Electron-impact excitation of quadrupole-allowed transitions in positive ions

Marita C. Chidichimo and Susan P. Haigh
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
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An asymptotic analysis of the partial collision strength for quadrupole-allowed transitions, in the limit of large angular momentum, is presented. It is shown that the infinite sum over partial collision strengths, which gives the total collision strength, is asymptotic to a geometric series of ratio $k_<^2/k_>^2$, where $k_>^2$ and $k_<^2$ are the energies of the free electron before and after the collision.

I. INTRODUCTION

Accurate calculations of the total collision strength for quadrupole-allowed transitions by electron impact, when the quadrupole moment is large, require, even at low electron energies, the inclusion of contributions from a large number of partial waves.[1–8] In this paper we derive the asymptotic behavior of the partial collision strength for large values of the total angular momentum. This analysis shows that the infinite sum over the partial collision strengths, is asymptotic to a geometric series of ratio $k_<^2/k_>^2$, where $k_>^2$ and $k_<^2$ are the energies of the colliding electron before and after the collision. The collision problem has been formulated in atomic units, except for energies where we have used rydbergs units.

II. THEORY

The total collision strength for the transition, induced by electron impact, from an initial atomic state specified by quantum numbers nl_a to a final state $n'l'_a$, in the coupled angular momentum representation, is given by

$$\Omega(n'l'_a,nl_a)=\sum_{l=0}^{\infty}\sum_{l'}\sum_{L}(2L+1)|T(n'l'_ak'l'L,nl_aklL)|^2 , \tag{1}$$

where k and l are, respectively, the wave number and the orbital angular momentum of the incident electron, k' and l' are those of the scattered electron, and L is the total orbital angular momentum of the system. $T(\gamma',\gamma)$ is an element of the transmission matrix which is related to the reactance matrix $\mathbf{R}$ by

$$\mathbf{T}=-\frac{2i\mathbf{R}}{1-i\mathbf{R}} \tag{2}$$

(strong-coupling case). If $R(\gamma'\gamma)\ll 1$ for all γ',γ (weak-coupling case, referred to as approximation I) the $\mathbf{T}$ matrix can be written as

$$\mathbf{T}=-2i\mathbf{R} . \tag{3}$$

We refer to the strong-coupling case [Eq. (2)] as approximation II. For electric quadrupole transitions, the partial wave contributions to the total collision strength,

$$\Omega_{l'l}=\sum_{L}(2L+1)|T(n'l'_ak'l'L,nl_aklL)|^2 , \tag{4}$$

make the sum over l slowly convergent. Therefore approximate methods should be used to estimate the sum for high angular momenta. We have split the infinite sum in Eq. (1) into two parts,

$$\Omega(n'l'_a,nl_a)=\Omega_{l_0}+\tilde{\Omega}_{l_0+1} , \tag{5}$$

where

$$\Omega_{l_0}=\sum_{l=0}^{l_0}\sum_{l'}\Omega_{l'l} , \tag{6}$$

and

$$\tilde{\Omega}_{l_0+1}=\sum_{l=l_0+1}^{\infty}\sum_{l'}\Omega_{l'l}^{\text{CBeI}} . \tag{7}$$

The sum from $l=0$ to $l=l_0$ may be evaluated in any desired approximation, e.g., close-coupling, Coulomb-distorted wave, etc. Here, the sum from $l=l_0+1$ to ∞ will be estimated using the weak-coupling Bethe approximation to the transmission matrix (CBeI). By choosing a high enough value of l_0, the ratio $\Omega_{l'l}/\Omega_{l'l}^{\text{CBeI}}$ can be made as near to unity as required. For quadrupole transitions ($\lambda=2$) the partial collision strength in the CBeI approximation is given by

$$\Omega_{l'l}^{\text{CBeI}}=16\sum_{L}(2L+1)|\langle l'l'_aL|P_2|ll_aL\rangle|^2|\mathcal{R}_2^{\text{CBe}}|^2 , \tag{8}$$

where the angular factor

$$\langle l'l'_aL|P_2|ll_aL\rangle=f_2(l_al,l'_al';L)$$

has been tabulated by Percival and Seaton,[9] and

$$\mathcal{R}_\lambda^{\text{CBe}}=\int_0^\infty F_{k'l'}(r)r^{-\lambda-1}F_{kl}(r)dr\int_0^\infty P_{n'l'_a}(r)r^\lambda P_{nla}(r)dr . \tag{9}$$

