example, the excess above unity of normalized moments for individual modes has been used to explain excess multiphoton ionization rates[20-22] and excess second (and higher) -harmonic generation efficiencies for multimode lasers[22,23(a)] (as compared with single-mode lasers of the same average intensity). Sometimes the logic has been inverted and the excess rates have been used to infer the number of independent degrees of freedom (quasimodes or mode clusters) in the multimode laser operation.[19,21] However, the connection between number of modes and the excess rate depends on assumptions about the constraints on the modal fluctuations. Though cumulants may provide high-resolution characterizations of fluctuating signals, moments are the appropriate measures for such calculations or inferences when the physical phenomena rely on a particular power of the instantaneous intensity.

In the following sections we consider the four different models outlined above. We apply our results to the analysis of fluctuating ASE signals as an example.

II. MAXIMUM ENTROPY FLUCTUATIONS WITHIN A CONSTANT TOTAL INTENSITY

Systems for which the total intensity is nearly constant include broadband and multimode amplified spontaneous emission[1-3] (including astrophysical masers), nonresonant feedback lasers,[1,4] and some cases of multimode laser action.[5-24,26] In all of these cases the total intensity is stabilized by the common interaction of the modes with the gain of the medium and the resulting mutual cross saturation arising from the saturation of the toal energy extracted from the medium. Specifically for ASE in a laser amplifier, it has been shown that for low saturation (short lengths) the total intensity fluctuates, but for high saturation (long lengths) the sum of the modal intensities becomes more nearly constant.[3,6]

By renormalizing the total intensity of multimode Gaussian radiation to have a constant value ("energy-stabilized Gaussian radiation"), Masalov[19] derived the intensity probability distribution function (IPDF) for the intensity of a single mode,

$$p_n(I_i)=(n-1)C^{1-n}(C-I_i)^{n-2} , \tag{1}$$

where I_i is the intensity of the ith mode ($I_i \geq 0$, $\forall i$), n is the number of modes, and C is the constant total intensity. Masalov and others have found that when the number of modes interacting under the constraint is large, the distribution of each individual mode approaches a negative exponential distribution[1,5,12,15,21] (the result for thermal light and spontaneous emission). This result indicates that the degree of gain saturation cannot be inferred from the fluctuations of a single mode if the total number of modes is large. This contrasts sharply with our intuition developed for single-mode lasers and single-mode ASE because in those cases the normalized variance of the fluctuations is readily reduced to zero with increasing gain saturation.[32-36]

In this section we assume a constraint on the intensities given by

$$\sum_{i=1}^{n} I_i=C . \tag{2}$$

We make the further assumption that the variations in the modal intensities have the maximum entropy distribution within the constraint of Eq. (2). This will be shown to agree with Masalov's "energy-stabilized Gaussian radiation" IPDF for a single mode.

A. Derivation of intensity probability distribution functions

The derivation of the probability distribution functions can proceed along two apparently different paths. In both methods the IPDF is sought which maximizes the entropy under the constraint of Eq. (2). The first method uses the conceptual idea that the maximum entropy distribution results when all points on the section of the $(n-1)$-dimensional hyperplane defined by Eq. (2) are equally probable.

The second method is to consider the constraint of Eq. (2) as subdividing a line segment of length C into n parts.

By taking partitions as equally likely, one can compute the desired probability distributions. Partitioning the line segment is the continuum limit of partitioning a fixed number of boson particles into n "compartments."[1,37]

For n modes the uniform distribution on an $(n-1)$-dimensional hyperplane has the probability distribution function

$$p_n(I_1,\ldots,I_n)=\hat{N}\delta(I_1+\cdots I_n-C)\;, \tag{3}$$

where $\hat{N}$ is a normalization constant. Repeated integration yields the normalization constant,[38]

$$\hat{N}=(n-1)!C^{1-n}\;, \tag{4}$$

so that

$$p_n(I_1,I_2,\ldots,I_n)=(n-1)!C^{1-n}\delta(I_1+\cdots+I_n-C)\;. \tag{5}$$

By integrating this joint IPDF over $(n-k)$ modal intensities, we obtain $p_n(I_1,I_2,\ldots,I_k)$, the joint distribution for k out of n coupled variables. Geometrically this is equivalent to projecting the hyperplane onto k dimensions and calculating the fractional area of the hyperplane that is projected onto each point.

