# Squeezing and quantum-noise quenching in phase-sensitive optical systems

J. Bergou,* M. Orszag,† M. O. Scully, and K. Wódkiewicz‡
*Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131*
*and Max-Planck-Institut für Quantenoptik, D-8046 Garching, Federal Republic of Germany*
(Received 13 June 1988; revised manuscript received 6 February 1989)

We present a systematic account of the different relations between squeezing, correlated-emission-laser (CEL) quantum-noise quenching, and general phase-diffusion noise expressed in terms of quadrature amplitudes. For a wide class of optical devices as described by a general Fokker-Planck equation, we establish conditions that lead to squeezing and CEL quantum-noise quenching.

## I. INTRODUCTION

In classical physics the complex amplitude $\varepsilon$ of the electric field can always be decomposed into its real amplitude $|\varepsilon|$ and its phase $\phi$. As a result of this decomposition it is possible to study separately different physical properties of the electric field fluctuations associated with the intensity and/or the phase noise. The situation is much more complex in quantum mechanics where a well-behaved (Hermitian) phase operator does not exist. Because of this difficulty, simple phase and amplitude descriptions of quantum fluctuations often lead to fundamental physical difficulties or mathematical inconsistencies. The mathematical situation can be improved considerably if instead of using the phase operator, one uses a trigonometric function of the phase. This approach has been carefully investigated in the past and these is an excellent review article devoted to just this problem.[1]

The recent interest in various physical problems associated with phase-sensitive quantum devices[2] has revived the question of quantum phase and amplitude fluctuations. In the context of squeezed states and reduced quantum fluctuations, the Hermitian components $\hat{a}_1$ and $\hat{a}_2$ of the quantized electric field annihilation operator $\hat{a}$ have been used in order to describe a phase-sensitive decomposition of the light signal.[3] As a result of this decomposition, squeezed quantum fluctuations in the in- or out-of-phase quadratures have been formulated and predicted.[4] In a series of recent experiments these predictions have been confirmed and light with nonclassical statistics has been generated and observed.[5]

In laser physics[6] and quantum optics, we frequently encounter a situation in which a mixture of semiclassical and quantum pictures can be used in order to describe the physical properties of the system. For example, for a laser operating above threshold the number of emitted photons is so large that it is often justifiable to replace the electric field amplitude by $\sqrt{\bar{n}}$, where $\bar{n}$ is the mean steady-state number of radiated photons. On the other hand, the intrinsic laser linewidth associated with the electric phase diffusion caused by spontaneous-emission noise (the Schawlow-Townes linewidth) is a quantum-mechanical effect and the semiclassical approach cannot really treat the problem.

In related work[7(a)] we have recently investigated the possibility of obtaining two-mode laser action in which the individual spontaneous-emission events are correlated via an appropriate preparation of the lasing media. Such a correlated-emission laser (CEL) has been predicted to have a phase-diffusion noise below the Schawlow-Townes limit. These predictions have been confirmed by recent experiments.[7(b)]

This combination of semiclassical and quantum features of the laser radiation motivates us to formulate in this paper a straightforward connection between the phase and amplitude fluctuations and the quantum fluctuations of the $\hat{a}_1$ and $\hat{a}_2$ components. In this way we avoid the mathematical complications related to the use of the quantum-mechanical phase operator and obtain a rather simple picture and description of amplitude and phase fluctuations in terms of quantum-mechanical $\hat{a}_1$ and $\hat{a}_2$ components.

Although some elements of our discussion can be found in the literature,[8,9] we believe we give here the first systematic account of the different relations between squeezing, phase-sensitive spontaneous emission in CEL operation, and phase-diffusion noise expressed in terms of quadrature amplitudes.

In Sec. II of this paper we give a heuristic description of amplitude and phase fluctuations, based on simple geometric arguments, in terms of the $\hat{a}_1$ and $\hat{a}_2$ components. In Sec. III we establish the connection between amplitude-phase fluctuations and quadrature fluctuations using the general properties of a Fokker-Planck equation satisfied by the Glauber's $P$ quasidistribution. We find conditions under which we can obtain phase-sensitive diffusion coefficients such as those that occur in CEL operation and/or squeezing. Our conclusions are to a large extent model independent and cover a broad range of possible applications including correlated-emission lasers,[7] linear amplifiers,[9] and two-photon lasers,[10] etc. In Sec. IV we derive general conditions for squeezing and phase-sensitive noise reduction in terms of the general diffusion coefficients of the Fokker-Planck equation. Finally some concluding remarks are presented.

 

## II. QUANTUM PHASE AND AMPLITUDE FLUCTUATIONS

Squeezing quantum fluctuations of the radiation field is associated with the decomposition of the electric field amplitude into its "$\cos\omega t$" and "$\sin\omega t$" phases. This suggests that the electric field annihilation operator be written as $\hat{a}=\hat{a}_1+i\hat{a}_2$, where $\hat{a}_1$ and $\hat{a}_2$ are the Hermitian amplitudes of the two quadrature phases. The quantum-mechanical properties of these amplitudes imply the uncertainty relation

$$\Delta\hat{a}_1\Delta\hat{a}_2\geq\tfrac{1}{4}\ . \tag{2.1a}$$

Squeezed states of light are those for which

$$(\Delta\hat{a}_1)^2<\tfrac{1}{4}\ \text{or}\ (\Delta\hat{a}_2)^2<\tfrac{1}{4}\ . \tag{2.1b}$$

These are the standard definitions of squeezing.[3]

In laser physics the fundamental quantities of interests are related to amplitude and phase fluctuations. It is well known that far above threshold the amplitude fluctuations are quite small, and occur around a constant value which, in a semiclassical approximation, is given by $\sqrt{\bar{n}}$, where $\bar{n}$ is the steady-state number of emitted photons. In the following we will relate the laser phase and amplitude fluctuations to the Hermitian operators $\hat{a}_1$ and $\hat{a}_2$. We present our arguments using first a simple semiclassical picture of the laser radiation.

