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Singularities in Rényi information as phase transitions in chaotic states

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Chaotic dynamical systems are investigated, with the help of the Rényi information concept, both in their phase space and in their history space. Several phases are distinguished and their characteristics are discussed. Emphasis is put on two particular situations representing borderline cases of chaos: when an unstable periodic orbit exists in the system with a zero or an infinite Lyapunov exponent.

I. INTRODUCTION AND SUMMARY

Chaotic motion, which is the subject of the present paper, is a phenomenon very often observed in dynamical systems.[1] Due to its inherent randomness its description requires statistical methods. Shannon information has played a central role in random systems long since answering the question how much information an observer gets, on average, while making a single measurement, if the measurement deals with a finite number of results.[2,3] Later, Rényi generalized the concept of information,[3] introducing the order-q information. ($q=1$ corresponds to Shannon information and when we refer to Rényi information in the following, Shannon information is included as a special case.) Dynamical systems have continuous variables in general, so it is necessary first to partition the phase space, next to make measurements, and finally to investigate the information as the resolution ϵ becomes infinitely fine. Unfortunately, Rényi information tends to infinity in this limit as a rule, and thus finding meaningful quantities to characterize the system becomes an important task.

To deal with the continuous limit, Rényi has introduced the concept of the order-q dimension $D(q)$,[3] which tells us the speed of the divergence. His more detailed investigations showed that usually only one part of the order-q information diverges, and one can specify a remaining part $I_{q,D}$,[3] which does not, by subtracting $D(q)\ln(1/\epsilon)$ and taking the $\epsilon \to 0$ limit. This phenomenon is well known from classical equilibrium statistical physics, where the order-1 Rényi dimension agrees with the dimension of the phase space. It should be mentioned that by investigating systems far from equilibrium Haken has found that Shannon information for such systems can also be split suitably into a divergent and a nondivergent part.[4,5]

In the case of chaotic dynamical systems a large number of papers has been devoted to the properties of the different dimensions in the last few years (for a review see Ref. 6) but to the best of our knowledge the quantity $I_{q,D}$, which will be called reduced Rényi information of order q, has not yet been investigated. Such an investigation is the first aim of the present paper. In particular, we wish to point out that one can use it for describing systems showing phase transitions in the spectrum of generalized dimensions,[7–9] when $D(q)$ is finite and nonzero, thus giving a more complete picture of the nature of such phase transitions. We shall present an example in which it diverges approaching the phase-transition point on both sides, while a break point can be seen in the spectrum of dimensions at the same time. Furthermore, we shall discuss such a situation where $D(q)$ is infinite (zero) in one part of the $D(q)$ spectrum. In such cases $I_{q,D}$ is minus (plus) infinity in the same region of q. We shall investigate how the phase transition appears approaching these anomalous phases. Two examples will be given where the behavior of $I_{q,D}$ shows the influence of a nearby phase-transition point.

Our second goal is to apply this approach when the central quantity is the dynamical Rényi information,[10] which tells us about the information obtained by an observer who makes a series of measurements. This problem can be best treated by constructing a new abstract space, the so-called history space, and mapping the problem to the first one, defined in phase space.[11] The dynamical counterparts of the Rényi dimensions turn out to be the generalized order-q Rényi entropies $K(q)$, apart from a trivial factor. $K(1)$ is by definition the Kolmogorov-Sinai entropy and as usual, we take as the definition of chaos that it is nonzero. In complete analogy with $I_{q,D}$, one can introduce in the history space the reduced dynamical Rényi information of order q denoted by $I^*_{q,D}$.

The discussion will include dynamical phase transitions defined by the appearance of nonanalytic points in the Rényi entropies as a function of q. We shall distinguish three phases. We shall denote as the chaotic chaos phase (CCP) the region of q values $q_1 \le q \le q_2$, where $K(q)$ is finite and nonzero. It is also characterized by a finite reduced dynamical Rényi information of order q between q_1 and q_2, which is found to diverge when approaching the phase-transition points q_1 or q_2. Two anomalous phases are defined as follows. In the region of the regular chaos phase (RCP) $K(q)$ is zero, while in the stochastic chaos phase (SCP) $K(q)$ is infinity. Furthermore, $I^*_{q,D}$ behaves anomalously in these phases, namely, it is plus infinity in RCP and minus infinity in SCP. The former anomalous phase was found previously in chaotic dynamical systems exhibiting weak intermittency.[12,13] The latter is a new phase, and we have observed it in systems in which the trajectory gets close to an unstable periodic orbit with an infinite Lyapunov number. It will be shown that the dynamical multifractal spectrum $g(\Lambda)$, which is the Legendre transform of $(1-q)K(q)$, is not a single-humped function for systems having the stochastic chaos phase, but that only the increasing branch of $g(\Lambda)$ appears.

Furthermore, we shall prove that the reduced dynamical Rényi information of order 1, $I^*_{1,D}$ is equal to the effective-measure complexity introduced by Grassberger.[14]

The general considerations will be illustrated throughout the paper using one-dimensional chaotic maps and carrying out both numerical and analytical calculations.

The paper is organized as follows. Section II will give some short definitions on the static quantities. Section III will discuss the "static" phase transitions. Section IV is devoted to the dynamical quantities and phase transitions in them, followed by examples and numerical evidence.

II. DESCRIPTION OF PHASE SPACE

Informationlike and dimensionlike quantities

Shannon's definition of information is not the only one possible. It fulfills a number of important conditions (the Khinchin axioms[3]) but if one relaxes these conditions somewhat, a number of other informationlike quantities can be defined. The most important of these generalized information quantities are the order-q Rényi information quantities defined in the phase space as[3]

$$I_\epsilon(q)=\frac{1}{1-q}\ln\sum_{i=1}^{n}P_i^q,\quad q\neq 1\ . \tag{1a}$$

One can define $I_\epsilon(q=1)$ as the Shannon information

$$I_\epsilon(1)=-\sum_{i=1}^{n}P_i\ln P_i\ . \tag{1b}$$

Here ϵ refers to the size of uniform boxes introduced by partitioning the phase space and n is the number of boxes with nonzero probability P_i according to an invariant distribution. Similar to the Shannon information, $I_\epsilon(q)$ diverges as $\epsilon\to 0$ as $\sim\ln(1/\epsilon)$, thus the dimension of order q is given by[3,15–17]

$$D(q)=\lim_{\epsilon\to 0}\frac{I_\epsilon(q)}{\ln(1/\epsilon)}\ . \tag{2}$$

The generalized dimensions $D(q)$ can be expressed through the spectrum $f(\alpha)$ as[8]

$$D(q)=\frac{1}{q-1}[q\alpha(q)-f(\alpha(q))]\ , \tag{3}$$

where $\alpha(q)$ is defined by the relation

$$\alpha(q)=\frac{d}{dq}(q-1)D(q)\ . \tag{4}$$

Consequently, $f(\alpha)$ is related to $(q-1)D(q)$ by means of a Legendre transformation.

