Stochastic resonance in a bistable ring laser

Gautam Vemuri and Rajarshi Roy
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332
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Stochastic resonance was recently observed in a bistable ring laser. It is a phenomenon in which the response of a bistable system to a periodic modulation is enhanced by the injection of noise along with the modulation. So far, comparisons of the experimental observations with theory have been made only for models that are much simplified compared with the more realistic two-mode equations that describe the ring laser. We report here the results of numerical simulations of the stochastic differential equations for the laser with periodic modulation of the asymmetry between the two modes and injected noise. The simulations reveal the phenomenon of stochastic resonance in a manner closely resembling the experimental observations. The appearance of second-harmonic frequencies in the output power spectrum and the role of the colored nature of the pump laser noise are studied. Finally, we discuss the relation of the recent experimental measurements of hysteresis in a ring dye laser by Gage and Mandel [J. Opt. Soc. Am. B (to be published)] to the observation of stochastic resonance.

I. INTRODUCTION

The response of a bistable system to a periodic modulation may be enhanced by the injection of random noise. The measure of the system's response to the modulation is the signal-to-noise ratio at the modulation frequency obtained from the power spectrum of the output of the bistable system. This phenomenon, called stochastic resonance, was invoked by Benzi *et al.* to explain the more or less periodic occurrence of the earth's ice ages.[1] The effect of a small periodic change in the eccentricity of the earth's orbit, by itself insufficient to cause large temperature changes, was shown to be enhanced in the presence of random perturbations. The mathematical problem of a dynamical system subject to both a periodic forcing and stochastic noise was examined by other authors as well; see, for example, Refs. 2 and 3. Experiments on a Schmitt trigger,[4] and more recently, on a bistable ring laser,[5] demonstrated this phenomenon clearly in the laboratory. The problem has since received intense theoretical attention.[6-10] Yet the models studied in Refs. 6-10 have always been much simplified so as to allow analytic solutions to be obtained. Thus, though some features of the experiments have been explained, other interesting aspects of the observations have not received much attention. Digital and analog simulations have also been performed,[7-10] but these, too, have been restricted to the simplified one-dimensional models. The only study of the two-mode laser equations that relates to stochastic resonance is the recent interesting work of Gage and Mandel on hysteresis in a ring dye laser.[11] They do not study the effect of injected noise, however, or the question of enhancement of the signal-to-noise ratio. The emphasis in our work is on these latter issues, and we will elucidate the connection between the observation of hysteresis and the phenomenon of stochastic resonance in this paper.

Our aim here is to obtain numerically the solution of the stochastic differential equations for the two-mode laser model with periodic modulation and injected noise. We will examine and interpret the experimental observation of stochastic resonance on the basis of equations that actually model the bistable laser accurately. These equations are analytically intractable and no analytic solutions are known at present.

Before we present the equations for the two-mode laser and describe their solutions, we summarize the basic features of stochastic resonance and the conditions for its occurrence. The appropriateness of the nomenclature has been discussed by a number of authors,[1-10] and we will not enter into a further discussion here. The basic scheme for observation of this phenomenon is shown in Fig. 1(a). A periodic modulation signal and noise are simultaneously input into a dynamical system. The output of the system, which could be a voltage, light intensity, displacement, or any other suitable variable, is spectrally analyzed. The response of the system to the periodic signal is measured by the height of the peak at the signal frequency in the power spectrum. The noise level is determined as shown in the figure. A linear system would then show the behavior for the signal-to-noise ratio shown in Fig. 1(b) as the noise level (measured, for example, by the standard deviation of the noise fluctuations) is increased, the signal level being maintained constant. In contrast, the phenomenon of stochastic resonance is characterized by the behavior shown in Fig. 1(c), where the qualitatively different behavior of a bistable, nonlinear system is depicted in a schematic fashion. The surprising feature is that the signal-to-noise ratio actually increases as noise is injected into the system, experiences a maximum, and then decreases with further increase of the noise strength. Obviously, this type of behavior cannot occur in a linear system.

