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Coulomb-diamagnetic problem in two dimensions

S. C. Chhajlany and V. N. Mal'nev*
Physics Department, Addis Ababa University, Box 1176, Addis Ababa, Ethiopia

N. Kumar†
International Centre for Theoretical Physics, Trieste, Italy
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Based on a conjecture for the four-step recursion relation occurring in the series solution of the Coulomb-diamagnetic problem in two space dimensions, the energy eigenvalue spectrum that reproduces the known limits in full is obtained exactly. The resulting nonperturbative spectrum is a deceptively simple combination of the purely Landau and Coulombic spectra but gives the quasi-Landau levels with $(3/2)\hbar\omega_c$ spacing near the ionization threshold. The conjecture is made plausible in terms of an adiabatic continuation of solutions in the parameter ratio $\hbar\omega_c/\mathcal{R}$.

The three-dimensional Coulomb diamagnetic problem remains an important unsolved problem.[1] The full problem is nonseparable, and no theoretical technique is yet available for handling such potentials exactly. In the vicinity of zero energy, i.e., for highly excited or Rydberg atoms, we expect a mixing of highly degenerate and closely packed hydrogen levels due to the full potential. Irrespective of the strength of the magnetic field B, this is, therefore, a nonperturbative regime. There is thus no *a priori* ground to expect a simple spectrum around $E=0$. And yet recent experiments have belied this expectation.[2] The reported spectrum reveals nearly equally spaced resonances with spacings approximately in multiples of $\hbar\omega_c/2$, ω_c being the cyclotron frequency. The most dominant of these is the $\frac{3}{2}\hbar\omega_c$ spacing. This has inevitably led to the speculation that nonseparability may have interesting and unsuspected consequences.[3] It has even been suggested that the hydrogen atom in a magnetic field may be a case of quantum chaos realizable in the laboratory.[4,5] Studies based on classical trajectory calculations support the dominant spacing of $\frac{3}{2}\hbar\omega_c$.[6] A complete analytical quantum calculation is however still lacking.

In this paper we examine the two-dimensional analogue of this problem, where the potential is Coulombic $(-e^2/\rho)$ and the electron moves in the plane $z=0$. This problem is physically interesting in its own right in that it describes the energy levels of an ionized impurity center in the two-dimensional systems realizable in the laboratory, such as inversion layers and heterostructures in the presence of a transverse magnetic field. It also approximates well the Rydberg states of the three-dimensional Coulomb-diamagnetic problem.[6]

Our model is a two-dimensional hydrogen atom in the $z=0$ plane subjected to a constant uniform magnetic field B of arbitrary strength and pointing in the z direction. Using cylindrical coordinates (ρ,ϕ), the principal task is to solve the radial equation

$$R''+\frac{1}{\xi}R'+\left[\gamma-\frac{m^2}{\xi 2}+\frac{\alpha}{\xi}-\xi^2\right]R=0 \ . \tag{1}$$

The primes denote derivatives with respect to the dimensionless radial variable ξ, where

$$\rho\equiv\nu\xi,\quad \nu=\sqrt{2\hbar/\mu\omega_c},\quad \omega_c\equiv\frac{|e|B}{\mu c}\ , \tag{2}$$

and the other dimensionless variables are

$$\gamma\equiv\frac{4E}{\hbar\omega_c}-2m,\quad \alpha^2\equiv\frac{8\mu e^4}{\hbar^3\omega_c}=\frac{16R}{\hbar\omega_c}\ . \tag{3}$$

Factoring out the limiting behavior corresponding to the separate Coulomb and Landau problems we take

$$R(\xi)=\xi^{|m|}e^{-\xi^2/2}e^{-\beta\xi}\nu(\xi)\ , \tag{4}$$

we find

$$\xi\nu''+(p-2\beta\xi-2\xi^2)\nu' +[(\alpha-\beta p)+(\delta+\beta^2)\rho+2\beta\xi^2]\nu=0\ , \tag{5}$$

where

$$p\equiv2|m|+1,\quad \delta\equiv\gamma-p-1\ . \tag{6}$$

Setting

$$\nu=\sum_{n=0}^{\infty}a_n\xi^n,\quad a_0\neq0 \tag{7}$$

we get a four-step recursion relation

$$n(n+p-1)a_n+[\alpha-\beta(p+2n-2)]a_{n-1} +(\delta+\beta^2-2n+4)a_{n-2}+2\beta a_{n-3}=0\ . \tag{8}$$

No standard technique exists for analyzing such a relation. We propose the following conjecture to be discussed later.