From Fig. 1 it is clear that the fluctuation $\delta a_{\parallel}$ is associated with pure amplitude fluctuations of $a$. Simple trigonometry leads to the following relations (see Fig. 1):

$$\delta a_{\parallel}=|\delta a|\cos(\varphi-\varphi_0)=\frac{\delta a e^{-i\varphi_0}+\delta a^* e^{i\varphi_0}}{2}\ , \tag{2.2a}$$

$$\delta a_{\perp}=|\delta a|\sin(\varphi-\varphi_0)=\frac{\delta a e^{-i\varphi_0}-\delta a^* e^{i\varphi_0}}{2i}\ , \tag{2.2b}$$

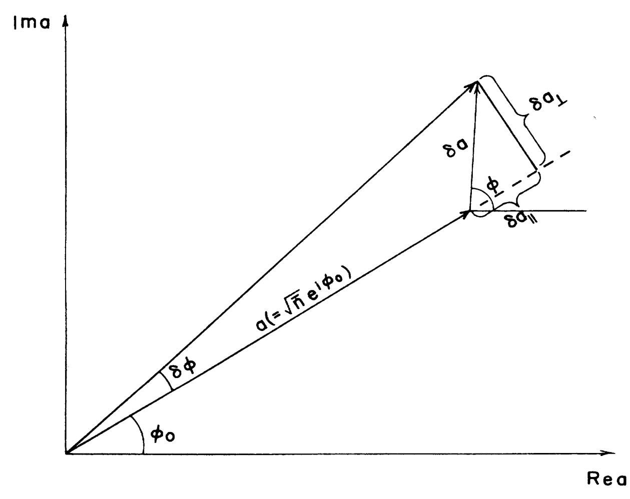


FIG. 1. Amplitude $\delta a_{\parallel}$ and phase fluctuation $\delta\varphi$ of the complex amplitude $a$. From this figure we see that $\delta a_{\perp}=|\delta a|\sin(\varphi-\varphi_0)=\delta a_2(\varphi_0)$ and $\delta a_{\parallel}=|\delta a|\cos(\varphi-\varphi_0)=\delta a_1(\varphi_0)$ with $\delta a=|\delta a|e^{i\varphi}$. For small phase fluctuations $\delta a_{\perp}\sim\sqrt{\bar{n}}\,\delta\varphi$.

where $\delta a=|\delta a|e^{i\varphi}$. In these formulas $\varphi_0$ is the instantaneous phase of the semiclassical electric field amplitude and $\varphi$ is the phase of the fluctuating displacement $\delta a$. From Eqs. (2.2) it is clear that $\delta a_{\parallel}$ and $\delta a_{\perp}$ are related to the following $\varphi_0$-dependent amplitudes:

$$a_1(\varphi_0)=\frac{ae^{-i\varphi_0}+a^* e^{i\varphi_0}}{2}\ , \tag{2.3a}$$

$$a_2(\varphi_0)=\frac{ae^{-i\varphi_0}-a^* e^{i\varphi_0}}{2i}\ . \tag{2.3b}$$

We recognize in these amplitudes the standard $a_1$ and $a_2$ quadratures but rotated by an angle $\varphi_0$ towards the direction fixed by the electric amplitude $a$. For fluctuations leading to a small change of the phase $\delta\varphi$ and amplitude $\delta r$ (see Fig. 1), we have

$$\delta a_{\parallel}=\delta a_1(\varphi_0)=\delta r\ , \tag{2.4a}$$

$$\delta a_{\perp}=\delta a_2(\varphi_0)=a\tan\delta\varphi\sim\sqrt{\bar{n}}\,\delta\varphi\ . \tag{2.4b}$$

Here in the last step we have replaced $a$ by the semiclassical expression $\sqrt{\bar{n}}$ since in the rotated frame its phase is zero. Also we have approximated the tangent by the arc. These relations allow us to approximately identify amplitude and phase fluctuations with the fluctuations of the phase-dependent quantities $a_1(\varphi_0)$ and $a_2(\varphi_0)$.

Up to this point our arguments have been semiclassical. We now extend our conclusions to the quantum case. In this case we simply replace in Eqs. (2.3) the classical amplitudes by boson annihilation and creation operators $\hat{a}$ and $\hat{a}^{\dagger}$ and associate the phase $\varphi_0$ with a rotation of Hermitian observables. Such a unitary transformation leaves the commutation relation intact, i.e., $[\hat{a}_1(\varphi_0),\hat{a}_2(\varphi_0)]=i/2$ and the uncertainty relation (2.1a) is seen to be invariant under arbitrary rotations by the angle $\varphi_0$. This means that by generalizing the semiclassical amplitude and phase fluctuations we obtain the following quantum-mechanical expressions:

$$\langle(\delta a_{\parallel})^2\rangle=\Delta\hat{a}_1(\varphi_0)^2\ , \tag{2.5a}$$

$$\langle(\delta\varphi)^2\rangle=\frac{1}{\bar{n}}\Delta\hat{a}_2(\varphi_0)^2\ , \tag{2.5b}$$

where now the right-hand side of Eqs. (2.5) are the quantum-mechanical variances of the $\hat{a}_1(\varphi_0)$ and the $\hat{a}_2(\varphi_0)$ operators. From these definitions and the relation (2.1) we obtain the following phase-amplitude uncertainty relation:

$$\langle(\delta\varphi)^2\rangle\langle(\delta a_{\parallel})^2\rangle\geq\frac{1}{16\bar{n}}\ . \tag{2.6}$$

Note that this definition is free from problems associated with attempts to construct a quantum phase operator. It has a clear physical interpretation and relates in a simple way the phase and the amplitude fluctuations to the well behaved quantum mechanical observables $\hat{a}_1$ and $\hat{a}_2$.