These definitions assume a special set of measuring instruments with uniform resolution ϵ. In general, this is not the case; the sizes of measuring boxes may be nonuniform. Furthermore, there is nothing in the definitions that need to be specific to attractors of dynamical systems. For more precise and technical definitions we refer to the literature.[11,15]

Further characteristics of the system can be obtained if one investigates the D-dimensional information of order q

$$I_{q,D}=\lim_{\epsilon\to 0}[I_\epsilon(q)-D(q)\ln(1/\epsilon)] \tag{5}$$

introduced by Rényi.[3] This quantity, which will be referred to in the following as reduced Rényi information of order q, is the next leading term to $D(q)\ln(1/\epsilon)$ contained in the order-q Rényi information $I_\varepsilon(q)$, when $D(q)$ and $I_{q,D}$ both exist. In the case of an absolutely continuous distribution with density function $P(x)$, $I_{q,D}$ is equal to $I^P_{q,D}$ in a certain region of q,[3] where

$$I^P_{q,D}=\frac{1}{1-q}\ln\int[P(x)]^q dx,\quad q\neq 1 \tag{6a}$$

$$I^P_{1,D}=-\int P(x)\ln P(x)dx\ . \tag{6b}$$

The region of q where this equality holds is defined by the condition that the integrals in (6a) and (6b) exist, at least as improper ones. (6b) can be interpreted as the Shannon information for continuous random variables and (6a) as its generalization, or alternatively the finite part of Rényi information, i.e., the nondivergent part of Rényi information as the resolution ϵ tends to zero.

The condition of a nonzero and finite $D(q)$ and $I_{q,D}$ can be formulated so that the partition function in (1a) should scale in the limit $\epsilon\to 0$ as

$$\sum_{i=1}^{n}P_i^q\cong A(q)\epsilon^{\tau(q)}\ , \tag{7}$$

where

$$\tau(q)=(q-1)D(q)\ , \tag{8a}$$

$$\ln A(q)=(q-1)I_{q,d}\ . \tag{8b}$$

The validity of (7) can generally be traced back to the basic assumption that the probability in a box scales as a power law with the size of the box,[8]

$$P_i\sim\epsilon^{\alpha_i}\ . \tag{9}$$

When, in the following, we will find phases where (7) does not hold, it will be the consequence of the violation of (9) in certain boxes.

III. PHASE TRANSITION IN FUNCTION OF q

The spectrum of generalized dimensions,[16,17] and the static multifractal spectrum $f(\alpha)$ have attracted recent interest,[8,18,19] and in certain cases nonanalytic behavior has been pointed out. Such behavior might be interpreted as a phase transition in the framework of different kinds of thermodynamic formalism worked out for dynamical systems.[20–22,12]

We want to point out that nonanalytic behavior in $D(q)$ is not the only sign of a phase transition. Qualitatively one can expect that near to the phase-transition points one needs more precise measurement of $I_\epsilon(q)$, i.e., one should choose smaller ϵ, to measure $D(q)$ with the same error bar. In other words , one expects that the reduced Rényi information $I_{q,D}$ diverges as q approaches the phase-transition points.

As an example let us investigate a simple dynamical system, namely, a family of one-dimensional, piecewise parabolic maps[23] defined by

$$x'=f_{\mathrm{PP}}(r,x)=\{r+1-[(r-1)^2+4r|1-2x|]^{1/2}\}/2r\ , \quad -1\le r\le 1,\ \ 0\le x\le 1\ . \tag{10}$$

The parameter r is the control parameter that measures the deviation of the map from the tent map. As the probability density generated by the map is known,[23] namely,

$$P_{\mathrm{PP}}(r,x)=1+r(1-2x),\quad -1\le r\le 1,\ \ 0\le x\le 1\ , \tag{11}$$

analytic calculation can be performed. For $r\neq\pm 1$ one obtains

$$D(q)=1\ , \tag{12}$$

$$I_{q,D}=I^P_{q,D}=\frac{1}{1-q}\ln\frac{(1+r)^{q+1}-(1-r)^{q+1}}{2r(1+q)}\ ,\quad q\neq\pm 1 \tag{13}$$

$$I_{1,D}=I^P_{1,D}=\tfrac{1}{2}-\frac{1}{4r}[(1+r)^2\ln(1+r)-(1-r)^2\ln(1-r)]\ , \tag{14}$$

$$I_{-1,D}=I^P_{-1,D}=\tfrac{1}{2}\ln\left[\frac{1}{2r}\ln\left[\frac{1+r}{1-r}\right]\right]\ . \tag{15}$$

It can be seen that no phase transition occurs if $r\neq\pm 1$. Phase transition in the $D(q)$ spectrum can be found, however,[7,8,12] if the control parameter r is equal to ± 1, namely (for $r=\pm 1$),

$$D(q)=\begin{cases}\dfrac{2q}{q-1} & \text{if } q\le -1\\ 1 & \text{if } q\ge -1\ .\end{cases} \tag{16}$$

We show below how this phase transition is reflected in the properties of $I_{q,D}$. The calculation of $I_{q,D}$ yields for $q>-1$ and $r=\pm 1$

$$I_{q,D}=I^P_{q,D}=\begin{cases}\dfrac{1}{1-q}[q\ln 2-\ln(q+1))],\quad q\neq 1\\ \tfrac{1}{2}-\ln 2,\quad q=1\end{cases} \tag{17}$$

which can be obtained from Eqs. (13) and (14) in the limit $r\to\pm 1$. For $q<-1$, where Eq. (6a) does not apply, a direct calculation gives

$$I_{q,D}=\frac{1}{1-q}\ln[(1-2^q)\zeta(-q)],\quad r=\pm 1,\quad q<-1 \tag{18}$$

where $\zeta(z)$ denotes the Riemann ζ function. Approaching the phase-transition point $q=-1$, $I_{q,D}$ diverges logarithmically on both sides of it.

The nonanalyitic point at $q_c=-1$ defines the phase-transition point, and the two phases for $q>-1$ and $q<-1$, respectively, are characterized by finite and nonzero $D(q)$ and $I_{q,D}$. We call this type of transition a second-rank phase transition. In addition, we point out below that stronger types of nonanalicity can also occur.