F_{kl} is a regular Coulomb function, P_{nl_a} and $P_{n'l'_a}$ are the radial wave functions of the initial and final states of the atomic system, and λ satisfies the selection rules $|l-l'|\le\lambda\le l+l'$ and $|l_a-l'_a|\le\lambda\le l_a+l'_a$. The Coulomb functions F_{kl} are subject to the boundary conditions

$$F_{kl}(0)=0 \tag{10}$$

and

$$F_{kl}(r) \sim k^{-1/2}\sin\left[kr+\frac{(z-1)}{k}(2kr)-\frac{l\pi}{2}+\sigma_l\right] \quad \text{for } r\to\infty\ , \tag{11}$$

with

$$\sigma_l = \arg\Gamma[l+1-i(z-1)/k]\ , \tag{12}$$

where $(z-1)$ is the ion charge.

A. Bound wave functions

In this work the bound state wave functions are single orbital wave functions with no explicit correlation or configuration interaction terms. The radial orbitals $P_i(r)$ for the target ion satisfy the equation

$$\left[\frac{d^2}{dr^2}-\frac{l_i(l_i+1)}{r^2}+\frac{2z}{r}+2\frac{Z_i^{\text{eff}}(\alpha|r)}{r}-\varepsilon_i\right]P_i(r)=0\ , \tag{13}$$

where $\varepsilon_i = I - E_i$ is the observed binding energy of an electron in subshell i. I is the ionization energy of the valence electron and E_i are the term energies of the target ion. The experimental energies quoted are weighted means of terms belonging to the same configuration and are given by

$$E_i = \frac{1}{2(2l_i+1)}\sum_{j=l_i-1/2}^{l_i+1/2}(2j+1)E_{ij}\ . \tag{14}$$

Nonlinear scaling parameters in the function $Z_i^{\text{eff}}(\alpha|r)$ are adjusted so that the correct experimental energy is obtained as part of the solution of Eq. (13). It is assumed that each electron moves independently in a potential which is generated by the nuclear charge Z_0 and the charge distribution of the other electrons. For an N-electron ion with nuclear charge Z_0, N_S atomic subshells, and with q_j electrons in each subshell j, the potential of an electron in subshell i is given in terms of $Z_i^{\text{eff}}(\alpha|r)$ by the equation

$$\frac{Z_0}{r}-\sum_{j=1}^{N_S}(q_j-\delta_{ij})y_0(P_{n_j}^{\text{STO}}|r)=\frac{z}{r}+\frac{Z_i^{\text{eff}}(\alpha|r)}{r}\ , \tag{15}$$

where α stands for the set of scaling parameters,

$$y_0(P|r)=\frac{1}{r}\int_0^r P^2(\xi)d\xi+\int_r^\infty P^2(\xi)\frac{d\xi}{\xi}\ , \tag{16}$$

where

$$P_{n_j}^{\text{STO}}(r)=\frac{1}{\sqrt{(2n_j)!}}\left[\frac{2Z_j\alpha_j}{n_j}\right]^{n_j+1/2} r^{n_j}\exp\left[\frac{Z_j\alpha_j r}{n_j}\right] \tag{17}$$

are Slater-type orbitals (STO's) which are used to compute the average screening by electrons in subshell j, with

$$Z_j = Z_0-\sum_{t=1}^{j-1}q_t-\tfrac{1}{2}(q_j-1)\ , \tag{18}$$

n_j is the principal quantum number of subshell j, and α_j is an adjustable scaling parameter. $Z_i^{\text{eff}}(\alpha|r)$, which is short range, has the following analytical form:[10]

$$Z_i^{\text{eff}}(\alpha|r)=(N-1)-\sum_{j=1}^{N_S}(q_j-\delta_{ij})\left[1-\frac{\exp(-\rho_j)}{2n_j}\sum_{m=0}^{2n_j-1}\frac{(2n_j-m)}{m!}\rho_j^m\right]\ , \tag{19}$$