To what extent the transition from a semiclassical to a quantum picture, as given by Eqs. (2.5), can be justified by a rigorous quantum-mechanical theory will be the subject of Sec. III. It turns out that the relationship between amplitude-phase fluctuations and quadrature fluctuations, as suggested by Eqs. (2.5), is indeed an intrinsic property of a broad class of quantum optical devices. Its validity is related to the existence of a $\varphi_0$ which, at least instantaneously, determines the direction of $\langle \hat{a} \rangle$ on the phase plane of Fig. 1. Also, the fluctuations should not be too large in order that the approximation of the tangent by the arc remains valid. If such a $\varphi_0$ cannot be defined uniquely or the fluctuations are too large the description based on phase-amplitude variables has to be replaced by a description using quadrature components.

For the purpose of this paper we will rewrite the quantum-mechanical variances $\Delta\hat{a}_i^2(\varphi_0)$ $(i=1,2)$ in a form that contains only the normally ordered operators (normally ordered variances) and the commutator contribution

$$\Delta\hat{a}_i^2(\varphi_0) = :\Delta\hat{a}_i^2(\varphi_0): + \tfrac{1}{4} \, . \tag{2.7}$$

This formulation of quantum-mechanical fluctuations has been seen in Ref. 11 to be extremely useful in the discussion of CEL operation when squeezing is also present.

We note that squeezed states of light are those for which $:\Delta\hat{a}_1^2(\varphi_0):$ or $:\Delta\hat{a}_2^2(\varphi_0):$ is less than zero.

Using (2.7) we may rewrite Eqs. (2.5) in the following form:

$$\langle(\delta\varphi)^2\rangle = \frac{1}{4\bar{n}} + \langle :(\delta\varphi)^2: \rangle \, , \tag{2.8a}$$

$$\langle(\delta a_{\parallel})^2\rangle = \tfrac{1}{4} + \langle :(\delta a_{\parallel})^2: \rangle \, , \tag{2.8b}$$

where the symbolical expressions: $\langle :(\delta\varphi)^2: \rangle$ and $\langle :(\delta a_{\parallel})^2: \rangle$ denote the normally ordered variances of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ as per (2.5). For squeezed states these "normally ordered" phase and amplitude fluctuations become negative.

The appearance of normally ordered operators in our equations is useful in that whenever a normally ordered quantum expectation value is involved we naturally use Glauber's $P$ representation.[6] For example, the normally ordered moments of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ are

$$\langle :\hat{a}_1^2(\varphi_0): \rangle = \int r\,dr\,d\varphi\,P(r,\varphi)r^2\cos^2(\varphi-\varphi_0) \, , \tag{2.9a}$$

$$\langle :\hat{a}_2^2(\varphi_0): \rangle = \int r\,dr\,d\varphi\,P(r,\varphi)r^2\sin^2(\varphi-\varphi_0) \, , \tag{2.9b}$$

where the quantum-mechanical average of $\hat{a}$ reduces to a statistical average of a classical complex amplitude $re^{i\varphi}$ with the quasidistribution function $P(r,\varphi)$. The quantum-mechanical expressions (2.9) have a simple semiclassical picture in which the two quantum observables $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ can be viewed as $r\cos(\varphi-\varphi_0)$ and $r\sin(\varphi-\varphi_0)$ with a fixed value of $\varphi_0$ and random orientation $\varphi$. The randomness of $\varphi$ is described by the statistical properties of the quantum state characterized by $P(r,\varphi)$.

## III. FOKKER-PLANCK TREATMENT

Having introduced the relevant definitions of the phase and amplitude fluctuations we proceed in this section with the investigation of the dynamical properties of these fluctuations. The dynamics of the quantum state is given by the time evolution of the Glauber-Sudarshan quasidistribution function $P(\alpha,\alpha^*,t)$. This time evolution is determined by the following general Fokker-Planck equation:

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*,t) = -\frac{\partial}{\partial\alpha}(d_\alpha P) - \frac{\partial}{\partial\alpha^*}(d_{\alpha^*}P) + 2\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}(D_{\alpha^*\alpha}P) + \frac{\partial^2}{\partial\alpha^2}(D_{\alpha\alpha}P) + \frac{\partial^2}{\partial\alpha^{*2}}(D_{\alpha^*\alpha^*}P) \, , \tag{3.1a}$$

with

$$d_\alpha = A + R\alpha + G^*\alpha^*, \quad d_{\alpha^*} = (d_\alpha)^* \, , \tag{3.1b}$$

and

$$D_{\alpha^*\alpha^*} = (D_{\alpha\alpha})^* \, . \tag{3.1c}$$

Here $A$, $R$, and $G$ are completely arbitrary complex parameters. Due to the fluctuation-dissipation theorem the diffusion coefficients $D$ are not completely independent of $R$ and $G$. For the purpose of this work we shall simply assume that all the parameters ( $A$, $R$, $G$, and $D$) are constant and their mutual relations are given by the particular physical models behind the (general) Fokker-Planck equation (3.1). We recognize in this equation the most general Fokker-Planck equation used in the theory of Brownian motion[12] or in the theory of linear quantum optical devices.[6] The first (second) term of this equation describes the drift of $\alpha$ ($\alpha^*$), while the remaining terms correspond to diffusion. At this point we simply note that by a specific choice of the parameters a broad class of quantum optical devices, for example, lasers,[6] linear amplifiers and attenuators,[9] correlated-emission lasers,[7] two-photon lasers[10] and amplifiers,[13] the two-photon correlated emission laser,[11] etc., are described by a Fokker-Planck equation of the type given by Eq. (3.1a).