It is a general assumption that the partition function scales in the whole region of q, $-\infty<q<\infty$ as in (7). But for certain systems this assumption does not hold in the whole region. Instead, it obeys

$$\sum_{i=1}^{n} P_i^q \sim B(q)(\ln 1/\epsilon)^{-\delta(q)} \tag{19a}$$

or

$$\sum_{i=1}^{n} P_i^q \sim B(q)C(q)^{-1/\epsilon^{\delta(q)}}\ , \tag{19b}$$

where $B(q)$, $C(q)$, and $\delta(q)$ are q-dependent constants. Possibly other non-power-law scaling forms also may have some relevances, but we restrict ourselves to the simplest cases.

For the occurrence of the form (19a) it is sufficient that there exists at least one box probability P_s, which decreases to zero slower than any power law as a function of the size of boxes ϵ, e.g.,

$$P_s>\frac{1}{\ln^\delta 1/\epsilon},\quad \delta>0\ . \tag{20}$$

In that case for $q>1$ an upper bound of $D(q)$ is given by

$$0\le D(q)\le\lim_{\epsilon\to 0}\frac{q}{1-q}\frac{\ln P_s}{\ln 1/\epsilon}\le\lim_{\epsilon\to 0}-\frac{q\delta}{1-q}\frac{\ln\ln 1/\epsilon}{\ln 1/\epsilon}=0\ ,\quad q>1\ . \tag{21}$$

Thus, if some box probabilities decay slower than a power law, $D(q)$ is zero for $q>1$, and the partition function scales as in (19a). Obviously by definition (5), the reduced Rényi information is plus infinity in that region. Concerning the $f(\alpha)$ spectrum $\alpha(q)=f(\alpha(q))=0$ for q values greater than 1, while $\alpha(q=1)=f(\alpha(q=1))$ can take any value between 0 and $D(1)$. Thus the phase transition at $q_c=1$ gives a straight line in $f(\alpha)$

$$f(\alpha)=\alpha,\quad \alpha\in[0,D(1)]\ . \tag{22}$$

The q values less than 1 contribute to the remaining part of $f(\alpha)$.

Similarly, if there exists at least one box probability P_f decaying to zero faster than any power law, the partition function for negative q values is dominated by the terms containing the anomalously fast decaying box probabilities. Let us suppose, for example, that there exists a box probability P_f for which

$$P_f < C^{-1/\epsilon^\delta}, \quad C>1, \delta>0 . \tag{23}$$

Estimating $D(q)$ from below we get for negative q values

$$D(q) \geq \lim_{\epsilon\to 0} \frac{q}{1-q} \frac{\ln P_f}{\ln 1/\epsilon} \geq \lim_{\epsilon\to 0} \frac{-q}{1-q} \frac{\ln C}{\epsilon^\delta \ln 1/\epsilon} = \infty, \quad q<0 . \tag{24}$$

The $f(\alpha)$ spectrum also has a strange behavior. From the trivial inequality

$$f(\alpha) \leq \alpha \tag{25}$$

it follows for $q<1$ that

$$\alpha(q) \geq D(q) . \tag{26}$$

Comparing this inequality with (24), one obtains

$$\infty = D(q) \leq \alpha(q), \quad q<0 \tag{27}$$

which says that there is no contribution to $f(\alpha)$ from the negative q values, and thus it contains only the monotonically increasing branch. In the case of (23) the partition function scales as in (19b), and the reduced Rényi information is minus infinity for $q<0$.

Usually both anomalous scaling forms (19a) and (19b) are restricted to a subset of $q \in [-\infty, \infty]$, which defines the anomalous phases. Approaching the border points q_c of these phases, a phase transition occurs, which will be called a first-rank phase transition, as contrasted to the one discussed earlier. According to our definition, at a first-rank phase transition the behavior (7) becomes violated and gives way to the one like (19a) or (19b), while in case of a second-rank phase transition the change results in a new exponent of the power law (7).

As an example, let us consider a one-dimensional chaotic dynamical system, namely, a map having an invariant distribution with support [0,1] according to

$$\mu(x) = \frac{1}{1-\ln x}, \quad x \in [0,1] . \tag{28}$$

In Ref. 23 the authors gave a straightforward way to construct a one-dimensional single-humped map that has a prescribed invariant distribution. By this method we get for the form of the map corresponding to (28)

$$f(x) = \exp\left[-\frac{|\ln x - \ln(1-x)|}{(1-\ln x)[1-\ln(1-x)] - |\ln x - \ln(1-x)|} \right] . \tag{29}$$

Anomalous scaling occurs in the leftmost box, where

$$P_s \simeq \frac{1}{\ln 1/\epsilon}, \quad \epsilon \to 0 \tag{30}$$

which corresponds to (20) with $\delta=1$. A direct calculation shows that a first-rank transition can be observed at $q_c=1$. The phase for $q>1$ is described by

$$D(q)=0, \quad q>1 \tag{31}$$

$$I_{q,D}=\infty, \quad q>1 \tag{32}$$

while below q_c the usual finite behavior can be found. Its characteristics are given by

$$D(q)=1, \quad q<1 \tag{33}$$

$$I_{q,D} = I^P_{q,D} = \frac{1}{1-q}[(1-2q)+(2q-1)\ln(1-q) + \ln\Gamma(1-2q, 1-q)], \quad q<1 \tag{34}$$

where $\Gamma(x,y)$ denotes the

$$\Gamma(x,y) = \int_y^\infty t^{x-1} e^{-t} dt \tag{35}$$

function. Approaching q_c, the reduced Rényi information shows the sign of a nearby phase-transition point, while $D(q)$ does not. Namely, the former one becomes singular

$$I_{q,D} \simeq -\ln \frac{1}{1-q} , \tag{36}$$

which can be extracted from (34). The Legendre transform of $(q-1)D(q)$ [see Eqs. (3) and (4)] gives the $f(\alpha)$ spectrum, which is

$$f(\alpha)=\alpha \quad \text{if } \alpha \in [0,1] . \tag{37}$$

The normal and the anomalous phases in $D(q)$, i.e., $q<1$ and $q>1$, respectively, reduce to one point in $f(\alpha)$

$$f(\alpha(q)) = \alpha(q) = \begin{cases} 1, & q<1 \quad (38) \\ 0, & q>1 \quad (39) \end{cases}$$

and the straight line between comes from the first-rank phase transition at $q_c=1$.