A qualitative picture of the effect of a periodic modulation signal on a bistable system in the absence of injected

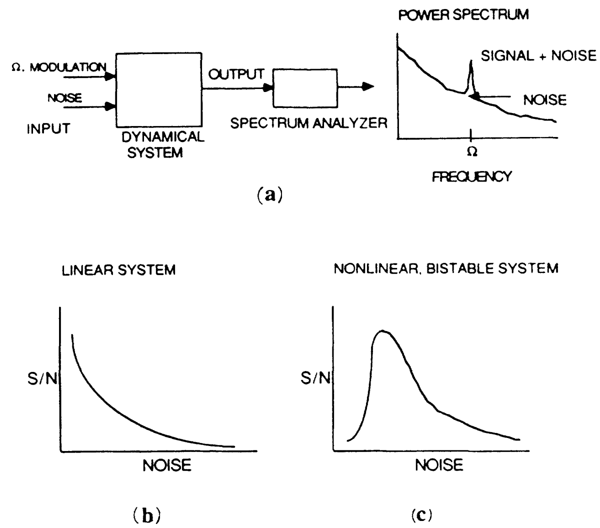

FIG. 1. (a) The effect of input modulation and noise on the output of a dynamical system is observed through spectral analysis. (b) Signal-to-noise ratio as a function of input noise for a linear system. (c) Signal-to-noise ratio as a function of input noise for a bistable, nonlinear system.

noise is presented in Fig. 2. for the case of overdamped motion of a particle in a double-well potential. The periodic signal distorts the shape of the generic bistable double-well potential in the sequence as shown. The position of the particle in the potential well determines the output of the system. When the particle is in the left well, the system is "off," and when it is in the right well, the system is "on." As the signal increases, the potential well distorts until the particle finally rolls over from the left well to the right one. The signal reaches its maximum value and then decreases. The particle is now trapped in the right well, however, and stays there until

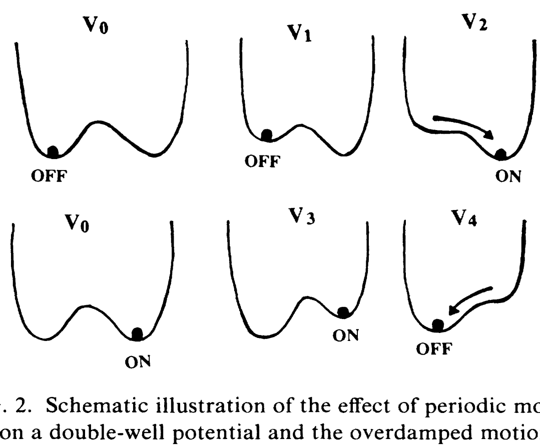

FIG. 2. Schematic illustration of the effect of periodic modulation on a double-well potential and the overdamped motion of a particle in the absence of noise. The modulation amplitude is large enough to cause transitions of the particle from the off to the on state and back. The input modulation values $V_0 - V_4$ correspond to those shown in Fig. 3 where the hysteresis cycle associated with the modulation of the bistable system is depicted.

the signal reaches the opposite extreme value, when it returns to the off state. We translate the sequence of pictures to an input-output characteristic of the bistable system in Fig. 3. The presence of hysteresis is clear from these qualitative considerations. The width of the hysteresis loop in the absence of noise is determined by the barrier height between the two wells of the potential.

In these considerations, the amplitude of the periodic signal was large enough to cause transitions from the on to the off state and back during a cycle. What happens if the amplitude is too small to cause these transitions? The answer is clear; the particle stays trapped in either the off or on state forever. Thus, for a small enough signal, the output of the bistable system displays no response to the input modulation. It is easily seen that if the signal amplitude is less than the width of the hysteresis loop, this is indeed the situation. A bistable system is unable to detect the presence of a modulation of amplitude smaller than the width of the hysteresis loop associated with the system if there is no noise present in the system. We have assumed in this discussion that local motion of the particle in a single well is not detectable. We will see later that this is not strictly true.