with

$$\rho_j = 2Z_j\alpha_j r/n_j\ . \tag{20}$$

In Eq. (20), we have adopted a more compact notation which does not show the i dependence of α_j and ρ_j. This potential has the important property that the right-hand side of Eq. (19) can be expressed as a power series in r with an infinite radius of convergence. We have assumed that the core is frozen during the collision and therefore we only require orbitals for the outer valence electron. All of the orbitals nl_a have been calculated using the same α set, for a given target ion, and therefore the same $Z^{\text{eff}}(r)$.

B. Introduction of reduced variables

Following the procedure of Burgess *et al.*[11] and defining

$$\mathcal{F}(\kappa l|\rho)=(z-1)^{1/2}F_{kl}(r)\ , \tag{21}$$

where

$$\kappa=\frac{k}{(z-1)},\quad \rho=(z-1)r\ , \tag{22}$$

we can rewrite Eq. (9) as

$$\mathcal{R}_\lambda^{\text{CBe}}=(z-1)^{\lambda-1}I(\kappa l,\kappa' l';\lambda)B(nl_a,n'l_a';\lambda)\ , \tag{23}$$

where

$$I(\kappa l,\kappa' l';\lambda)=\int_0^\infty \mathcal{F}(\kappa l|\rho)\mathcal{F}(\kappa' l'|\rho)\rho^{-\lambda-1}d\rho \tag{24}$$

and

$$B(nl_a,n'l_a';\lambda)=\int_0^\infty P_{nl_a}(r)r^\lambda P_{n'l_a'}(r)dr\ . \tag{25}$$

The conservation of angular momentum and parity in the excitation process implies that the only integrals I occurring in the total collision strength are those for which

$$l-l'=-\lambda,-\lambda+2,\ldots,\lambda\ . \tag{26}$$

In Eq. (8) the sum over the total angular momentum L can be carried out,[9,12] giving for the case $|l-l'|=\lambda=2$

$$\sum_L (2L+1)|\langle l'l_a'L|P_2|ll_aL\rangle|^2 = \frac{1}{5}\left[\frac{3}{2}\right]^2 \frac{l_{a>}(l_{a>}-1)(2l_{a<}+1)}{(2l_{a>}-1)(2l_{a>}-3)} \frac{(l_<+1)(l_<+2)}{(2l_>-1)} , \tag{27}$$

and for the case $l=l'$

$$\sum_L (2L+1)|\langle l'l_a'L|P_2|ll_aL\rangle|^2 = \frac{1}{5}\left[\frac{3}{2}\right] \frac{l_{a>}(l_{a>}-1)(2l_{a<}+1)}{(2l_{a>}-1)(2l_{a>}-3)} \frac{l(l+1)(2l+1)}{(2l+3)(2l-1)} , \tag{28}$$

where $l_{a>}$ and $l_>$ are the greater of l_a and l_a' and of l and l', respectively. $\tilde{\Omega}_{l_0+1}$ [Eq. (7)] then reduces to

$$\tilde{\Omega}_{l_0+1} = \tfrac{16}{5}(z-1)^2 B^2(nl_a,n'l_a';2)\frac{l_{a>}(l_{a>}-1)(2l_{a<}+1)}{(2l_{a>}-1)(2l_{a>}-3)}\frac{3}{2} \times \sum_{l=l_0+1}^{\infty}\left[\frac{3}{2}\frac{l(l-1)}{(2l-1)}I^2(\kappa l,\kappa' l-2;2)+\frac{l(l+1)(2l+1)}{(2l+3)(2l-1)}I^2(\kappa l,\kappa' l;2)+\frac{3}{2}\frac{(l+1)(l+2)}{(2l+3)}I^2(\kappa l,\kappa' l+2;2)\right] . \tag{29}$$