Our second example is when the invariant measure of the interval $[0,x]$ is given by

$$\mu(x) = e^{1-1/x}, \quad x \in [0,1] . \tag{40}$$

The map which generates this invariant distribution is given as follows:

$$f(x) = 1/[1-\ln(1-|e^{1-1/x} - e^{1-1/(1-x)}|)] . \tag{41}$$

Anomalous scaling occurs also in the leftmost box

$$P_f = e e^{-1/\epsilon} , \tag{42}$$

which corresponds to (23) with $\delta=1$. The result of a calculation as above is

$$D(q)=\infty, \quad q<0 \tag{43}$$

$$I_{q,D} = -\infty, \quad q<0 \tag{44}$$

while in the normal phase

$$D(q)=1, \quad q>0 \tag{45}$$

$$I_{q,D}=\frac{1}{1-q}[(2q-1)+(1-2q)\ln q+\ln\Gamma(2q-1,q)] , \quad q>0, q\neq 1 \tag{46}$$

$$I_{1,D}=\lim_{q\to 1} I_{q,D}=1+2e \ \mathrm{Ei}(-1) . \tag{47}$$

Approaching the phase transition point $q_c=0$, $I_{q,D}$ behaves like

$$I_{q,D}\simeq -q\ln 1/q, \quad q>0 . \tag{48}$$

The $f(\alpha)$ spectrum has quite a strange behavior

$$f(\alpha)=1 \quad \text{if } \alpha\in[1,\infty] . \tag{49}$$

The contribution of negative q values cannot be seen in $f(\alpha)$ because

$$\alpha(q)=\infty \quad \text{if } q<0 . \tag{50}$$

The normal phase reduces to $f(\alpha(q>0))=\alpha(q>0)=1$, and the horizontal line from $\alpha=1$ to $\alpha=\infty$ comes from the first-rank phase transition of $D(q)$.

The two examples of first-rank phase transition can seem somewhat untypical because the anomalous behavior of the measure was built in, and subsequently the map, i.e., the dynamical system, was constructed. In Sec. IV, however, we shall investigate the so-called history space where such first-rank phase transitions show up in a natural way in borderline situations of chaos.

IV. DESCRIPTION OF DYNAMICS

A. History space

Until now, we have studied the information obtained in a single measurement. Next, we consider the information gained by an observer in a series of measurements, which follow the time development of the dynamical system. Let us give first the definition of the measurement. Take a record of a signal x_t with $t=1,2,\ldots,n$ and partition the phase space into m pieces labeled from 0 to $m-1$, considering only which box is visited by the trajectory at specified moments. The sequence $O_n=(i_1,i_2,\ldots,i_n)$, $i_j=0,\ldots, m-1$; $j=1,\ldots,n$ gives a possible characterization of the trajectory, where i_j are attached to the boxes and the symbols $i_1,i_2,\ldots,i_n$ are taken at subsequent sampling times. It is assumed that there exists a stationary probability distribution on the chaotic attractor. Accordingly, the sequences O_n form a stationary process. The probability that a given sequence occurs is $P(O_n)$, which can be calculated from the invariant distribution. The symbol sequence probabilities should be conserved when the system makes transitions to new states,

$$\sum_{i_{n+1}=0}^{m-1} P(i_1,\ldots,i_n,i_{n+1})=P(i_1,\ldots,i_n) . \tag{51}$$

Following Farmer,[11] a new space can be introduced. A given sequence of length n specifies the number

$$z_n^{(j)}=\sum_{k=1}^{n} i_k m^{-k}, \quad j=0,\ldots,m^n-1 . \tag{52}$$

The numbers $z_n^{(j)}$ are in [0,1), and this encoding results a one-to-one correspondence between the numbers $z_n^{(j)}$ and the sequences $(i_1,\ldots,i_n)$. Next, let us construct a new probability-density function by the relation

$$P_n(z)=P(i_1,\ldots,i_n)m^n \quad \text{if } z_n^{(j)}\le z<z_n^{(j)}+m^{-n} . \tag{53}$$

In the limit $n\to\infty$ the resulting $P(z)=\lim_n P_n(z)$ probability-density function is normalized and induces a well-defined measure. Furthermore, the conservation laws (51) ensure that integrating over the intervals $[z_n^{(j)},z_n^{(j)}+m^{-n})$ gives the corresponding symbol-sequence probabilities $P(i_1,\ldots,i_n)$

$$P(i_1,\ldots,i_n)=\int_{z_n^{(j)}}^{z_n^{(j)}+m^{-n}} P(z)dz . \tag{54}$$

This new space, where z is a point and $P(z)$ is defined as above will be referred to in the following as the history space.

Let $\kappa=(\mathcal{A}_0,\ldots,\mathcal{A}_{m-1})$ be a finite partition of the phase space (i.e., $U_{j=0}^{m-1}\mathcal{A}_j$ is equal to the full phase space; $\mathcal{A}_k\cap\mathcal{A}_l=0$ for $k\neq l$), and $d(x,y)$ some metric. The diameter of a set C is $d(C)=\sup_{x,y}d(x,y)$, where the supremum is over any two points x and y in C. The diameter of a partition $\kappa=\{\mathcal{A}_i\}$ is $d(\kappa)=\sup_{\mathcal{A}_i}d(\mathcal{A}_i)$. Let the system evolve in one time unit as $x_{t+1}=f(x_t)$. For every piece $\mathcal{A}_j$ we write $f^{-k}\mathcal{A}_j$ for the set of points mapped by f^k to $\mathcal{A}_j$. The partition κ^n is the partition generated by κ in a time interval of length n

$$\kappa^n=\{\mathcal{A}_{i_1}\cap f^{-1}\mathcal{A}_{i_2}\cap\cdots\cap f^{-n+1}\mathcal{A}_{i_n}\}_{i_1,\ldots,i_n} .$$

This partition has the following property. If two trajectories start from the same partition element they give the same symbol sequences up to the length n. Starting from different partition elements the symbol sequences will be different. Finally, the partition is a generating one if $\lim_{n\to\infty}d(\kappa^n)=0$.[15]. In the following we will assume that a finite generating partition exists, i.e., m is finite, and we will always be dealing with such partitions throughout the paper.

B. Generalized information in history space

Information-theoretically, the observer who makes a series of measurements, which was discussed earlier, obtains

$$I_n(1)=-\sum_{O_n}{}' P(O_n)\ln P(O_n) \tag{55}$$

information, where n refers to the time interval and the prime means here and in the following that the summation is taken over the sequences with nonzero probabilities. If the process is chaotic $I_n(1)$ goes to infinity as $\sim(\text{const})\times n$. Following Sinai,[24] the mean rate of creation of information, which is called Kolmogorov-Sinai entropy, can be defined as

$$K(1)=\lim_{n\to\infty} I_n(1)/n . \tag{56}$$

To describe the dynamical properties of the system, other informationlike quantities can be defined by

$$I_n(q)=\frac{1}{1-q}\ln\sum_{O_n}{}' P(O_n)^q, \quad q\neq 1 \tag{57}$$

analogously to the "static" order-q Rényi information. The dynamical information of order q, $I_n(q)$ also goes to infinity as $\sim(\text{const})\times n$ and the rate

$$K(q)=\lim_{n\to\infty} I_n(q)/n \tag{58}$$

is called in the literature the order-q Rényi entropy. (For a review see Ref. 6.)