The general result for the k-variable joint probability distribution of a system of n coupled variables is given by

$$p_n(I_1,\ldots,I_k)=\int_0^\infty dI_{k+1}\cdots\int_0^\infty dI_n p_n(I_1,\ldots,I_n)\;, \tag{6}$$

for $k<n$; therefore,

$$p_n(I_1,\ldots,I_k)=\frac{(n-1)!}{C^{n-1}}\frac{(C-I_1-I_2-\cdots-I_k)^{n-k-1}}{(n-k-1)!}\;. \tag{7}$$

For the case $k=1$ and arbitrary n, we obtain Eq. (1), and for $k=2$,

$$p_n(I_1,I_2)=(n-1)(n-2)C^{1-n}(C-I_1-I_2)^{n-3}\;. \tag{8}$$

These results may also be obtained using the second method in which the interval $[0,C]$ is partitioned into n subintervals. A still more fundamental quantum derivation starts from the expression for the probability of finding k indistinguishable bosons in a given cell of phase space (or mode of the field),

$$p_k=\frac{\begin{bmatrix} n+r-k-2\\ r-k\end{bmatrix}}{\begin{bmatrix} n+r-1\\ r\end{bmatrix}}\;, \tag{9}$$

where r is the total number of bosons and n is the number of cells (modes). By letting r and k tend to infinity, using Stirling's formula for those factorials involving large quantities, changing variables to $C=r\alpha$ and $I=k\alpha$, where α has units of intensity, and normalizing the integral of the resulting IPDF, we recover Eq. (1). A similar procedure will recover Eq. (8). From the standard derivation of p_k one sees that this is the quantum (discrete) equivalent of partitioning the interval $[0,C]$.

B. Moments, cumulants, and cross correlations

From the general results for n modes, the following results are found:

$$\langle I_1\rangle=\frac{C}{n}\;, \tag{10a}$$

$$\frac{\langle I_1^q\rangle}{\langle I_1\rangle^q}=\frac{n^q q!(n-1)!}{(n+q-1)!}\;, \tag{10b}$$

$$k_2\equiv\frac{\langle (I_1-\langle I_1\rangle)^2\rangle}{\langle I_1\rangle^2}=\frac{(n-1)}{(n+1)}\;, \tag{10c}$$

$$k_3\equiv\frac{\langle (I_1-\langle I_1\rangle)^3\rangle}{\langle I_1\rangle^3}=\frac{2(n-1)(n-2)}{(n+2)(n+1)}\;, \tag{10d}$$

$$k_4\equiv\frac{\langle (I_1-\langle I_1\rangle)^4\rangle}{\langle I_1\rangle^4}-3k_2^2$$
$$=\frac{6(n^4-5n^3+3n^2+7n-6)}{(n+3)(n+2)(n+1)^2}\;, \tag{10e}$$

$$c_{12}\equiv\frac{\langle I_1I_2\rangle-\langle I_1\rangle\langle I_2\rangle}{\langle I_1\rangle\langle I_2\rangle}=\frac{-1}{(n+1)}\;, \tag{10f}$$

$$c'_{12}=\frac{\langle I_1I_2\rangle-\langle I_1\rangle\langle I_2\rangle}{\langle (I_1-\langle I_1\rangle)^2\rangle}=\frac{-1}{(n-1)}\;, \tag{10g}$$

$$\langle (I_1-\langle I_1\rangle)^q\rangle=q!(n-1)!C^q\sum_{m=0}^{q}\frac{(-1)^m}{m!(n-m+q-1)!n^m}\;. \tag{10h}$$

Some of these results, notably Eqs. (10c), (10f), and (10g), have been reported earlier by others who have considered stationary solutions for n-mode systems in the limit of heavily saturated gain.[5] The distributions $p_n(I_i)$ are plotted for $n=2$, 3, 4, 8, and ∞ (for the arbitrary choice of $\langle I_i\rangle=10$) in Fig. 1.

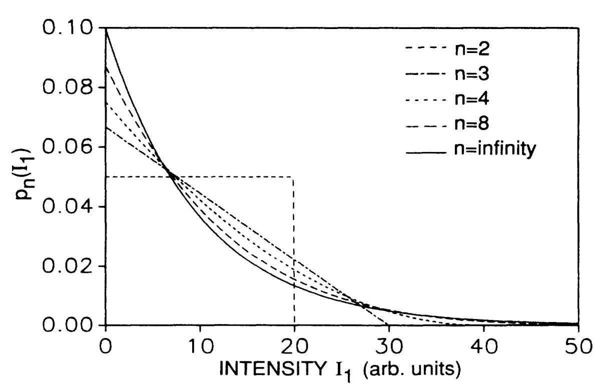

FIG. 1. Plot of the single-mode intensity probability distribution function $p_n(I_1)$ vs I_1 for $n=2$, 3, 4, 8, and ∞ for maximum entropy fluctuations within the constraint $\sum_i I_i=C$. $\langle I_1\rangle$ is arbitrarily set to 10 in each case.