The question now is whether the injection of noise could possibly make the bistable system respond more or less periodically to the input signal, so that its presence could be detected in the power spectrum of the output. It is shown in Fig. 4 how this is in fact the case. The periodic signal amplitude is now insufficient to cause transitions from one well to another, but the injected noise induces transitions with good likelihood at the extremes of the periodic modulation, when the barrier height is reduced to a minimum. This would result in a peak at the modulation frequency in the output power spectrum. Of course, if the noise introduced is extremely strong, transitions may occur not only at the extremes of the modulation, but at other times as well, reducing the strength of the signal in the output.

With this qualitative picture of stochastic resonance and the connection between modulation of a double-well potential and hysteresis, we will proceed to the analysis of the two-mode laser equations. These are introduced in

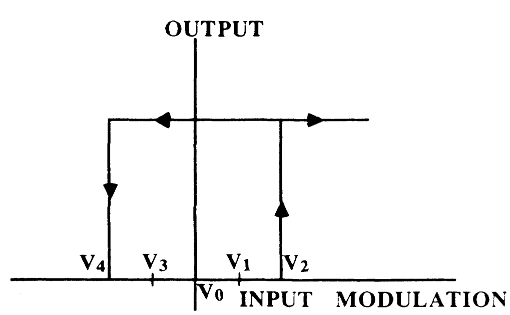

FIG. 3. Hysteresis cycle associated with the modulation of the double-well potential and the motion of the particle are shown. Motion internal to the wells is assumed negligible. The width of the hysteresis loop is determined by the height of the barrier between the wells.

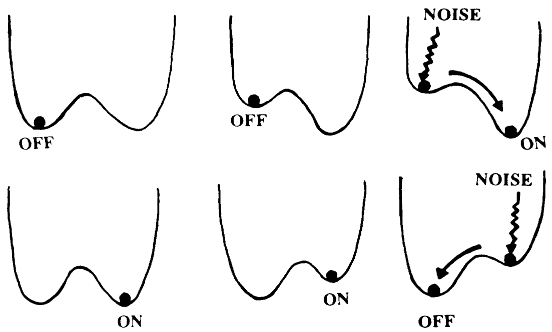

FIG. 4. Modulation amplitude is now insufficient to cause transitions of the particle from one well to another in the absence of noise. When noise is introduced, it is possible for the particle to make more or less periodic noise-induced transitions from the on to the off state and back. These transitions would be detected in the power spectrum of the output as an enhancement of signal-to-noise ratio.

Sec. II, and the method of numerical simulation is outlined. The results of the numerical simulations are presented in Sec. III and compared with experimental results. In Sec. IV we discuss the role of pump noise, and its colored nature, and issues that should be investigated in future experiments and theoretical treatments.

II. TWO-MODE LASER EQUATIONS WITH INJECTED NOISE AND MODULATION

The experiments of Ref. 5 were performed on a two-mode ring laser, with a homogeneously broadened dye medium. Such a laser is bistable in operation, since strong coupling exists between the waves (of the same frequency) traveling in opposite directions. This bistability manifests itself in either one of the beams (clockwise or counterclockwise) being on at a given time. Random switching of the beam intensities occurs, initiated by spontaneous emission in the active medium. The average dwell time of the laser in a given mode or direction increases rapidly with pumping, and the laser will remain on in a given direction for tens or hundreds of seconds even when it is pumped at a few percent above threshold. The experiments of Ref. 5 were performed for such a situation, where the bistable nature of the laser is clearly defined, and random switching plays a minimal role.

Associated with this bistable behavior is a double-well potential model of the laser that will be described shortly. The connection of the qualitative models described in Sec. I to the behavior of the two-mode laser can thus be put on a mathematical footing. The asymmetric modulation of the double-well potential corresponds now to the introduction of a periodically variable anisotropic loss element into the laser. The introduction of noise corresponds to a random variation of this anisotropic loss. Both periodic modulation and noise were introduced by means of an acousto-optic modulator in the experiments of Ref. 5, while in Ref. 11 a Faraday rotator device was used to create a periodic modulation of the potential.