By constructing the history space using a generating partition, the relationship of the order-q dimension to the order-q Rényi entropy becomes clear. As we mentioned before, studying symbol sequences of length n is equivalent to examining the symbol-sequence probability density $P(z)$ at a scale of resolution $\epsilon=m^{-n}$. This mapping allows the order-q Rényi entropy to be rewritten in terms of the order-q dimension in history space. Since $\ln(1/\epsilon)=n\ln m$, the order-q Rényi entropy $K(q)$ can be written as

$$K(q)=\lim_{n\to\infty}\left[\frac{I_n(q)}{n\ln m}\right]\ln m ,$$
$$K(q)=D^*(q)\ln m , \tag{59}$$

where we introduced the notation $D^*(q)$ for the order-q dimension in the history space. These equalities are generalizations of Farmer's results for $q=0$ and 1.[11]

There exists the dynamical counterpart of the $f(\alpha)$ spectrum denoted by $g(\Lambda)$, which is basically the $f(\alpha)$ spectrum in the history space apart from some trivial factors. It is defined via[25–27]

$$\Lambda=\alpha^*\ln m , \tag{60}$$

$$g(\Lambda)=f^*(\alpha^*)\ln m , \tag{61}$$

where we used the notations α^* and $f^*(\alpha^*)$ in the history space instead of the usual notations α and $f(\alpha)$, respectively.

Going on with this analogy, we can define the D^*-dimensional information of order q in history space as

$$I_{q,D^*}=\lim_{n\to\infty}[I_n(q)-D^*(q)n\ln m]$$
$$=\lim_{n\to\infty}[I_n(q)-nK(q)] , \tag{62}$$

which we denote as the reduced dynamical Rényi information of order q. $I^*_{q,D}$ has an important meaning at the particular value $q=1$. To the characterization of the complexity generated by the patterns of the symbolic dynamics, Grassberger has introduced the quantity called effective measure complexity (EMC) (Ref. 14):

$$\sum_{n=1}^{\infty} n(h_{n-1}-h_n) , \tag{63}$$

where h_n is defined by

$$h_n=I_{n+1}(1)-I_n(1) . \tag{64}$$

Let us consider

$$C(N)=\sum_{n=1}^{N} n(h_{n-1}-h_n) , \tag{65}$$

which is by the definitions (64) and (65) equivalent to

$$C(N)=I_N(1)-Nh_N . \tag{66}$$

From (62), (64), and (66) it follows that in the limit when N goes to infinity

$$I_N(1)=NK(1)+I_{1,D^*}+O(1/N^\sigma), \quad \sigma>0 \tag{67}$$

$$h_N=K(1)+O(1/N^{\sigma+1}), \quad \sigma>0 \tag{68}$$

$$C(N)=I_{1,D^*}+O(1/N^\sigma), \quad \sigma>0 . \tag{69}$$

Thus the EMC becomes

$$\lim_{N\to\infty} C(N)=I^*_{1,D} . \tag{70}$$

Consequently, the order-1 reduced dynamical Rényi (Shannon) information is equal to the effective measure complexity.

It can be shown that for the unique definition of (62) it is not enough that the partition be a generating one, since (62) can take different values for different generating partitions. For example, if we use the partition $\kappa'=\kappa^k$, which is also a generating one, I'_{q,D^*} will be shifted by $D^*(q)k\ln m$.

To define I_{q,D^*} unambiguously, we require that the partition chosen should be a generating one and

$$I_{1,D^*}=\inf_\kappa I_{1,D^*}(\kappa) , \tag{71}$$

i.e., we use that generator which minimizes I_{1,D^*}.

C. Dynamical phase transitions

Dynamical phase transitions show up as nonanalytic points in the spectrum of Rényi entropies, and we shall point out that they are also exhibited in the singular behavior of the reduced dynamical Rényi information. The assumption that $K(q)$ is nonzero and finite is equivalent to that the partition function $\sum'[P(i_1,\ldots,i_n)]^q$ (where the summation is over $i_1,\ldots,i_n$) scales as $\sim A(q)\lambda^n(q)$ for large n, where the q-dependent $\lambda(q)$ is $\infty>\lambda(q)>1$ for $q<1$, and $0<\lambda(q)<1$ for $q>1$, respectively. But, for certain chaotic dynamical systems this scaling assumption does not hold in the whole region of q, $-\infty<q<\infty$. The investigation of two possibilities in detail leads us to anomalous scaling.

First, let us suppose that using a generator partition in phase space and measuring the sequence probabilities, our chaotic system has one (or more) symbol sequence $\tilde{O}_n$ decaying slower than exponentially, say

$$P(\tilde{i}_1,\ldots,\tilde{i}_n)=P(\tilde{O}_n)\geq An^{-s} , \tag{72}$$

with positive A and s, and this relation holds for large enough n. For $q>1$ a nontrivial upper bound of $K(q)$ is given by

$$K(q) \le \lim_{n\to\infty} \frac{q}{1-q}\frac{1}{n}\ln P(\tilde{O}_n)$$
$$\le \lim_{n\to\infty} \frac{q}{1-q}\frac{1}{n}(\ln A - s\ln n) = 0 , \quad (73)$$

thus $K(q)$ should be zero for $q > 1$.[12] Recall that our system is chaotic, i.e., $K(1)$ is nonzero by definition. Knowing the monotonically decreasing property of the Rényi entropies, it can be seen that a phase transition occurs at $q_c = 1$. This type of phase transition was found in some one-dimensional chaotic maps in the weak intermittent state,[12,28,29] and later, in the Lorenz-model[13] at parameters $\sigma = 3.929$, $b = 1.032$, and $r = 16.49$, which also showed weak intermittency. The intermittency is a possible mechanism for the power-law decay (72) because Eq. (72) means that some symbol sequence decays slower than the usual exponential decay; in other words, this symbol sequence is much more probable than the others. This is the case when the trajectory gets close to a marginally stable periodic orbit, i.e., the system is in weak intermittent state.