The moment-generating function defined by

$$M_n(\lambda)\equiv\int_0^C \exp(\lambda I)P(I)dI \tag{11}$$

yields the moments by differentiation

$$\langle I^q\rangle=\left[\frac{d^q}{d\lambda^q}M_n(\lambda)\right]_{\lambda=0} . \tag{12}$$

Evaluating the integral in Eq. (11) yields

$$M_n(\lambda)=\sum_{q=0}^{\infty}\frac{(\lambda C)^q(n-1)!}{(n+q-1)!} . \tag{13}$$

As this cannot be written in closed form, the cumulant generating function,

$$Q_n(\lambda)=\ln M_n(\lambda) , \tag{14}$$

cannot be evaluated directly. However, the cumulants can be found from the moments with the usual relations implicit in this formula.[39] Higher accuracy in the numerical calculations (by avoiding cumulative errors) is obtained using the analytical expressions for the central moments [Eq. (10h)] to calculate the cumulants because they include differences of the moments. Note that the cumulants in Table I have a much more sensitive dependence than the moments on the number of modes. This arises from the fact that each cumulant is independent of the values of the other cumulants, unlike the dependence, in part, of each moment on all lower-order moments.

Normalized cumulants are compiled in Table I for $n=2$–10, 15, 20, 25, 100, 500, and ∞. In ac-coupled measurements it is often difficult to determine $\langle I_i\rangle$, so the cumulants normalized to $k_2^{q/2}$, defined to be k_q' and given in Table I, are of considerable practical value. Comparisons of measured statistical fluctuations in experimental systems with Eqs. (10) and Table I may aid in identifying the number of interacting modes or in confirming whether the model is applicable.

As an example of the usefulness of these results, we consider recent experimental measurements of one polarized component of the output of an ASE source[3,29–31] summarized in Fig. 2.[30] For discharge lengths above 160 cm, the values of the variance ($\equiv K_{02}$) and higher-order cumulants (K_{03} and K_{04}) remained nearly constant [see Fig. 2(a)] while the cross correlation of orthogonally polarized copropagating beams c_{12}' [see Fig. 2(b)] continued to decrease, reaching a value of -0.145 ± 0.002 at the longest length measured. (For clarity we use K_{0j}, as in the notation of Ref. 30, as the experimentally determined values of k_j in arbitrary units.) The values of k_3' and k_4' for the longest lengths are 1.79 ± 0.08 and 4.9 ± 0.3, respectively. Using Eqs. (10c)–(10e) and the values of k_3' and k_4' as if they were asymptotic values, we find the estimated number of modes to be 27.8 and 41, respectively, with the range of $n(k_3')$ being [19.5,46.9] and the range of $n(k_4')$ being [30,61]. However, other measurements[3(a),31(b)], for more heavily saturated ASE signals give values for k_3' and k_4' consistent with the number of modes being in the range 7–12. Using Eq. (10g) as if $c_{12}'=-0.145\pm0.002$ were an asymptotic limit suggests eight (7.92 ± 0.07) modes interacting in the inhomogene-