C. Limit of large orbital angular momentum

In the limit of large angular momentum ($l \gg 1$) we have[13]

$$I(\kappa l,\kappa' l';\lambda) \sim \exp\left[\frac{\pi}{2}|\eta-\eta'|\right] I_0(\kappa l,\kappa' l';\lambda) , \tag{30}$$

where

$$\eta=\frac{1}{\kappa},\quad \eta'=\frac{1}{\kappa'} , \tag{31}$$

and $I_0(\kappa l,\kappa' l';\lambda)$ is the integral in Eq. (24) evaluated in the Born approximation. The radial wave functions $\mathcal{F}(\kappa l|\rho)$ are in this case related to the spherical Bessel functions. The connection is given through the relation

$$\mathcal{F}(\kappa l|\rho)=\sqrt{\kappa}\rho j_l(\kappa\rho) . \tag{32}$$

The integral I_0 thus takes the form[14]

$$I_0(\kappa l,\kappa' l';\lambda)=\int_0^\infty j_l(\kappa\rho)j_{l'}(\kappa'\rho)\rho^{-\lambda+1}d\rho . \tag{33}$$

By employing the further relation

$$j_l(\kappa\rho)=\left[\frac{\pi}{2\kappa\rho}\right]^{1/2} J_{l+1/2}(\kappa\rho) , \tag{34}$$

between the spherical Bessel functions and ordinary Bessel functions one obtains

$$I_0(\kappa l,\kappa' l';\lambda)=\frac{\pi}{2}\int_0^\infty J_{l+1/2}(\kappa\rho)J_{l'+1/2}(\kappa'\rho)\rho^{-\lambda}d\rho . \tag{35}$$

This integral is of the Weber-Schafheitlin type and has a discontinuity in the expressions for I_0 at $\kappa=\kappa'$. From Watson[15] we have

$$I_0(\kappa l,\kappa' l';2)=\frac{\pi}{8}\left[\frac{\kappa'}{\kappa}\right]^{l'+1/2}\frac{\kappa\Gamma\left[\frac{l+l'}{2}\right]}{\Gamma(l'+3/2)\Gamma\left[\frac{3}{2}+\frac{l-l'}{2}\right]}\,{}_2F_1\left[\frac{l+l'}{2},\frac{l'-l}{2}-\frac{1}{2};l'+3/2;\frac{\kappa'^2}{\kappa^2}\right],\quad 0<\kappa'<\kappa \tag{36}$$

and

$$I_0(\kappa l,\kappa' l';2)=\frac{\pi}{8}\left[\frac{\kappa}{\kappa'}\right]^{l+1/2}\frac{\kappa'\Gamma\left[\frac{l+l'}{2}\right]}{\Gamma(l+3/2)\Gamma\left[\frac{3}{2}+\frac{(l'-l)}{2}\right]}\,{}_2F_1\left[\frac{l+l'}{2},\frac{l-l'}{2}-\frac{1}{2};l+3/2;\frac{\kappa^2}{\kappa'^2}\right],\quad 0<\kappa<\kappa' , \tag{37}$$

where ${}_2F_1$ is the usual Gauss hypergeometric function. The radius of convergence of the series expansion of the function ${}_2F_1$ is the unit circle $\kappa_<^2/\kappa_>^2=1$ and the series is absolutely convergent in the open interval of convergence. $\kappa_<$ is the smaller of κ and κ', and $\kappa_>$ is the greater of κ and κ'. The analytic continuation of ${}_2F_1$ can be expressed by[16]

$${}_2F_1(a,b;c;z)=(1-z)^{-b}\,{}_2F_1\left[c-a,b;c;\frac{z}{z-1}\right] , \tag{38}$$

and inserting Eq. (38) into Eqs. (36) and (37) one obtains

$$I_0(\kappa l,\kappa' l+2;2)=\frac{\pi}{8}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+1/2+p}\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{3/2-p}\frac{\Gamma(l+1)}{\Gamma(l+3/2+p)\Gamma(5/2-p)}\times{}_2F_1\left[\frac{1}{2}+p,-\frac{3}{2}+p;l+\frac{3}{2}+p;\frac{\kappa_<^2}{\kappa_<^2-\kappa_>^2}\right], \tag{39}$$