Next let us investigate the case when there exists at least one symbol sequence $\tilde{O}_n$ decaying faster than exponentially. For example,

$$P(\tilde{i}_1, \ldots, \tilde{i}_n) = P(\tilde{O}_n) \le B^{-C^n} , \quad (74)$$

with constants $B > 1$ and $C > 1$. A lower bound of $K(q)$ for $q < 0$ is given by keeping only this symbol sequence in the partition function

$$K(q) \ge \lim_{n\to\infty} \frac{1}{1-q}\frac{1}{n}\ln P(\tilde{O}_n) \ge \lim_{n\to\infty} \frac{q}{q-1}\frac{C^n}{n}\ln B = \infty , \quad q<0 \quad (75)$$

which shows that $K(q)$ is infinite for negative q values. However, the topological entropy $K(0)$ is smaller or equal to $\ln m$, where m is the number of partition elements; consequently, we have a phase transition again, with infinite jump at $q = 0$. Because the Rényi entropies are monotonic and positive, $K(q)$ is finite for positive q values. Equations (72) and (74) are the dynamical counterparts of Eqs. (20) and (23), respectively, and the transitions associated with them are first-rank phase transitions in the history space.

It is interesting to ask the question, for which dynamical systems do the singular behavior (75) of the $K(q)$ spectrum arise? In some sense the second possibility (75) occurs in the chaotic state opposite to intermittency.

Usually the chaotic dynamics have (infinitely many) unstable periodic orbits that repel the trajectory. The measure of the instability of a periodic orbit is the Lyapunov number along the periodic orbit

$$L = \lim_{n\to\infty} \frac{1}{n}\ln\|f'(x_n)f'(x_{n-1})\cdots f'(x_1)\|$$
$$x_{i+1} = f(x_i), \quad x_0 \epsilon \text{ periodic orbit} , \quad (76)$$

where $f'(x_i)$ denotes the Jacobian matrix at point x_i. Qualitatively, the larger the Lyapunov number, the stronger the repelling property of the periodic orbit. A periodic orbit can be called marginally unstable if the Lyapunov number is zero, and superunstable if L is infinite. Anomalous scaling may occur if the dynamical system has one of these singular periodic orbits, causing the trajectory to stay for an anomalously long or short time, respectively, in the corresponding boxes of the n-level partition κ^n. The first situation is related to intermittency as discussed above, while the second one is responsible for the behavior (74) and (75).

The usual classification of dynamical systems uses the following terminology. If the system has nonzero Kolmogorov-Sinai entropy $K(1)$ it is called chaotic. For systems exhibiting regular motion (i.e., being in a fixed point or in a limit cycle) $K(1)$ is zero and the systems are called regular. Finally, a system is called stochastic if it has infinite Kolmogorov-Sinai entropy associated with the stochastic noise coming from outside of the system.

Let us suppose that the system is chaotic specified by a nonzero Kolmogorov-Sinai entropy $K(1)$. On the basis of the above terminology we introduce the following classification for phases in chaotic dynamical systems. The chaotic chaos phase (CCP) is that part of the spectrum of the Rényi entropies where $K(q)$ is nonzero and finite. The region of q where $K(q)$ is zero will be called the regular chaos phase (RCP). Finally, the stochastic chaos phase (SCP) is characterized by infinite Rényi entropies.

We are now in a position to sketch the qualitative form of the $g(\Lambda)$ spectrum. For systems that have only CCP, $g(\Lambda)$ is usually a single-humped function.[25] [We note it can be highly degenerate as, e.g., for the logistic map $f(x) = 4x(1-x)$, where $K(q) = \ln 2$ independently of q. Here the single-humped function shrinks to a point.] It is zero at $\Lambda(\pm\infty) = K(\pm\infty)$, positive between them, touches the line $g(\Lambda) = \Lambda$ at $\Lambda(1)$, and attains its maximum at $\Lambda(0)$ where $g(\Lambda)$ is equal to $K(0)$. The shape of $g(\Lambda)$ for systems having RCP was discussed in Ref. 11. Namely, because of the jump in $K(q)$ at $q=1$, when one performs the Legendre transform of $(1-q)K(q)$ one finds that $\Lambda(q=1)$ can take any value between $K(1)$ and O. Thus $g(\Lambda) = \Lambda$ in the range $0 \le \Lambda \le K(1)$. This part of the $g(\Lambda)$ curve then joins a single-humped curve with a continuous first derivative.

It can be easily proved that

$$g(\Lambda(q)) \le \Lambda(q) , \quad (77)$$

from which

$$\Lambda(q) \ge K(q), \quad q<0 \quad (78)$$

follows. Thus, in the SCP $\Lambda(q) = \infty$, i.e., the negative q values do not contribute to the $g(\Lambda)$ spectrum. However, there still remains two possibilities. If the limit of $\Lambda(q)$ is finite when q tends to zero from performing the Legendre transformation one finds that $\Lambda(q=0)$ can take any value between $\Lambda(0+0)$ and ∞. Thus, in that case $g(\Lambda) = K(0)$ in the range $\Lambda(0+0) \le \Lambda \le \infty$. The second possibility occurs when $\Lambda(0+0)$ is infinite, i.e., $g(\Lambda)$ is a strictly monotonically increasing function of Λ, and its maximum, $K(0)$, is reached at infinity.

Next, let us discuss the behavior of the reduced dynamical Rényi information in the different phases. In the

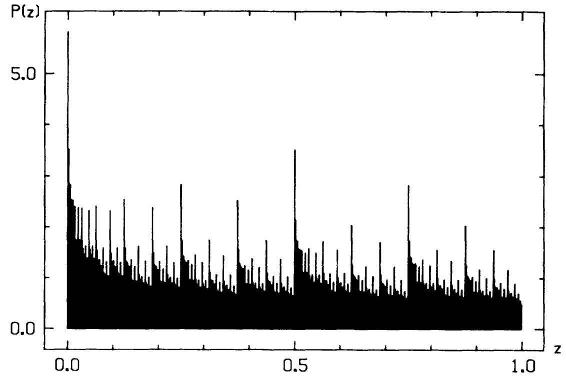

FIG. 1. History-space probability of the map (10) at the resolution $\epsilon=2^{-9}$. The control parameter r is 0.25.

RCP $I_{q,D^*}=\lim_n I_n(q)$ [according to (62) and with $K(q)=0$], which is bounded by

$$\lim_{n\to\infty} I_n(q) \geq \lim_{n\to\infty} I_n(\infty) = -\lim_{n\to\infty} \ln \max P(i_1,\ldots,i_n) \tag{79}$$

from below. Thus, I_{q,D^*} is plus infinity for $q>1$, provided that there is no $P(i_1,\ldots,i_n)$ which remains finite if $n\to\infty$. In the SC phase it directly follows that $I_{q,D^*}=-\infty$ using the definition (62) and the fact that $K(q)=\infty$. Finally I_{q,D^*} is finite in the CCP and expected to tend to infinity approaching the phase-transition points. For finite n $I_n(q)/n \sim K(q)+I_{q,D^*}/n$; consequently, I_{q,D^*} gives the first correction to the "finite-time-scaling" calculations. Furthermore, it shows the phenomenon of critical slowing down in calculating the Rényi entropies for finite times n when q approaches the phase-transition points.