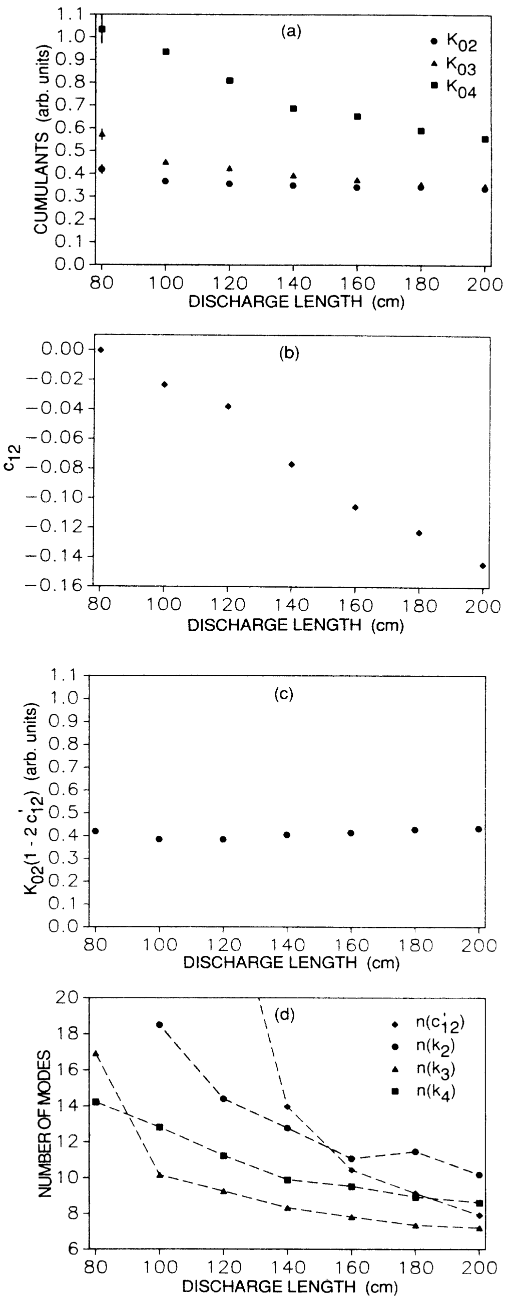

FIG. 2. (a) The cumulants K_{02}, K_{03}, and K_{04} (in arbitrary units) of the intensity of a single polarization component of the output from the ASE source in Ref. 30 plotted vs discharge length. The partial pressure of Xe was 182 mTorr and the partial pressure of He was 1.9 Torr. (b) The cross correlation c_{12}' (normalized to the square root of the product of the variances of the two signals) of orthogonally polarized copropagating beams from the source in (a). (c) The quantity $K_{02}(1-2c_{12}')$ vs discharge length. (d) The calculated number of modes using the cumulants in (a), renormalized by dividing by 0.41 ± 0.02 and substituted into Eqs. (10c)–(10e), and the cross correlation in (b) substituted into Eq. (10g) vs discharge length. The uncertainties for $n(k_2)$ range from 10–16%. The uncertainties for $n(k_3)$, $n(k_4)$, and $n(c_{12}')$ are the size of the data point or smaller. The dashed lines serve as a guide to the eye.

TABLE I. Normalized cumulants $k'_q(\equiv k_q/k_2^{q/2})$ and cross correlation c'_{12} for the distributions given by Eq. (1).

n	k'_3	k'_4	k'_5	k'_6	k'_7	c'_{12}
2	0.0000	−1.2000	0.0000	6.8571	0.0000	−1.0000
3	0.5657	−0.6000	−2.4244	−0.3429	20.3647	−0.5000
4	0.8607	0.0952	−2.4590	−6.4198	3.8252	−0.3333
5	1.0498	0.6964	−1.6622	−9.1454	−17.2379	−0.2500
6	1.1832	1.2000	−0.5679	−9.4036	−33.6119	−0.2000
7	1.2830	1.6222	0.6065	−8.0636	−43.4984	−0.1667
8	1.3607	1.9792	1.7671	−5.7385	−47.3886	−0.1429
9	1.4230	2.2841	2.8733	−2.8338	−46.3654	−0.1250
10	1.4741	2.5470	3.9092	0.3871	−41.5339	−0.1111
15	1.6350	3.4538	8.0398	17.0161	12.3271	−0.0714
20	1.7203	3.9854	10.8495	31.1344	79.2821	−0.0526
25	1.7733	4.3340	12.8431	42.3191	141.4732	−0.0417
100	1.9409	5.5374	20.6337	94.0725	503.2345	−0.0101
500	1.9880	5.9047	23.2897	114.3620	671.1297	−0.0020
∞	2.0000	6.0000	24.0000	120.0000	720.0000	0.0000

ously broadened ASE source. The contradictory results for the data in Ref. 30 suggest that the constant total intensity model is not appropriate for this length, perhaps because the gain medium is not yet fully saturated (as is suggested by the continued decrease in c'_{12} and K_{04} with increasing length of the ASE source). Noninteger numbers of modes (and values larger than 4—two directions, two polarizations) may be explained for ASE as resulting from an inhomogeneously broadened medium in which there is incomplete cross-spectral coupling leading to a poor spectral definition of modes, or by off-axis waveguided modes. We return to ASE data in Sec. III, where we consider the case of a fluctuating total intensity.

C. Limit of a large number of modes (n)

Taking the limit $n\to\infty$ in Eq. (1) and using the relation $\langle I_1\rangle = C/n$, we have after substituting for C,

$$p_\infty(I_1)=\frac{1}{\langle I_1\rangle}\lim_{n\to\infty}\frac{n-1}{n}\left[1-\frac{I_1}{n\langle I_1\rangle}\right]^{n-2}$$
$$=\frac{1}{\langle I_1\rangle}\exp(-I_1/\langle I_1\rangle)\ , \qquad (15)$$

which is the negative exponential intensity probability distribution function characteristic of the Gaussian amplitude statistics of spontaneous emission and thermal light. This is the well-known maximum entropy result for a positive random variable with a given mean. This result can also be seen in the limits of Eqs. (10b)–(10e). The correlation functions vanish in the limit of $n\to\infty$, giving the same result as for n uncoupled negative exponential distributions. Thus the "thermal" or "spontaneous-emission" negative exponential probability distribution applies equally well to a mode with heavily saturated gain interacting (via the gain) with a large number of other modes.