The physical meaning of the different terms in (3.1b) is the following. $A$ corresponds to an injected external signal, $R$ to the usual (linear) gain, and $G$ (as we will see later) is responsible for the phase sensitivity of the system. First we note that by a simple transformation, we can eliminate $A$ from (3.1b). Indeed, if we introduce

$$\tilde{\alpha} = \alpha - \alpha_0 \, , \tag{3.2a}$$

with

$$\alpha_0 = \begin{cases} \dfrac{G^*A^* - R^*A}{RR^* - GG^*} & \text{if } RR^* \neq GG^* \\ At \quad \text{if } RR^* = GG^*, \end{cases} \tag{3.2b}$$

then in terms of $\tilde{\alpha}$ we obtain a Fokker-Planck equation which is identical to (3.1) with only $A$ is missing from the definition of $d_\alpha$ in (3.1b). In the following we always assume that this is the case, i.e., in (3.1b) we set $A=0$.

As it is often the case we are generally interested in fluctuations around a nonlinear steady state. In this case we can linearize the problem in question around a nonlinear steady state. Alternatively, one can develop a linear theory of the particular quantum optical process under study. In both of these cases we find an equation of the form of (3.1) where the parameters $R$ and $G$ (and, consequently, $D_{\alpha\alpha^*}$, $D_{\alpha\alpha}$, and $D_{\alpha^*\alpha^*}$) become (usually complex) constants. For the investigation of fluctuation dynamics around a linear steady state (threshold) or nonlinear steady state (above threshold) we can, thus, confine ourselves to the case of a Fokker-Planck equation with constant $R$ and $G$. The unitarity of the physical models requires that the diffusion constants $D_{\alpha\alpha^*}$ and $D_{\alpha\alpha}=D^*_{\alpha^*\alpha^*}$ are related to the drift coefficients via the generalized Einstein relations.[6] From these relations it follows that if $D_{\alpha\alpha}\neq 0$ we have that $G_d$, the dissipation-related part of $G$, is nonzero. $G$ still can vanish if its gain-related part $G_g$ satisfies $G_g=-G_d$ since then $G=G_g+G_d=0$. This is precisely the case with an injected squeezed vacuum (and, in general, in systems with zero detuning where emission, i.e., gain, coincides with absorption, i.e., loss, and infinite detuning where both of them are zero). As we shall see, $G$ is directly responsible for phase locking. Thus in the case of a laser with squeezed input vacuum[14] we find no locking at all in the quadrature components [cf. Eqs. (4.1) with $G=0$] and a small locking term for the phase [cf. Eq. (3.5b) with $G=0$ in (3.4b), the locking is then proportional to $n^{-1}$]. In systems with $G\neq 0$, however, the locking plays an essential role in quantum-noise quieting.[11] Keeping the restriction, imposed on $D_{\alpha\alpha}$, in mind we have just arrived at the most general Fokker-Planck equation with constant coefficients one can have. This represents the starting point of the following treatment of fluctuation dynamics.

In order to derive the relationship between phase-amplitude variances and quadrature variances we transform our Fokker-Planck equation (3.1) into phase-amplitude variables (polar coordinates) and quadrature component variables (orthogonal coordinates). First we introduce polar coordinates as $\alpha=re^{i\varphi}$ to obtain the drift and diffusion coefficients associated with phase and amplitude variables $\varphi$ and $r$. The Fokker-Planck equation (3.1) with $A=0$ and $R$ and $G$ constant reads in terms of these variables as

$$\frac{\partial P}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}(r\,d_r P)-\frac{\partial}{\partial\varphi}(d_\varphi P)+\frac{1}{r}\frac{\partial}{\partial r}\left[r\,D_{rr}\frac{\partial P}{\partial r}\right]+\frac{\partial^2(D_{\varphi\varphi}P)}{\partial\varphi^2}+\frac{2}{r}\frac{\partial^2}{\partial r\,\partial\varphi}(r\,D_{r\varphi}P)\ . \qquad (3.3)$$

Here

$$d_r=r[\mathrm{Re}R+|G|\cos(\theta_g+2\varphi)]-\frac{2|D|}{r}\cos(2\varphi-\theta_\alpha)\ , \qquad (3.4a)$$

$$d_\varphi=\mathrm{Im}R-|G|\sin(\theta_g+2\varphi)+2\frac{|D|}{r^2}\sin(2\varphi-\theta_\alpha)\ , \qquad (3.4b)$$

$$D_{rr}=\frac{D_{\alpha\alpha^*}+|D|\cos(2\varphi-\theta_\alpha)}{2}\ , \qquad (3.4c)$$

$$D_{\varphi\varphi}=\frac{D_{\alpha\alpha^*}-|D|\cos(2\varphi-\theta_\alpha)}{r^2}\ , \qquad (3.4d)$$