Physical solutions are obtained as follows. The integer $k\,(=0,1,2\ldots,)$ will be seen to have the nature of a principal quantum number. Choose a particular value of k. Then set $n=k+1$ in the recurrence relation (8) and require that the coefficient of a_k be zero, so that a_k does not contribute to a_{k+1} via (8). In addition, set $n=2k+2$ in (8) and then require that the coefficient of a_{2k} be zero, so that a_{2k} does not contribute to a_{2k+2} via (8). This procedure gives a mathematically legitimate class of solutions.

Setting $n=k+1$ and $n=2k+2$ successively in Eq. (8) and equating to zero the coefficients of a_k and a_{2k}, we immediately obtain

$$\beta=\alpha/(2k+p) \tag{9}$$

and

$$\delta+\beta^2=4k\ . \tag{10}$$

These can be solved to give the energy eigenvalues

$$E\equiv E_{km}=\frac{\hbar\omega_c}{2}(2k+m+|m|+1)-\mathcal{R}/(k+|m|+\tfrac{1}{2})^2\ , \tag{11}$$

with

$$\beta\equiv\beta_{km}=\alpha/(2k+2|m|+1)\ . \tag{12}$$

Equations (11) and (12) are our principal results. Notice that the conjecture does not alter the infinite series nature of the solution $\nu(\xi)$ but fixes it up to an overall normalization factor.

Let us now discuss the implications of our results. The present problem has three standard limits that are readily recovered in full from Eqs. (11) and (12). First, the Coulomb bound states ($E<0$) followed by $\omega_c\to 0$. Our results give $\beta=\alpha/(n+\frac{1}{2})$ and $E_{km}=-\mathcal{R}/(n+\frac{1}{2})^2$, where $n=k+|m|$ is the conventional principal quantum number of the two-dimensional Coulomb problem. The recursion relation (8) then reproduces the Coulomb functions. Next, letting $\beta\to 0$, $\alpha\neq 0$, $k+|m|\to\infty$, and $\omega_c\to 0$, we get the scattering states of the Coulomb problem with E having any positive value. These limits are best seen by returning to the ρ variable. Finally, by letting $\alpha\to 0$, the Landau spectrum is obtained in full. Thus the set of two conditions imposed by the conjecture exactly reproduces the infinite set of data on the limiting-energy eigenvalues and eigenfunctions. Yet another striking feature of the spectrum is that it generates the quasi-Landau levels with spacing $\frac{3}{2}\hbar\omega_c$ in the Rydberg limit, i.e., near the zero-field ionization threshold. To see this, introduce the "principal" quantum number $n\equiv k+|m|$ and define other quantum number $m'\equiv k+m$. Thus we rewrite Eq. (11) as

$$E=\tfrac{1}{2}\hbar\omega_c(n+m'+1)-\frac{\mathcal{R}}{(n+\frac{1}{2})^2}\ . \tag{11$'$}$$

We now evaluate $(\partial E/\partial n)/\hbar\omega_c$ in the limit $n\gg m'$, $E\simeq 0$ for a fixed value of m', which is readily seen to equal $\frac{3}{2}$. We have not, however, calculated the oscillator strength for the corresponding σ-polarization transition.

Now let us turn to the other general features of the spectrum. Introducing again the principal quantum number $n\equiv k+|m|$, we have the following level scheme. The level E_{nm} is $(n+1)$-fold degenerate for $m>0$, while for $m<0$ there are no degeneracies barring accidental ones which may appear for tuned values of the parameter α. One notices that such a qualitative feature is very desirable for the full Coulomb-diagmagnetic problem. It is readily seen that, in general, there is level crossing which, however, may not allow dipole-induced transition.