where

$$p=\begin{cases}0 & \text{if } \kappa<\kappa' \\ 2 & \text{if } \kappa>\kappa'\end{cases},$$

$$I_0(\kappa l,\kappa' l-2;2)=\frac{\pi}{8}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+1/2-p}\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{-1/2+p}\frac{\Gamma(l-1)}{\Gamma(l+3/2-p)\Gamma(1/2+p)}\times{}_2F_1\left[\frac{5}{2}-p,\frac{1}{2}-p;l+\frac{3}{2}-p;\frac{\kappa_<^2}{\kappa_<^2-\kappa_>^2}\right], \tag{40}$$

where

$$p=\begin{cases}0 & \text{if } \kappa<\kappa' \\ 2 & \text{if } \kappa>\kappa'\end{cases}$$

and

$$I_0(\kappa l,\kappa' l;2)=\frac{\pi}{8}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+1/2}\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{1/2}\frac{\Gamma(l)}{\Gamma(l+3/2)\Gamma(3/2)}\,{}_2F_1\left[\frac{3}{2},-\frac{1}{2};l+\frac{3}{2};\frac{\kappa_<^2}{\kappa_<^2-\kappa_>^2}\right]. \tag{41}$$

The behavior of ${}_2F_1(a,b;c;z)$ for fixed a,b,z and large $|c|$ is described by[17]

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{m}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n+O(|c|^{-m-1}). \tag{42}$$

Using the previous asymptotic expansion and the limit property of the Γ function[18]

$$\lim_{n\to\infty} n^{t-s}\frac{\Gamma(n+s)}{\Gamma(n+t)}=1, \tag{43}$$

it is possible to show that

TABLE I. Partial collision strength Ω_l in the CDWII approximation for the $5s$-$4d$ transition in Sr^+. Incident electron energy $k_>^2$ (Ry). The excitation energy is 0.134 27 Ry. Numbers in brackets denote powers of ten.

l	$k_>^2=0.456\,27$	l	$k_>^2=4.0$	l	$k_>=22.443$
10	7.162[−2]	23	3.757[−2]	40	3.436[−2]
11	4.901[−2]	24	3.310[−2]	41	3.198[−2]
12	3.38[−2]	25	2.906[−2]	42	2.982[−2]
13	2.342[−2]	26	2.607[−2]	43	2.783[−2]
14	1.629[−2]	27	2.330[−2]	44	2.602[−2]
15	1.135[−2]	28	2.091[−2]	45	2.431[−2]
16	7.928[−3]	29	1.884[−2]	46	2.279[−2]
17	5.544[−3]	30	1.703[−2]	47	2.137[−2]
18	3.881[−3]	31	1.446[−2][a]	48	2.008[−2]
19	2.720[−3]	32	1.321[−2][a]	49	1.890[−2]
20	1.907[−3]	33	1.210[−2][a]	50	1.781[−2]
21	1.349[−3]	34	1.111[−2][a]	51	1.678[−2]
22	9.473[−4]	35	1.022[−2][a]	52	1.531[−2]
23	6.653[−4]	36	9.430[−3][a]	53	1.454[−2][a]
24	4.675[−4]	37	8.717[−3][a]	54	1.374[−2][a]
25	3.286[−4]	38	8.075[−3][a]	55	1.286[−2][a]
26	2.310[−4]	39	7.496[−3][a]	56	1.233[−2][a]
27	1.625[−4]	40	6.981[−3][a]	57	1.185[−2][a]

[a]Ω_l calculated in the CBeI approximation.

$$I_0(\kappa_< l,\kappa_> l+2;2)\approx \frac{(\pi l)^{1/2}}{3(2l+1)}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+1/2} \times\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{3/2}, \quad l \gg \frac{\kappa_<^2}{\kappa_>^2-\kappa_<^2}, \tag{44}$$

$$I_0(\kappa_> l,\kappa_< l+2;2)\approx \frac{(\pi l)^{1/2}}{(2l+1)}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+5/2} \times\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{-1/2} \tag{45}$$

$$\times\frac{1}{(2l+3)(2l+5)}, \quad l \gg \frac{\kappa_<^2}{\kappa_>^2-\kappa_<^2}, \tag{46}$$

and

$$I_0(\kappa_> l,\kappa_< l;2)\approx \frac{(\pi l)^{1/2}}{2(2l+1)}\left[\frac{\kappa_<}{\kappa_>}\right]^{l+1/2} \times\kappa_>\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{1/2}\frac{1}{l}, \quad l \gg \frac{\kappa_<^2}{\kappa_>^2-\kappa_<^2}. \tag{47}$$