III. MAXIMUM ENTROPY FLUCTUATIONS WITH A FLUCTUATING TOTAL INTENSITY

The Langevin equations for the modal amplitudes of a two-mode laser perturbed by noise can be solved by Fokker-Planck equations, with the result that the intensity and phase fluctuations are decoupled. The steady-state intensities for the two modes have the joint probability distribution function[5,40]

$$p(I_1,I_2)\propto\exp\left[-\frac{(I_1+I_2-A)^2}{2}\right]\ , \qquad (16)$$

where the intensities are normalized to dimensionless units. This is equivalent to the condition $I_1+I_2=C$, where C satisfies a Gaussian distribution with a mean value of A and a normalized variance of A^{-2}. In the limit $A\gg 1$ the result approaches the δ-function distribution $\delta(I_1+I_2-A)$.

If a multimode laser or a multimode ASE source is not highly saturated, then C is not a constant and varies according to a distribution function [such as represented in Eq. (16)]. With the further assumption that the fluctuations in C occur at a much slower rate than those of the modal intensities, the results for constant C can be generalized by simply averaging over C. The effect on a few of the statistics is as follows:

$$\frac{\langle I_1^q\rangle}{\langle I_1\rangle^q}=\frac{n^q q!(n-1)!}{(n+q-1)!}\frac{\langle C^q\rangle}{\langle C\rangle^q}\ , \qquad (17)$$

$$k_2=\frac{2n}{(n+1)}\frac{\langle C^2\rangle}{\langle C\rangle^2}-1 \qquad (18a)$$

$$=\frac{n-1}{n+1}+\frac{2n}{(n+1)}\frac{\sigma_C^2}{\langle C\rangle^2}\ , \qquad (18b)$$

and

$$c_{12}=\frac{n}{(n+1)}\frac{\langle C^2\rangle}{\langle C\rangle^2}-1 \tag{19a}$$

$$=-\frac{1}{n+1}+\frac{n}{(n+1)}\frac{\sigma_C^2}{\langle C\rangle^2}, \tag{19b}$$

where $\sigma_C^2=\langle C^2\rangle-\langle C\rangle^2$. The result in (19b) is 0 if $\sigma_C^2=\langle C\rangle^2/n$, which is the condition for the gamma distribution. (The gamma distribution results for the sum of n-independent negative exponential random variables, i.e., the case of completely thermal, randomly phased, noninteracting modes.)

Surprisingly, combining the expressions for k_2 and c_{12} to eliminate $\langle C^2\rangle/\langle C\rangle^2$ yields a result independent of n. From Eqs. (18b) and (19b), we find that

$$k_2=1+2c_{12} \tag{20a}$$

or

$$k_2(1-2c'_{12})=1 . \tag{20b}$$

Though it may appear that this relation depends on the strong coupling of the modes or on the assumption that the system evolves on two time scales [a rapid rate of modal fluctuations within a particular value of C and a (relatively) slow rate for fluctuations of C itself] it results, instead, from another assumption, which is outlined in the next paragraph. First, however, it is worth noting that Eqs. (20) are also valid for independent thermally fluctuating modes. In addition, they can be derived[41] (for any degree of saturation or number of modes) using a set of intensity rate equations of the form

$$\frac{dI_i}{dz}=\frac{\gamma I_i}{1+\sum_{k=1}^{n} I_k}, \tag{21}$$

where (i) each mode begins with an independent negative exponential (thermal) distribution and (ii) the modes are treated as having equal gains and equal cross-saturation effects.

A quite general result for any number of equally distributed modes (regardless of the specifics of their coupling) is, in our notation,[42]

$$n\frac{\sigma_c^2}{\langle C\rangle^2}=k_2+(n-1)c_{12} , \tag{22}$$

which is satisfied by Eqs. (18) and (19). The special result of Eq. (20) [satisfied by the maximum entropy cases of constant C and fluctuating C, and by all solutions of Eq. (21), as described above] appears to result from the radial symmetry of the multivariable probability distribution in the space of the complex modal amplitudes. This radial symmetry in the "amplitude variable space" is equivalent to the uniform probability density on planes of constant total intensity in the variable space of modal intensities. When this radial symmetry exists there is a more general equivalence of the fluctuations of any (normalized) linear combination of modes.

Violation of Eq. (22) indicates that the modes are not statistically equivalent. Violation of Eq. (20) without violating Eq. (22) indicates that the multivariable amplitude distribution is not radially symmetric (or, equivalently, that all points on a plane of constant total intensity in the variable space of modal intensities are not equally probable). (An example of the latter is given in Sec. IV.)