$$D_{r\varphi}=-\frac{|D|}{2r}\sin(2\varphi-\theta_\alpha)\ , \qquad (3.4e)$$

where we have $G=|G|e^{i\theta_g}$ and $D_{\alpha\alpha}=|D|e^{i\theta_\alpha}$. The key feature of the diffusion and drift coefficients is the fact that they explicitly introduce a phase dependence into the driven random-walk process described by (3.1). Depending on the relative value of $\varphi$ with respect to the phase $\theta_\alpha$ of $D_{\alpha\alpha}$ the diffusion rate may be accelerated or decelerated. Also, depending on the relative value of $\varphi$ with respect to the phase $\theta_g$ of $G$ the drift rate may be larger or smaller. The Brownian motion of such a system is not rotationally invariant. From the Fokker-Planck equation (3.3) we derive the following equation of motion for the amplitude and phase:

$$\frac{d}{dt}\langle r\rangle=\langle d_r\rangle\ , \qquad (3.5a)$$

$$\frac{d}{dt}\langle\varphi\rangle=\langle d_\varphi\rangle\ , \qquad (3.5b)$$

where $d_r$ and $d_\varphi$ are given by Eqs. (3.4a) and (3.4b). Furthermore, as noted at the end of Sec. II the expectation value of any normally ordered operator expression can be obtained with the help of the Fokker-Planck equation by simply replacing $\hat{a}$ with $\alpha$ and $\hat{a}^+$ with $\alpha^*$ in that expression and taking the average of the resulting expression with $P(\alpha,\alpha^*)$ as the distribution function. We thus obtain for the normally ordered variances

$$:\dot{v}_r:\equiv\frac{d}{dt}(\langle r^2\rangle-\langle r\rangle^2)=2\langle\delta r\,d_r\rangle+2\langle D_{rr}\rangle\ , \qquad (3.6a)$$

$$:\dot{v}_\varphi:\equiv\frac{d}{dt}(\langle\varphi^2\rangle-\langle\varphi\rangle^2)=2\langle\delta\varphi\,d_\varphi\rangle+2\langle D_{\varphi\varphi}\rangle\ , \qquad (3.6b)$$

where $\delta r=r-\langle r\rangle$ and $\delta\varphi=\varphi-\langle\varphi\rangle$, $v_r=\langle(\delta r)^2\rangle$, $v_\varphi=\langle(\delta\varphi)^2\rangle$, and $D_{rr}$ and $D_{\varphi\varphi}$ are given by (3.4c) and (3.4d). At this point in our calculations it is tempting to associate $D_{\varphi\varphi}$ and $D_{rr}$ with phase and amplitude diffusion of the components given by Eq. (2.4). We will show in Sec. IV that such an interpretation can be fully justified if one discusses more carefully the corresponding statistical properties of the two observables $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ and their relations to phase and amplitude fluctuations given by Eq. (2.5).

## IV. QUANTUM NOISE QUENCHING AND SQUEEZING

In Sec. III we have shown that with $D_{\alpha\alpha}$-dependent terms in our Fokker-Planck equation (3.1) we obtain phase sensitive diffusion coefficients $D_{\varphi\varphi}$, and $D_{rr}$. In order to support further the heuristic arguments of Sec. II that the fluctuation properties of the semiclassical phase and amplitude are closely related to the fluctuation dynamics of the quadrature components, we shall investigate the dynamical evolution of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$.

From the Fokker-Planck equation (3.3) we derive the following equations of motion for the quantum expectation values of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$:

$$\frac{d}{dt}\langle\hat{a}_1(\varphi_0)\rangle=R\langle\hat{a}_1(\varphi_0)\rangle+|G|\langle\hat{a}_1(-\theta_g-\phi_0)\rangle\ , \qquad (4.1a)$$

$$\frac{d}{dt}\langle \hat{a}_2(\varphi_0)\rangle = R\langle \hat{a}_2(\varphi_0)\rangle - |G|\langle \hat{a}_2(-\theta_g-\varphi_0)\rangle \ . \quad (4.1b)$$

In these equations we have assumed that $R$ is real, because the imaginary part of the gain corresponds to a change in the actual operating frequency of the system. We can always incorporate it in a frequency-pulling term and we shall assume here after that $\mathrm{Im}R=0$. Equations (4.1) together with the definitions of the $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ components correspond to a phase-locking equation. For an ordinary laser the phase-sensitive term vanishes[15,16] ($G=0$) and Eqs. (4.1) lead to the standard phase and amplitude drift with a rate given by $R$. This is a typical behavior of a phase-insensitive system. If $G\neq 0$ we see from Eq. (4.1) that with a proper choice of $\varphi_0$ we can have the situation that $\hat{a}_1(\varphi_0)$ is independent of $\hat{a}_2(\varphi_0)$ and one quadrature, e.g., $\hat{a}_2(\varphi_0)$, locks to zero. From Eq. (4.1) we see that this corresponds to $\varphi_0=-\theta_g/2$, where $\theta_g$ is the phase of $G=|G|e^{i\theta_g}$. In this case we obtain that the condition for a nontrivial steady state for the amplitude component is $R+|G|=0$. It is possible to obtain the locking condition of the phase $\varphi_0$ in a more standard way using the equation of motion for the quantum expectation values of the annihilation and creation operators:

$$\frac{d}{dt}\langle \hat{a}\rangle = R\langle \hat{a}\rangle + G^*\langle \hat{a}^\dagger\rangle \ , \quad (4.2a)$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\rangle = R\langle \hat{a}^\dagger\rangle + G\langle \hat{a}\rangle \ . \quad (4.2b)$$

Following the semiclassical replacement $\langle \hat{a}\rangle = \sqrt{\bar{n}}\,e^{i\varphi_0}$ we obtain from Eq. (4.2)

$$\dot{\varphi}_0 = -|G|\sin(2\varphi_0+\theta_g) \ . \quad (4.3)$$

This equation is the customary phase-locking equation obtained from the semiclassical decomposition of $\langle \hat{a}\rangle$ into an amplitude (fixed by a steady state number of photons $\bar{n}$) and the field phase $\varphi_0$. This equation exhibits the fundamental role of the $G$ term in the phase-locking problem. From Eq. (4.3) we obtain that the phase $\varphi_0$ is locked, i.e., the direction in Fig. 1 is fixed by $\varphi_0=-\theta_g/2$. This condition is equivalent to the condition of phase locking based on Eqs. (4.1). This example shows that the phase-locking condition for the quantum expectation values of the creation and annihilation operator can be fully formulated in terms of the $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ quadratures.