Thus, for a quadrupole transition, the partial collision strengths are proportional to

$$\tfrac{3}{4}lI^2(\kappa_< l,\kappa_> l+2;2)\approx \frac{\pi}{48}\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{3} e^{\pi|\eta_<-\eta_>|} \times\kappa_>^2\left[\frac{\kappa_<}{\kappa_>}\right]^{2l+1} \quad (l\ \text{large}), \tag{48}$$

$$\tfrac{1}{2}lI^2(\kappa_> l,\kappa_< l;2)\approx \frac{\pi}{32}\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right] e^{\pi|\eta_<-\eta_>|} \times\kappa_>^2\left[\frac{\kappa_<}{\kappa_>}\right]^{2l+1}\frac{1}{l^2} \quad (l\ \text{large}), \tag{49}$$

and

$$\tfrac{3}{4}lI^2(\kappa_> l,\kappa_< l+2;2)\approx \frac{3\pi}{256}\left[\frac{\kappa_>^2-\kappa_<^2}{\kappa_>^2}\right]^{-1} e^{\pi|\eta_<-\eta_>|} \times\kappa_>^2\left[\frac{\kappa_<}{\kappa_>}\right]^{2l+5}\frac{1}{l^4}, \quad (l\ \text{large}). \tag{50}$$

From Eqs. (48)–(50) we can see that the partial collision strength $\Omega_{l'l}^{\text{CBeI}}(\kappa',\kappa)=\Omega_{l+2,l}^{\text{CBeI}}(\kappa_>,\kappa_<)$ is large compared with $\Omega_{l,l}^{\text{CBeI}}(\kappa_<,\kappa_>)$ and $\Omega_{l+2,l}^{\text{CBeI}}(\kappa_<,\kappa_>)$ for $l\gg\kappa_<^2/(\kappa_>^2-\kappa_<^2)$. Therefore the main contribution to the sum over l, in Eq. (29), arises from the partial collision strength $\Omega_{l-2,l}^{\text{CBeI}}(\kappa_<,\kappa_>)$. Consequently the infinite sum in $\tilde{\Omega}_{l_0+1}$, which is slowly convergent for large incident energies ($\kappa_</\kappa_>\sim 1$), is asymptotic to a geometric series of ratio $\kappa_<^2/\kappa_>^2$. This greatly simplifies the completion of the summation over partial collision strengths and leads to the following result:

TABLE II. Partial collision strength Ω_l in the CDWII approximation for the $3s$-$3d$ and $4s$-$3d$ transitions in Mg^+. Incident electron energy $k_>^2$ (Ry). ΔE is the excitation energy (Ry). Numbers in brackets denote powers of ten.