The completely nonperturbative result in Eq. (11) can be understood in the following terms. The effective potential appearing in the radial equation (1) is a combination of the Coulombic and the harmonic potentials that control the asymptotic behavior of the radial function in the respective single-potential limit. It is, therefore, apt to factor out the asymptotic behavior, as in Eq. (4). This facilitates passage to the Coulombic and the Landau limits. Thus, for any nonzero diamagnetic term, however small, the asymptotic behavior is always dominated by the oscillator cutoff factor $e^{-\xi^2/2}$. The presence of such a dominant cutoff factor renders inadmissible the neglect of the divergent Coulomb solution as in a conventional treatment of the Coulomb problem. In fact, an expansion of $\nu(\xi)$ in the small parameter $1/\alpha$ reveals the presence of the conventionally discarded solution of the Coulomb problem in addition to the solution with the usual asymptotic behavior of $e^{-\beta\xi}$. Thus the perturbation theory based entirely on the standard Coulomb basis will be incapable of giving a meaningful result in any finite order, irrespective of the smallness of the diamagnetic term. Indeed, such a perturbation expansion will have to effectively incorporate this factor $e^{-\xi^2/2}$ to all orders. Perturbation theory may work, however, were the potential to have a finite support (short range) which is not the case in our model.

We now come to a discussion of the conjecture itself which can be rationalized in terms of the following continuity consideration. First let us recall that for any value of the parameter α we have a set of solutions that continuously flow into the complete set of exact limiting solutions as α is varied from zero to infinity. Thus, in the spirit of the adiabatic hypothesis, the solutions should map into solutions as α is varied. To gain confidence, however, we compare our results with those obtained by the exact treatment of the Wentzel-Kramers-Brillouin (WKB) approximation.[7] From Eq. (11′) we derive straightforwardly for the quasi-Landau level spacing

$$\frac{1}{\hbar\omega_c}\left[\frac{dE}{dn}\right]=\frac{3}{2}-\left[\frac{\mathcal{R}}{\hbar\omega_c}\right]\frac{2}{n^3}\ ,$$

with n given by

$$n^3-2\left[\frac{E}{\hbar\omega_c}\right]n^2-2\left[\frac{\mathcal{R}}{\hbar\omega_c}\right]=0,\quad n\gg 1;\ m'=0\ . \tag{13}$$

We readily see that the level spacing is a decreasing function of the energy ($E/\hbar\omega>0$) and is smaller for larger fields. The opposite is true for the negative energies and thus there is a crossover at the threshold $E=0$. This is in complete agreement with the results of Ref. 7. Quantitatively, our results agree exactly with the latter at the threshold, giving $(1/\hbar\omega_c)(dE/dn)=\frac{3}{2}$. Away from the threshold, however, our level spacings are somewhat smaller (larger) than theirs for positive (negative) energies. We are now in the process of applying our conjecture to other important classes of anharmonic potentials.

Finally, we would like to point out that the spectrum in Eq. (11) has a remarkable regularity and simplicity, in that cross terms do not appear. This, we believe, is due to the underlying symmetry of the problem that deserves further investigation. We suspect, specifically, that this may be related to a duality between the Coulomb and the harmonic potentials suggested by Schwinger's well-known transformation of the Coulomb problem into an oscillator problem. Inasmuch as the quadratic Zeeman correction must be a cross term $[\sim(\hbar\omega_c)^2/\mathcal{R}]$, its absence in our two-dimensional Coulomb-diamagnetic problem may be understandable in terms of such a duality.

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*Permanent address: Physics Department, Kiev State University, Kiev, U.S.S.R.

†On sabbatical leave from Physics Department, Indian Institute of Science, Bangalore 560012, India.

[1]D. Kleppner, M. G. Littman, and M. L. Zimmerman, in *Rydberg States of Atoms and Molecules,* edited by R. F. Stebbings and F. B. Dunning (Cambridge University Press, Cambridge, England, 1983), pp. 73–116.

[2]A. Holle, G. Wiebusch, J. Main, B. Hager, H. Rotte, and K. H. Welge, Phys. Rev. Lett. **56**, 2594 (1986); W. R. S. Garton and F. S. Tomkins, Astrophys. J. **158**, 839 (1969).

[3]A. R. P. Rau, Nature **325**, 577 (1987), and references cited therein.

[4]D. Wingten and H. Friedrich, Phys. Rev. Lett. **57**, 571 (1986).

[5]D. Delande and J. C. Gray, Phys. Rev. Lett. **57**, 2006 (1986).

[6]See, e.g., A. R. P. Rau, Phys. Rev. A **16**, 613 (1977); A. F. Starace, J. Phys. B **6**, 585 (1973); A. R. Edmonds, J. Phys. (Paris) Colloq. **4**, C-71 (1970); R. F. O'Connell, Astrophys. J. **187**, 275 (1974).

[7]J. A. C. Gallas and R. F. O'Connell, J. Phys. A **15**, L75 (1982); see also pp. L309 and L593.