As shown in Fig. 2(c), the values of K_{02} and c'_{12} in the ASE data referred to earlier obey an equation similar to Eq. (20b), $K_{02}(1-2c'_{12})=0.41\pm0.02$, over all the discharge lengths for which results were obtained. Since at sufficiently short lengths $c'_{12}=0$ and $k_2=1$, we have that $k_2(1-2c'_{12})=1$. If we assume that our system is composed of statistically equivalent modes satisfying Eqs. (20) and thus that the variance, when normalized to the actual mean intensity squared, is $k_2=K_{02}/(0.41\pm0.02)$ [in which case $k_2(1-2c'_{12})=1$ for all discharge lengths], then we can calculate $k_3=K_{03}/(0.41\pm0.02)^{3/2}$ and $k_4=K_{04}/(0.41\pm0.02)^2$ and use Eqs. (10d) and (10e) to determine the number of modes for the maximum entropy, constant total intensity model. The results, shown in Fig. 2(d), suggest that, if the uncertainties in the original cumulants were larger, then this may be a correct assumption because the different normalized cumulants and c'_{12} appear to be approaching values consistent with the same number of modes (given approximately by $n=7-9$). That these numbers differ from the results using k'_3 and k'_4, however, indicates that the model is not fully applicable.

IV. MAXIMUM ENTROPY MODE-HOPPING MODEL

One might also consider that under certain conditions all values of $\{I_i\}$ satisfying the constraint [Eq. (2)] may not be equally likely. If, as is true for the two-mode laser,[5,40,43] the preferred condition is for all modes to have zero intensity except for one mode, then the single-mode IPDF is

$$p(I_1)=\frac{n-1}{n}\delta(I_1)+\frac{1}{n}\delta(I_1-C) , \tag{23}$$

and the moment generating function is given by

$$M(\lambda)=\left[\frac{n-1}{n}+\frac{1}{n}\exp(\lambda C)\right] . \tag{24}$$

This distribution has the following characteristics:

$$\langle I_1\rangle=\frac{C}{n} , \tag{25a}$$

$$\frac{\langle I_1^q\rangle}{\langle I_1\rangle^q}=n^{q-1} , \tag{25b}$$

$$k_2=n-1 , \tag{25c}$$

$$k_3=n^2-3n+2 , \tag{25d}$$

$$k_4=n^3-7n^2+12n-6 , \tag{25e}$$

$$c_{12}=-1 , \tag{25f}$$

$$c'_{12}=-1/(n-1) , \tag{25g}$$

TABLE II. Normalized cumulants $k'_q(\equiv k_q/k_2^{q/2})$ and the cross correlation c'_{12} for the mode-hopping model.

n	k'_3	k'_4	k'_5	k'_6	k'_7	c'_{12}
2	0.0000	−2.0000	0.0000	16.0000	0.0000	−1.0000
3	0.7071	−1.5000	−5.3033	5.2500	77.9585	−0.5000
4	1.1547	−0.6667	−7.6980	−11.5556	79.0328	−0.3333
5	1.5000	0.2500	−8.6250	−28.4375	36.0938	−0.2500
6	1.7889	1.2000	−8.5865	−44.1600	−36.0633	−0.2000
7	2.0412	2.1667	−7.8248	−58.3056	−129.2219	−0.1667
8	2.2678	3.1429	−6.4794	−70.6939	−238.0713	−0.1429
9	2.4749	4.1250	−4.6404	−81.2344	−358.8180	−0.1250
10	2.6667	5.1111	−2.3704	−89.8765	−488.5597	−0.1111
15	3.4744	10.0714	14.1458	−103.8520	−1202.1233	−0.0714
20	4.1295	15.0526	37.3827	−68.3657	−1899.3335	−0.0526
25	4.6949	20.0417	65.9236	16.9184	−2461.6610	−0.0417

$$\langle (I_1-\langle I_1\rangle)^q\rangle = \left[\frac{-C}{n}\right]^q + C^q \sum_{m=1}^{q} \frac{q!}{m!(q-m)!} \times (-1)^{m+q} n^{m-q-1} , \qquad (25h)$$

$$k_2 c'_{12} = -1, \quad k_2(1-2c'_{12}) = n+1 . \qquad (25i)$$

Note that Eqs. (25c) and (25f) satisfy Eq. (22), but Eq. (25i) indicates a violation of Eq. (20) resulting from the breaking of the "radial symmetry" in this case.

Some of these characteristics are similar to those for the maximum entropy, equal probabilities case. In Eq. (25g) we see that there are the same relative anticorrelated fluctuations in the modes for this constraint as are found in Eq. (10g), where they are normalized to $\sigma^2 \equiv \langle I_i^2\rangle - \langle I_i\rangle^2$. We also see in Eq. (25c) that "superthermal" ($\sigma^2 > \langle I_i\rangle^2$) fluctuations result for $n>2$. It is clear that one must calculate several different characteristics of the fluctuations to unambiguously identify the underlying distribution and the physical process. Again, cumulants and cross correlations give the sharpest distinctions between models and provide better discrimination of the number of modes than do moments and central moments. The normalized cumulants of the distribution given by Eq. (23) are shown in Table II.