So far we have discussed only the dynamics of the mean values of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$. In order to derive amplitude and phase fluctuations we shall investigate the dynamical properties of the quantum variances of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$.

From the Fokker-Planck equation (3.3) we obtain the following equations of motion for the normally ordered variances of the two quadratures:

$$\frac{d}{dt}{:}\Delta\hat{a}_1^2(\varphi_0){:} = 2R{:}\Delta\hat{a}_1^2(\varphi_0){:} + 2|G|{:}\Delta[\hat{a}_1(\varphi_0)\hat{a}_1(-\varphi_0-\theta_g)]{:} + D_{\alpha\alpha^*} + |D|\cos(2\varphi_0-\theta_\alpha) \ , \quad (4.4a)$$

$$\frac{d}{dt}{:}\Delta\hat{a}_2^2(\varphi_0){:} = 2R{:}\Delta\hat{a}_2^2(\varphi_0){:} - 2|G|{:}\Delta[\hat{a}_2(\varphi_0)\hat{a}_2(-\varphi_0-\theta_g)]{:} + D_{\alpha\alpha^*} - |D|\cos(2\varphi_0-\theta_\alpha) \ . \quad (4.4b)$$

These two equations, together with Eqs. (4.1), give an exact description of the quantum fluctuations of the quadratures $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$.

In the introduction we have associated the variances of $\hat{a}_1(\varphi_0)$ and $\hat{a}_2(\varphi_0)$ with amplitude and phase fluctuations. The Fokker-Planck equation Eq. (3.4), gives an exact dynamical evolution of the normally ordered contributions of amplitude and phase fluctuations given by the expressions (2.8a) and (2.8b).

The remarkable feature of Eqs. (4.4) is the appearance of the diffusion coefficients (3.4c) and (3.4d), $\bar{n}D_{\varphi\varphi}$, and $D_{rr}$ as source terms, with $\varphi$ being replaced by $\varphi_0$ and $r$ being replaced by $\sqrt{\bar{n}}$. This means that the diffusion term of the $\hat{a}_1(\varphi_0)$ fluctuations is

$$D_{rr}(\varphi_0) = \frac{D_{\alpha\alpha^*} + |D|\cos(2\varphi_0-\theta_\alpha)}{2} \ , \quad (4.5a)$$

and the corresponding diffusion term of the phase associated with the variance of $\hat{a}_2(\varphi_0)$ is

$$D_{\varphi\varphi}(\varphi_0) = \frac{D_{\alpha\alpha^*} - |D|\cos(2\varphi_0-\theta_\alpha)}{2\bar{n}} \ . \quad (4.5b)$$

For an ordinary laser the phase sensitive terms vanish ($G=D_{\alpha\alpha}=0$) and we have the usual Schawlow-Townes diffusion. Note that in this case $D_{\varphi\varphi}$ is always positive, and for small $t$ from Eqs. (2.8a) and (4.4b) we obtain

$$\langle(\delta\varphi)^2\rangle \cong \frac{1}{4\bar{n}} + \frac{D_{\alpha\alpha^*}}{\bar{n}}t = \frac{1}{4\bar{n}} + 2D_{\varphi\varphi}t \ . \quad (4.6)$$

For this case Eq. (4.6) reproduces the Schawlow-Townes linewidth.[14]

If $G=0$ but $D_{\alpha\alpha}\neq 0$ from Eq. (4.4b) we obtain

$$\frac{d}{dt}{:}\Delta\hat{a}_2^2(\varphi_0){:} = 2R{:}\Delta\hat{a}_2^2(\varphi_0){:} + 2\bar{n}D_{\varphi\varphi}(\varphi_0) \ . \quad (4.7)$$

Depending on the relative phase of $\varphi_0$ and $\theta_\alpha$, the diffusion of the $\hat{a}_2(\varphi_0)$ variance can be reduced. There is an interesting feature of Eq. (4.7) in this case. If, initially, $\varphi_0=\theta_\alpha/2$ the phase fluctuations grow with a rate $D_{\varphi\varphi}=(D_{\alpha\alpha^*}-|D|)/2\bar{n}$. This growth rate is less than the one predicted by the Schawlow-Townes expression.[14] Note, however, that this is only a transient effect.[17]

The most interesting case is when $G\neq 0$ and $D_{\alpha\alpha}\neq 0$. Then from (4.3) the phase locks to

$$\varphi_0 = -\frac{\theta_g}{2} \ . \quad (4.8)$$

Depending on the difference $\theta_\alpha-\theta_g$ the phase sensitive contribution ($\sim|D_{\alpha\alpha}|$) can now be positive or negative. In particular, when $D_{\alpha\alpha^*}=|D_{\alpha\alpha}|$ we can obtain complete quenching of the quantum noise for $\theta_\alpha=\theta_g$. If $|D_{\alpha\alpha}|>D_{\alpha\alpha^*}$ the phase noise is below the vacuum-noise level for the same condition, i.e., the phase fluctuations are squeezed. The quantum nature of the squeezed state is exhibited in that the Fokker-Planck equation contains a negative diffusion constant. If $D_{\varphi\varphi} \gtrsim 0$ we can quench phase fluctuations down to the vacuum-noise level by phase-sensitive terms related to $D_{\alpha\alpha}$, as in the correlated emission laser (CEL).[7] If $D_{\varphi\varphi} < 0$ we can squeeze the phase fluctuations below the vacuum limit, as in the two-photon CEL.[11] For a negative diffusion the quasiprobability distribution $P(r,\varphi)$ has no classical analogy and the corresponding state of the radiation field exhibits purely quantum-mechanical effects.