D. Examples

The dynamical system generated by the one-dimensional family of maps (10) is particularly well suited for illustrating the phase transitions outlined above and for investigating the behavior of the dynamical quantities because at the control-parameter value $r=1$ the corresponding map has a marginally unstable fixed point in the origin, while for $r=-1$ this fixed point becomes superunstable. A generating partition corresponding to (71) consists of a bipartition $\mathcal{A}_1=[0,\hat{x})$, $\mathcal{A}_2=[\hat{x},1]$, where $\hat{x}$ denotes the maximum point of $f_{PP}(r,x)$. For the unique natural invariant distribution of the family, known[23] to be (11), the history space of the maps can be easily constructed numerically by the common refinement of the bipartition. Figure 1 shows the history space probability $P_9(z)$ for the map $f(0.2,x)$ at the resolution $\epsilon=2^{-9}$. [See Eq. (53).]

We have numerically determined $K(q)$, $g(\Lambda)$ and I_{q,D^*} in three situations. As a typical situation when only CC phase is present we have chosen $r=0.5$. To illustrate the CCP↔RCP and the SCP↔CCP phase transitions we had to choose $r=1$ and $r=-1$, respectively.

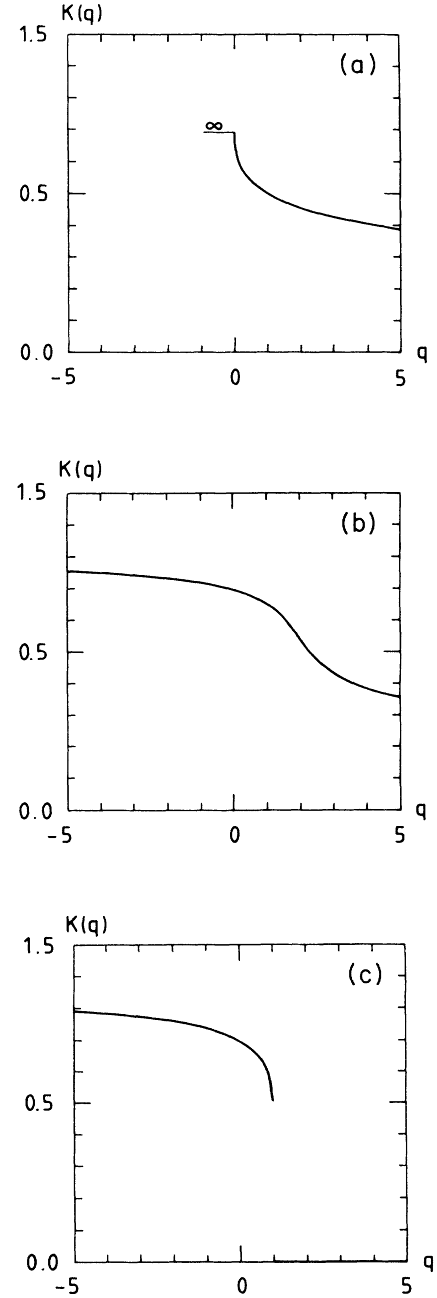

FIG. 2. Order-q Rényi entropies for the class of maps (10) in three situations. (a) $r=-1$ (SCP for $q<0$); (b) $r=0.5$ (only CCP); (c) $r=1$ (RCP for $q>1$).

In the CCP we used the method described in Ref. 12, which is based on the following: Starting from the points of one of the pieces of the n-level partition κ^n the trajectory generates the same symbol sequence. Consequently, integrating the probability-distribution function (11) over

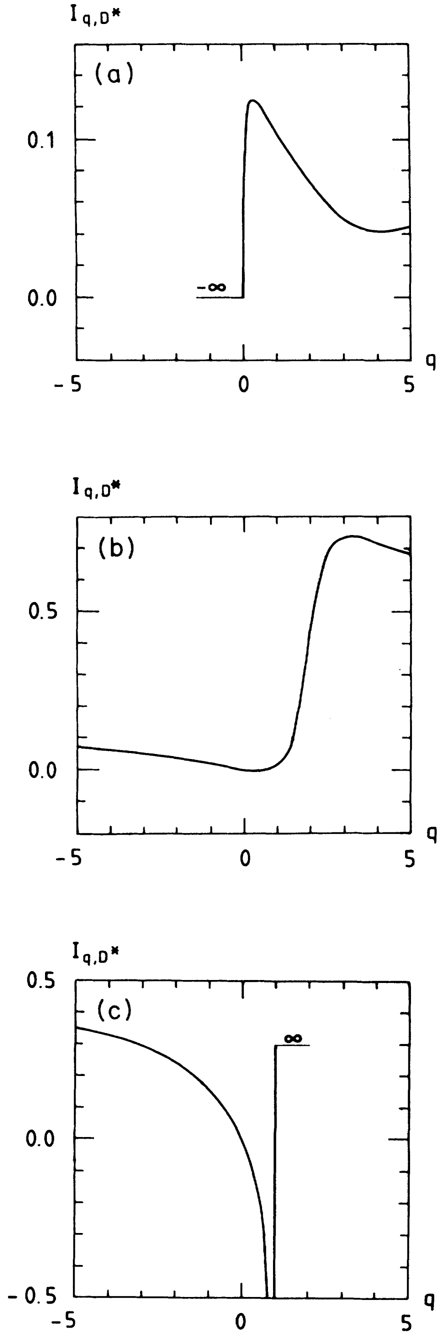

FIG. 3. q dependence of the reduced dynamical Rényi information for the class of maps (10) in the same situations (a)–(c) as in Fig. 2.