l	$3s$-$3d$ $\Delta E=0.651\,47$ $k_>^2=2.579\,48$	$4s$-$3d$ $\Delta E=0.016\,63$ $k_>^2=1.944\,64$	l	$3s$-$3d$ $\Delta E=0.651\,47$ $k_>^2=5.779\,48$	$4s$-$3d$ $\Delta E=0.016\,63$ $k_>^2=5.144\,64$	$3s$-$3d$ $\Delta E=0.651\,47$ $k_>^2=9.767\,70$	$4s$-$3d$ $\Delta E=0.016\,63$ $k_>^2=9.132\,86$
22	5.548[−3]	1.110[−1]	42	2.775[−3]	4.099[−2]	9.688[−3]	7.210[−2]
23	4.077[−3]	9.719[−2]	43	2.435[−3]	3.820]−2]	8.877[−3]	6.725[−2]
24	2.999[−3]	8.558[−2]	44	2.139[−3]	3.566[−2]	8.140[−3]	6.282[−2]
25	2.209[−3]	7.573[−2]	45	1.879[−3]	3.334[−2]	7.470[−3]	5.876[−2]
26	1.628[−3]	6.773[−2]	46	1.652[−3]	3.122[−2]	6.861[−3]	5.505[−2]
27	1.202[−3]	6.013[−2]	47	1.452[−3]	2.928[−2]	6.306[−3]	5.163[−2]
28	8.876[−4]	5.393[−2]	48	1.278[−3]	2.749[−2]	5.799[−3]	4.850[−2]
29	6.560[−4]	4.855[−2]	49	1.124[−3]	2.585[−2]	5.336[−3]	4.561[−2]
30	4.852[−4]	4.386[−2]	50	9.897[−4]	2.434[−2]	4.913[−3]	4.295[−2]
31	3.591[−4]	3.975[−2]	51	8.714[−4]	2.294[−2]	4.526[−3]	4.048[−2]
32	2.660[−4]	3.615[−2]	52	7.676[−4]	2.164[−2]	4.172[−3]	3.820[−2]
33	1.971[−4]	3.298[−2]	53	6.763[−4]	2.044[−2]	3.847[−3]	3.609[−2]
34	1.461[−4]	3.016[−2]	54	5.961[−4]	1.933[−2]	3.549[−3]	3.413[−2]
35	1.083[−4]	2.766[−2]	55	5.255[−4]	1.830[−2]	3.276[−3]	3.232[−2]
36	8.037[−5]	2.543[−2]	56	4.633[−4]	1.734[−2]		

TABLE III. Ω_l/Ω_{l-1} for the 5s-4d transition in Sr^+. $a=k_<^2/k_>^2$, $b=k_<^2/(k_>^2-k_<^2)$. $k_>^2$ (Ry) is the energy of the colliding electron before excitation, $k_<$ (Ry) is the energy of the colliding electron after excitation. The excitation energy is 0.134 27 Ry.

l	$a=0.706$ $b\sim 3$ $k_>^2=0.45627$	l	$a=0.966$ $b\sim 29$ $k_>^2=4.0$	l	$a=0.994$ $b\sim 167$ $k_>^2=22.443$
10	0.676	23	0.876	40	0.930
11	0.684	24	0.881	41	0.931
12	0.690	25	0.885	42	0.932
13	0.693	26	0.890	43	0.933
14	0.696	27	0.894	44	0.935
15	0.697	28	0.897	45	0.934
16	0.699	29	0.901	46	0.937
17	0.699	30	0.904	47	0.938
18	0.700	31	0.911[a]	48	0.940
19	0.701	32	0.914[a]	49	0.941
20	0.701	33	0.916[a]	50	0.942
21	0.707	34	0.918[a]	51	0.942
22	0.702	35	0.920[a]	52	0.943
23	0.702	36	0.923[a]	53	0.950[a]
24	0.703	37	0.924[a]	54	0.945[a]
25	0.703	38	0.926[a]	55	0.936[a]
26	0.703	39	0.928[a]	56	0.959[a]
27	0.703	40	0.931[a]	57	0.961[a]

[a]Ratio Ω_l/Ω_{l-1} calculated using the CBeI approximation.

$$\tilde{\Omega}_{l_0+1}\sim\Omega^{\mathrm{CBeI}}_{l_0-1,l_0+1}(\kappa_<,\kappa_>)/(1-a),\quad l_0 \gg \frac{\kappa_<^2}{\kappa_>^2-\kappa_<^2} \tag{51}$$

where

$$a=\frac{\kappa_<^2}{\kappa_>^2}=\frac{k_<^2}{k_>^2}, \tag{52}$$

and $k_>^2$ and $k_<^2$ are the energies of the free electron before and after the collision. For the particular case of an s-d transition we approximate $\tilde{\Omega}_{l_0+1}$ by

$$\tilde{\Omega}_{l_0+1}\sim\Omega(n'l_a',nl_a;L=l_0+1)/(1-a)\,. \tag{53}$$

The geometric series method might become impractical at energies for which $\kappa_<^2/(\kappa_>^2-\kappa_<^2)\sim 50$, since contri-