For the phase-locking condition (4.8), Eqs. (4.4a) and (4.4b) take the following form:

$$\frac{d}{dt}:\Delta\hat{a}\,_1^2(\varphi_0): = 2(R + |G|):\Delta\hat{a}\,_1^2(\varphi_0): + 2D_{rr}(\varphi_0) \ , \qquad (4.9a)$$

$$\frac{d}{dt}:\Delta\hat{a}\,_2^2(\varphi_0): = 2(R - |G|):\Delta\hat{a}\,_2^2(\varphi_0): + 2\bar{n}D_{\varphi\varphi}(\varphi_0) \ . \qquad (4.9b)$$

It is a remarkable feature of the system of Eqs. (4.4a) and (4.4b) that, under the stable locking condition (4.8), $\langle \hat{a}_1(\varphi_0)\rangle$ is independent from $\langle \hat{a}_2(\varphi_0)\rangle$ and one quadrature, e.g., $\hat{a}_2$, locks to zero [as given by Eqs. (4.9a) and (4.9b)]. A nontrivial value of the steady state for the "amplified" component $\hat{a}_1$ can be obtained if $R + |G| = 0$. In this case the steady-state value of the phase fluctuation is given by

$$\lim_{t\to\infty} \langle (\delta\varphi)^2\rangle = \frac{1}{4\bar{n}} + \frac{D_{\alpha\alpha^*} - |D|\cos(\theta_g - \theta_\alpha)}{2\bar{n}|G|} \ . \qquad (4.10)$$

One interesting question is under what conditions on the parameters is the uncertainty principle (2.1a) satisfied. It is quite straightforward to show, by integrating Eqs. (4.9), that the conditions $\bar{n}D_{\varphi\varphi}(\varphi_0) \ge -|G|/2$ and $D_{rr} + \bar{n}D_{\varphi\varphi}(\varphi_0) \ge 0$ have to be satisfied in the long- and short-time limits, respectively, in the phase-locked situation.

At this point we can compare phase-amplitude fluctuation dynamics with the dynamics of quadrature fluctuations. First we note that if $R = 0$, and $\varphi_0$ is chosen according to Eq. (4.8) then the driving term in leading order of $\bar{n}^{-1}$ for the amplitude, Eq. (3.5a) is the same as the driving term for the $\hat{a}_1$ quadrature, Eq. (4.1a). The small difference is the second term on the right-hand side of Eq. (3.4a) which corresponds to a diffusion induced drift. Under most practical circumstances (e.g., phase-insensitive systems, when $D_{\alpha\alpha} = 0$, or when $r^2 \approx \bar{n} \gg 1$) this term is negligible. The amplitude diffusion (3.4c) and the $\hat{a}_1$ quadrature diffusion (4.5a) are again very similar. Indeed, in phase-insensitive systems, where $D_{\alpha\alpha} = 0$, or in systems with locking, where $\varphi \approx \varphi_0 = -\theta_g/2$ [see Eq. (4.8)], they are the same. The only exception seems to be the case where $|G| = 0$, $D_{\alpha\alpha} \neq 0$, i.e., the case of no locking but phase-sensitive diffusion. In this case the diffusion process may slow down for a particular choice of $\varphi_0$ or accelerate for another but all these effects are transients only since there is no preferred value of $\varphi_0$. Thus the first two moments of $r$ and $\hat{a}_1$ obey very similar equations and for all practical purposes they can be considered the same. However, the use of $\hat{a}_1$ is advantageous since it is a Hermitian quadrature operator, whereas the operator $\hat{r}$ corresponding to the semiclassical amplitude $r$ is not defined unambiguously.

Next we consider the relationship between the phase $\varphi$ and the $\hat{a}_2$ quadrature. First we note that even if $\mathrm{Im}R = 0$ the driving terms of the phase, Eq. (3.4b), are quite different from the driving terms of the $\hat{a}_2$ quadrature component. If there are no locking terms $(G = D_{\alpha\alpha} = 0)$ $\langle\varphi\rangle$ is completely arbitrary, whereas $\langle\hat{a}_2\rangle$ is determined by $\mathrm{Re}R(\langle\hat{a}_1\rangle,\langle\hat{a}_2\rangle)$, i.e., by the same condition as $\langle\hat{a}_1\rangle$. When $G = 0$, $D_{\alpha\alpha} \neq 0$ the situation is not much different, $\varphi$ and $\hat{a}_2$ still appear to be unrelated. However, when there is a locking, i.e., $G \neq 0$, strong similarities arise. The phase locks to $\langle\varphi\rangle = \varphi_0$, where $\varphi_0$ is given by (4.8). In a frame rotated by $\varphi_0$ this means that $\varphi$ locks to zero. In this rotated frame $\hat{a}_2$ also locks to zero. The phase diffusion (3.4d) and the $\hat{a}_2$ quadrature diffusion (4.5b) are again very similar. But $\varphi$ in (3.4d) is a stochastic variable and $\varphi_0$ in (4.5b) is a fixed angle of rotation. In phase-insensitive systems, when (3.4d) is averaged over $\varphi$, we find the usual phase diffusion rate. In systems with locking $(\varphi \approx \varphi_0)$ the two expressions are the same. From here we may conclude that in systems with locking the phase and the $\hat{a}_2$ quadrature component play the same role (first moments describe the same physics, locking to $\varphi_0$, second moments are identical). This suggests that in locked systems $\varphi$ and $\hat{a}_2$ play the same role. Again note that the use of $\hat{a}_2$ is advantageous since it is related to a Hermitian operator, whereas $\varphi$ is not.