V. DYNAMICALLY COUPLED SWITCHING OF TWO MODES

Examples of dynamically coupled modes have recently been studied by Raymer and co-workers[24,25] and Atmanspacher, and Baev and co-workers,[26] among others. For comparison with the stochastic cases discussed above, we show an example of two coupled modes of a bidirectional ring laser.[43] Figure 3 shows the time-dependent behavior for this deterministic system (for the detuning parameter $\Delta=0.1$). It approximately fulfills the "mode-hopping" conditions of Eqs. (23) and (25f) [as shown in Fig. 3(a), the switching time is much less than the dwell time] and the constraint of a nearly constant total intensity [as shown by the plot of I_1 versus I_2 in Fig. 3(b)]. The probability distribution in Fig. 3(c) shows two sharp peaks, as given in Eq. (23). Note that the intensities are not periodic "square" waves. The variable duration of the "on"

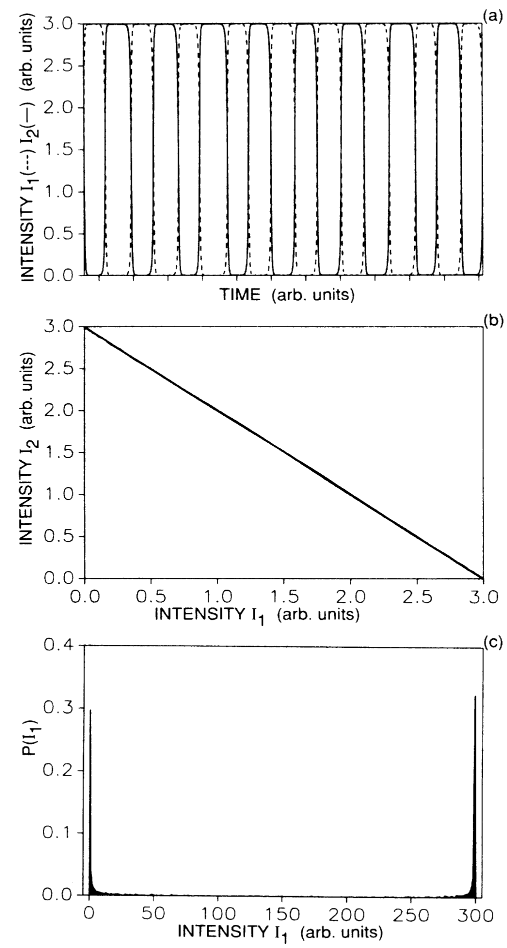

FIG. 3. (a) Intensity vs time for two modes in a dynamical bidirectional ring-laser model (Ref. 43). (b) Plot of I_1 vs I_2 for the data in (a). (c) Probability histogram of intensity values $P(I_1)$ for 300 equally spaced intervals.

TABLE III. Normalized cumulants k_q (normalized to $\langle I \rangle^q$) and the cross correlation c_{12} for the $n=2$ case of the mode-hopping model [cf. Eqs. (25)] and for numerical solutions for both modes of a two-mode bidirectional ring laser (Ref. 43) (Δ is a detuning parameter). The larger detuning cases have more intensity modulation in the "on" mode (cf. Ref. 43) than the $\Delta=0.1$ case shown in Fig. 3.

	k_2	k_3	k_4	k_5	k_6	k_7	c_{12}
$n=2$	1.000	0.000	−2.00	0.00	16.0	0.0	−1.0000
Mode 1							
$\Delta=0.1$	0.843±0.007	−0.059±0.005	−1.38 ±0.02	0.39±0.03	9.1 ±0.2	−5.6±0.4	−0.9104±0.0002
$\Delta=0.2$	0.870±0.002	0.013±0.002	−1.440±0.006	−0.08±0.01	9.73±0.06	1.2±0.2	−0.8561±0.0002
$\Delta=0.4$	0.78 ±0.01	0.12 ±0.02	−0.95 ±0.02	−0.47±0.08	5.2 ±0.1	5±1	−0.746 ±0.003
Mode 2							
$\Delta=0.1$	0.983±0.007	0.074±0.008	−1.88 ±0.03	−0.57±0.06	14.4 ±0.3	9±1	−0.9104±0.0002
$\Delta=0.2$	0.845±0.002	−0.009±0.001	−1.361±0.005	0.07±0.01	8.94±0.05	−0.9±0.1	−0.8561±0.0002
$\Delta=0.4$	0.83 ±0.01	0.20 ±0.01	−0.10 ±0.03	−0.86±0.07	5.4 ±0.3	10±1	−0.746 ±0.003

and "off" times for each mode are evident in Fig. 3(a). In this case the dynamical switching appears much like the stochastic switching within the mode-hopping constraint. One may view this as evidence that, in this case, the dynamical contraction of phase space is similar to the stochastic mode-hopping constraint.