TABLE IV. Ω_l/Ω_{l-1} for the 3s-3d and 4s-3d transitions in Mg^+. $a=k_<^2/k_>^2$, $b=k_<^2/(k_>^2-k_<^2)$. $k_>^2$ (Ry) is the energy of the colliding electron before excitation, $k_<^2$ (Ry) is the energy of the colliding electron after excitation. ΔE is the excitation energy (Ry).

l	$a=0.747$ $b\sim 3$ 3s-3d $\Delta E=0.65147$ $k_>^2=2.57948$	$a=0.991$ $b\sim 116$ 4s-3d $\Delta E=0.01663$ $k_>^2=1.94464$	l	$a=0.887$ $b\sim 8$ 3s-3d $\Delta E=0.65147$ $k_>^2=5.77948$	$a=0.997$ $b\sim 309$ 4s-3d $\Delta E=0.01663$ $k_>^2=5.14464$	$a=0.993$ $b\sim 14$ 3s-3d $\Delta E=0.65147$ $k_>^2=9.76770$	$a=0.998$ $b\sim 549$ 4s-3d $\Delta E=0.01663$ $k_>^2=9.13286$
22	0.734	0.871	42	0.877	0.931	0.916	0.932
23	0.735	0.876	43	0.877	0.932	0.916	0.933
24	0.736	0.881	44	0.878	0.934	0.917	0.934
25	0.737	0.885	45	0.878	0.935	0.918	0.935
26	0.737	0.889	46	0.879	0.936	0.918	0.937
27	0.738	0.893	47	0.879	0.938	0.919	0.938
28	0.738	0.897	48	0.880	0.939	0.920	0.938
29	0.739	0.900	49	0.879	0.940	0.920	0.940
30	0.740	0.903	50	0.881	0.942	0.921	0.942
31	0.740	0.906	51	0.880	0.942	0.921	0.942
32	0.741	0.909	52	0.881	0.943	0.922	0.944
33	0.741	0.912	53	0.881	0.945	0.922	0.945
34	0.741	0.914	54	0.881	0.946	0.923	0.946
35	0.741	0.917	55	0.882	0.947	0.923	0.947
36	0.742	0.916	56	0.882	0.948		

butions from high partial waves ($l_0 \gg 50$) should be included in the summation to obtain convergence of the partial-wave expansion. In such a case the analytic formula, derived in a previous paper,[2] to estimate the contribution to the total collision strength from large values of angular momentum should be more appropriate.

III. RESULTS

The asymptotic behavior of the collision strengths for large angular momenta were checked using the data given in Tables I and II. Table I contains partial collision strengths for the excitation of the 5*s*-4*d* transition in Sr^+ taken from a previous publication.[4] Similarly, Table II contains data for the excitation of the 3*s*-3*d* and 4*s*-3*d* transitions in Mg^+.[3] Both sets of data were calculated in the strong-coupling Coulomb distorted wave approximation. We have tabulated in Tables III and IV the ratio Ω_l/Ω_{l-1} as a function of the colliding electron angular momentum l, where $\Omega_l=\sum_{l'}\Omega_{l'l}$, for different incident electron energies. For large l, $\Omega_{l'l}^{\mathrm{CDWII}} \sim \Omega_{l'l}^{\mathrm{CBeI}}$ and the ratio Ω_l/Ω_{l-1} should tend to a constant $a=k_<^2/k_>^2$, where $k_>^2$ and $k_<^2$ are the energies of the free electron before and after the collision. Table III shows results for the excitation of the 5*s*-4*d* transition in Sr^+, and the excitation of the 3*s*-3*d* and 4*s*-3*d* transitions in Mg^+ is shown in Table IV. Also shown are the quantities $a=k_<^2/k_>^2$ and $b=k_<^2/(k_>^2-k_<^2)$. Tables III and IV illustrate the fact that for large values of the electron impact energy, and/or for transitions in which the atomic states are energetically close, e.g., the transition 4*s*-3*d* in Mg^+, the sum over the partial collision strengths [Eq. (29)] is slowly convergent, requiring contributions from a large number of partial waves.

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