The only remaining question is what happens in systems without locking? In several problems (the best known example being the laser) two different time scales arise quite naturally. The dynamics of the first moments (expectation values) are governed by a time scale set by the loss rate $\gamma$ (cavity losses, absorption loss, diffraction losses, etc.). The fluctuation dynamics are governed by the, usually much longer, time scale set by the diffusion rate. In this sense (3.5b) with $d_\varphi = 0$ yields $d\langle\varphi\rangle/dt = 0$, i.e., phase stability on a time scale $\gamma^{-1} < t \ll D_{\varphi\varphi}^{-1}$. Thus for measurement times satisfying the preceding inequality $\varphi$ and $\hat{a}_2$ still play a very similar role provided $\varphi$ in (3.4d) and $\varphi_0$ in (4.5b) are replaced by the (approximately constant) measured value (instantaneous or short-time average) of $\varphi$.

## V. CONCLUSIONS

We have investigated here the relationship between the phase and amplitude fluctuations and the $\hat{a}_1$ and $\hat{a}_2$ quadrature component fluctuations. Our treatment is based on a Fokker-Planck analysis using the Glauber-Sudarshan $P$ representation for the density operator. The main findings of the paper can be summarized in the following way. In systems where phase locking (of some kind) is present the dynamics of the first two momenta of the phase and the $\hat{a}_2$ quadrature component are the same. This gives an Hermitian operator related method for treating the dynamics of phase fluctuations. The relationship also implies that the notion of phase is sensible as long as, on the time scale of an actual measurement, the semiclassical replacement $\langle \hat{a} \rangle = |a| e^{i\varphi_0} = \sqrt{\bar{n}}\, e^{i\varphi_0}$ holds.

For a wide class of linear optical devices, described by our general Fokker-Planck equation, we have established conditions leading to quantum-noise quenching and squeezing of the phase fluctuations. We have shown that a particular set of terms in the Fokker-Planck equation can lead to a phase-sensitive diffusion. This diffusion can be completely suppressed for certain values of the radiating phase, in which case the phase noise is quenched to the vacuum level. We have shown that the diffusion can also become negative, leading to noise reduction below the vacuum level.

## ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of The Office of Naval Research. One of us (K.W.) thanks the Polish Program No. CPBP.01.03 for a travel grant.

*On leave from the Central Research Institute for Physics, H-1525 Budapest 114, P.O. Box 49, Hungary.

†On leave from Facultad de Fisica Pontificia Universidad Catolica de Chile, Casilla 6177, Santiago, Chile.

‡Permanent address: Institute of Theoretical Physics, Warsaw University, Warsaw 00681, Poland.

[1]P. Carruthers and M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

[2]See, for example, C.. M. Caves, in *Quantum Optics, Experimental Gravitation, and Measurement Theory,* edited by P. Meystre and M. O. Scully (Plenum, New Yórk, 1982).

[3]See, for example, the review article by D. Walls, Nature **306**, 141 (1983), and references therein.

[4]H. Yuen, Phys. Rev. A **13**, 2226 (1976).

[5]R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985); R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. De Voe, and D. F. Walls, Phys. Rev. Lett. **57**, 691 (1986); L. Wu, J. F. Kimble, J. L. Hall, and H. Wu, Phys. Rev. Lett. **57**, 2520 (1986).

[6]M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[7](a) M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985); M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987); J. Bergou and M. Orszag, J. Opt. Soc. Am. B **5**, 249 (1988). (b) P. E. Toschek and J. L. Hall, J. Opt. Soc. Am. B **4**, 124 (1987); M. Ohtsu and K Y. Liou, Appl. Phys. Lett. **52**, 10 (1988).

[8]See, for example, C. M. Caves, Phys. Rev. D **26**, 1817 (1982).

[9]See, for example, G. J. Millburn, M. L. Steyn-Ross, and D. F. Walls, Phys. Rev. A **35**, 4443 (1987), and references therein.

[10]M. D. Reid and D. F. Walls, Phys. Rev. A **28**, 332 (1983).

[11]M. O. Scully, K. Wódkiewicz, M. S. Zubairy, J. Bergou, Ning Lu, and J. Meyerter-Vehn, Phys. Rev. Lett. **60**, 1832 (1988).

[12]I. R. Senitzky, Phys. Rev. A **31**, 421 (1970).

[13]M. A. Dupertuis and S. Stenholm, Phys. Rev. A **37**, 1226 (1988).

[14]A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).

[15]M Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[16]As is well known (see, e.g., Ref. 15), the laser operating point $\bar{n}$ is determined by a nonlinear theory. However, once steady state is achieved the laser phase diffusion linewidth is determined by a linear analysis; see Ref. 15, Chap. 17, Eqs. (52) and (53).

[17]J. Gea-Banacloche, Phys. Rev. Lett. **59**, 543 (1987).