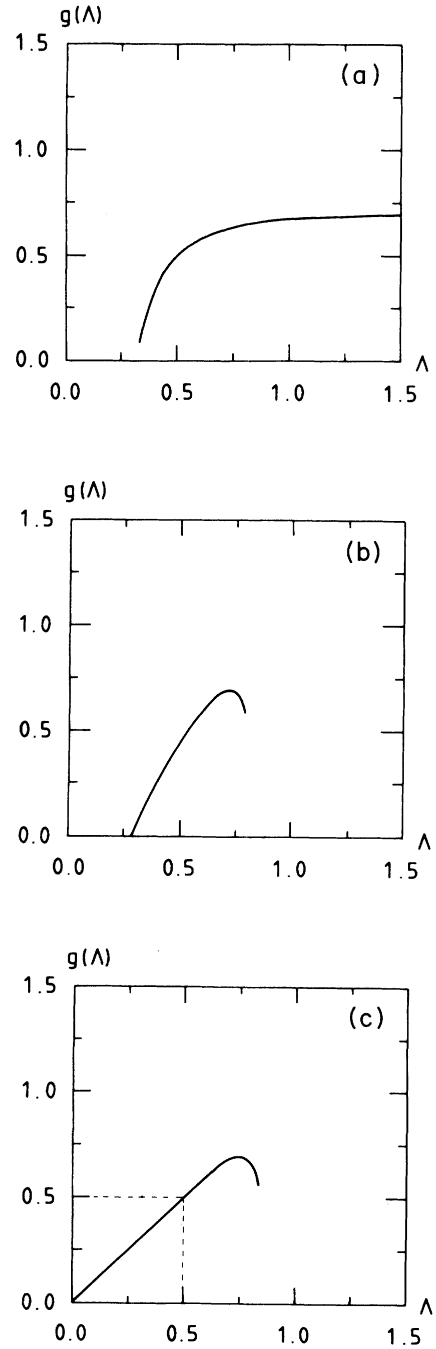

FIG. 4. $g(\Lambda)$ spectrums of the maps (10). The control parameters r were chosen as in Figs. 2(a)–2(c).

these intervals gives the corresponding symbol sequence probability. Knowing them, one gets the dynamical Rényi information $I_n(q)$ by (55) and (57). To obtain more accurate results we used the asymptotic form for one-dimensional maps[28]

$$I_n(q)/n \cong K(q)+[A(q)+B(q)\delta^n(q)]/n \ . \tag{80}$$

From (62) it directly follows that the constant $A(q)$ is

$$A(q)=I_{q,D^*} \ . \tag{81}$$

In this way, we obtain good results at moderate n apart from small neighborhood of the phase-transition points. Figures 2(a)–2(c) and 3(a)–3(c) show the values of $K(q)$

and I_{q,D^*}, respectively, obtained from $n=13,\ldots,16$ in the three situations. From these data we obtained the $g(\Lambda)$ spectrum using the Legendre transform of $(q-1)K(q)$. The present calculation was made in the range $-5\le q\le 5$; thus, in Figs. 4(a)–4(c) one can see that part of the spectrum $g(\Lambda)$ which is between $\Lambda(-5)$ and $\Lambda(5)$. For this map, however, in the presence of the superunstable fixed point $(r=-1)$ we could determine analitically that $\Lambda(0+0)$ is infinity. Consequently, this is an example where $g(\Lambda)$ consists of a strictly monotonically increasing part and nothing else.

One goes along a different thermodynamical path if one investigates the effective measure complexity I_{1,D^*} as a function of the control parameter r. This function, depicted in Fig. 5, shows that the tent map $(r=0)$ is less complex, and the complexity increases upon leaving this point in both directions. In one of the borderline cases of chaos $(r=-1)$ the effective measure complexity is finite, while in the opposite case $(r=1)$ it is an open question whether it is finite or not.

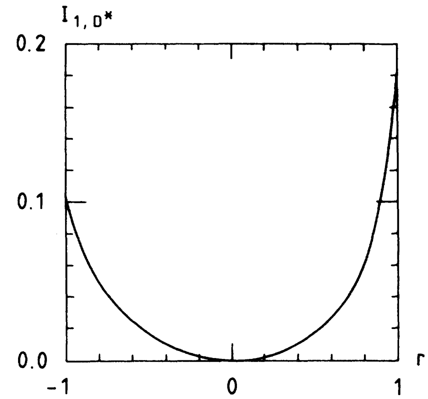

FIG. 5. r dependence of the effective measure complexity I_{1,D^*} for the maps (10).

The examples above belong to the family of fully developed chaotic maps. The borderline situations,[29] where the phase transition occurs can be formulated in a general way for this class of maps. Let us start by recalling the definition of the fully developed chaotic maps $x_i=f(x_i)$.[23,30] They map the interval $[a,b]$ onto itself, where $f(a)=f(b)=a$ and $f(x)$ is supposed to have a maximum at $\hat{x}$, where $f(\hat{x})$. Further specifications understood in the definition of $f(x)$ are that they are differentiable, except possibly at $a,\hat{x},b$, their slopes at the fixed points x^* are $|f'(x^*)|\ge 1$ and they are monotonically increasing and decreasing for $a<x<\hat{x}$ and $b>x>\hat{x}$, respectively. It is also required that the maps are ergodic for almost all initial values and they have a unique invariant measure μ, which is absolutely continuous (with respect to the Lebesgue measure), and have positive Kolmogorov-Sinai entropy. The general form of FDC maps is given by

$$\tilde{f}(x)=1-|1-2x|+v(\tilde{f}(x)) , \tag{82}$$

$$v(x)=v(1-x),\quad v(0)=0 \tag{83}$$

and by conjugates $f(x)=u[\tilde{f}(u^{-1}(x))]$ of $\tilde{f}(x)$.[23] Among these maps a phase transition of type CCP↔RCP occurs at $q=1$, if $v'(0)=1$, while for $v'(0)=-1$ an SCP↔CCP transition can be found at $q=0$. The map (10) corresponds to the choice $v(x)=rx(1-x)$. At the borderline situation further examples are the maps below:

$$f(s,x)=[1-|x^s-(1-x)^s|]^{1/s} , \tag{84}$$

$$P(s,x)=sx^{s-1} . \tag{85}$$

They have an SC phase if $s>1$ and $q<0$, while the maps

$$f(r,x)=1-|x^r-(1-x)^r|^{1/r} , \tag{86}$$

$$P(r,x)=r(1-x)^{r-1} \tag{87}$$

have an RC phase if $r<1$ and $q>1$.

As we have discussed earlier, the reduced order-q dynamical Rényi information $I_n(q)/n$ does not scale in the RC and SC phases as (80). For maps having a marginally unstable fixed point it was shown[12] that

$$I_n(q)/n\sim(\text{const})\ln n/n \tag{88}$$

in the RC phase. The scaling form for maps in the SC phase is quite different, namely,

$$I_n(q)/n\sim F(q)G(q)^n/n , \tag{89}$$

where $F(q)$ and $G(q)$ are q-dependent constants and $G(q)>1$. Finally, we mention that according to Eq. (80) one can define the generalized entropy decay rate by

$$\gamma(q)=-\ln|\delta(q)| , \tag{90}$$

which is also an interesting quantity.[28,31] It describes the characteristic inverse relaxation time of the order-q dynamical Rényi information. $\gamma(q)$ also becomes nonanalytic in the phase-transition points and is infinity, finite, and zero in the SC, CC, and RC phases, respectively.

ACKNOWLEDGMENTS

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