The deterministic intensity fluctuations shown in Fig. 3 have similar statistics to those of Eq. (23). Shown in Table III are the normalized cumulants k_n through order 7 and the cross correlation c_{12} for the $n=2$ case of the mode-hopping model and for single-mode intensity fluctuations in the bidirectional-ring-laser simulation for $\Delta=0.1$, 0.2, and 0.4 [as the detuning parameter is increased from 0.1, the intensity fluctuations increasingly deviate from the "on" and "off" character shown in Fig. 3(a) and from the constraint $I_1+I_2=C$]. Using the statistics of the mode-hopping model as benchmarks, it is clear that as Δ is decreased, the dynamical ring-laser system behaves more like the stochastic mode-hopping system. At the same time, the higher-order cumulants k_6 and k_7, being measures of more subtle differences, distinguish the ring-laser intensity fluctuations from those of the stochastic mode-hopping model.

VI. INDEPENDENT GAUSSIAN MODES

For purposes of comparison, we also present the general results when the total intensity is given by the sum of n randomly phased independent modes all having Gaussian amplitude statistics (negative exponential intensity statistics) with the same mean. In this case each individual mode's distribution has normalized qth-order moments (normalized to $\langle I_1 \rangle^q$) given by $q!$ and similarly normalized qth-order cumulants given by $(q-1)!$. The total intensity fluctuates with cumulants given to all orders by the sum of the cumulants of the contributing modes. The result is that the normalized qth-order cumulants of the total intensity are $(q-1)!/n^{q-1}$ and the IPDF for the total intensity is a gamma distribution. When n is large, the fluctuations in the total intensity will be small as in the case of coupled modes. However, for all n, there will be no modal cross correlations and no deviations of individual mode fluctuations from those satisfying a negative exponential distribution.

VII. CORRELATION FUNCTIONS

Kovalenko[13] has derived expressions for the correlation functions for coupled modes. Assuming that systems such as ours have diffusive motion with two diffusion rates, one for motion on the hyperplane of fixed C and the other for variation of C, then it is natural to write an expression of the same form as Kovalenko's for the time dependence of any second-order correlation function $f(\tau)$

$$f(\tau)=h_1(0)\exp(-D_1|\tau|)+h_2(0)\exp(-D_2|\tau|) . \quad (26)$$

In this equation $h_1(0)$ is the result of Eqs. (10), $h_2(0)$ is the correction from Sec. III (additional terms dependent on C) for the same moment or cumulant, and D_1 and D_2 are the two diffusion constants mentioned above. We believe that Kovalenko's correlation functions erroneously include a factor of $\frac{1}{2}$. Otherwise, his correlation functions for $\tau=0$ agree with the moments and cumulants derived here and by others.

VIII. DISCUSSION

Another application of these results would be comparison with the statistics for coupled-mode lasers. Reports of normalized variances of the fluctuations of individual modes reaching values larger than unity[24] ("superthermal") are not consistent with the assumptions of maximum entropy fluctuations and constant total intensity. However, they are consistent with the mode-hopping model (for $n>2$) and for some distributions for fluctuating C. It also may be that superthermal results occur if the modal intensities are taken to have different mean values (i.e., different modal parameters) without having to relax the constraint of constant total intensity or maximum entropy within the constraints of fixed mean values and fixed total intensity. Alternatively, superthermal fluctuations may be intrinsically related to the chaotic (deterministic) evolution of the multimode system. Careful analysis of each of these may explain the observed superthermal fluctuations and would be instructive. Work on extending the present models to incorporate these effects is presently underway.

Besides the application of these results to different optical (and non-optical) systems obeying constraints of the form $\sum_i I_i = C$, this research can be extended by considering the case of modes with unequal weighting (i.e., different mean values, as mentioned in the preceding paragraph). This is the case of different gain or loss coefficients for each mode (as in the differences in gain available to laser modes with different frequency detunings from the peak of the gain profile or as in different portions of the ASE spectrum interacting with different groups of atoms within an inhomogeneous gain profile). Another alternative is to consider the effect of unequal cross-saturation coefficients, such as typically appears for modes with different detunings from the atomic resonance. This is even more dramatic in an inhomogeneously broadened medium when the homogeneous linewidth is sufficiently less than the inhomogeneous linewidth that the interaction between the modes is limited to those that couple to the same homogeneously broadened groups of atoms.

In conclusion, statistics for fluctuations in a saturated multimode system can be calculated from relatively simple assumptions. For each model of the modal interactions, unique characteristics are derived that depend only on the number of interacting modes. However, the number of interacting modes cannot be determined from the measurement of a single normalized moment without an *a priori* assumption about which model governs the mode couplings. It is best to measure cross correlations between the modes and a variety of cumulants. From these it may be possible to determine the number of modes, the nature of their interactions, and the symmetries of the multivariable probability distribution. Characterization of fluctuating signals by their cumulants and cross correlations can now be more effectively used to complement the calculation of dimensions and entropies.

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