The semiclassical equations for a two-mode laser are well established, and their predictions have been tested experimentally for a number of years.[12–14] The equations are often augmented with noise sources to account for the effect of spontaneous emission and pump noise (which may arise from the fluctuations in the intensity of the pump laser) and take the form

$$dE_1/dt=[\bar{a}+\Delta a+p(t)+|E_1|^2-\xi|E_2|^2]E_1+q_1(t) \quad (1)$$

and

$$dE_2/dt=[\bar{a}-\Delta a+p(t)+|E_2|^2-\xi|E_1|^2]E_2+q_2(t)\ , \quad (2)$$

where E_1 and E_2 are complex, scaled, dimensionless fields of the two modes. The pump parameters of the two modes, a_1 and a_2, have been written as $(a+\Delta a)$ and $(a-\Delta a)$, respectively. The noise sources q_1 and q_2 are complex and Gaussian, with zero mean and correlation functions

$$\langle q_i^*(t)q_j(t')\rangle=4\delta_{ij}\delta(t-t')\quad (i,j=1,2)\ , \quad (3)$$

while the pump noise $p(t)$ is an Ornstein-Uhlenbeck process with zero mean and correlation function

$$\langle p(t)p(t')\rangle=(P\lambda)e^{-\lambda|t-t'|}\ . \quad (4)$$

P and λ specify the strength and time scale of the pump fluctuations. The laser is assumed to be homogeneously broadened, and the value of the coupling constant for the two modes is $\xi=2$ (strong coupling).

In the experiment of Ref. 5, an acousto-optic modulator was used to introduce an asymmetry between the net gains of the two counterpropagating modes, i.e., a periodic modulation of the acoustic frequency was converted to a modulation of the asymmetry of the pump parameters. The injected noise was added to the periodic modulation by a summing amplifier. The effect of both the periodic modulation and the injected noise are included in the laser equations as follows:

$$dE_1/dt=[\bar{a}+\Delta a(t)+r(t)+p(t)+|E_1|^2-\xi|E_2|^2]E_1+q_1(t) \quad (5)$$

and

$$dE_2/dt=[\bar{a}-\Delta a(t)-r(t)+p(t)+|E_2|^2-\xi|E_1|^2]E_2+q_2(t)\ , \quad (6)$$

where $\Delta a(t)=\Delta a\sin\Omega t$, Ω is the modulation frequency, and $r(t)$ is the injected white noise. This noise is Gaussian and δ correlated with strength $2R$. We note that though the pump noise enters the equations similarly, the injected noise differs in sign in the two equations. The modulation frequency Ω is assumed to be slow in comparison to growth and decay rates of the laser field in the cavity.

In the absence of the periodic modulation, injected noise, and pump noise a "potential" associated with the steady-state solution of the Fokker-Planck equation may be obtained. This potential $V(I_1)$ is determined by integration of the joint steady-state probability distribution

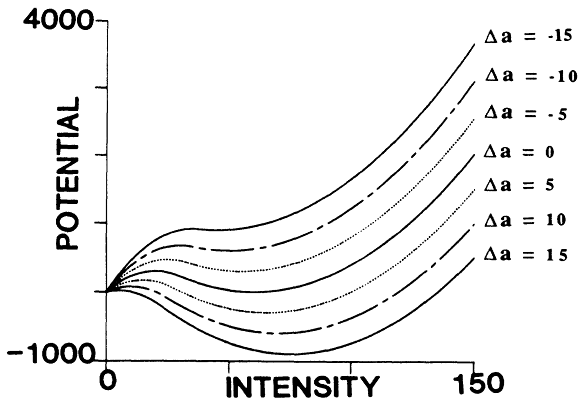

FIG. 5. "Potential" of Eq. (7) plotted as a function of the intensity of mode 1 for $\bar{a}=60$. It is seen that at the extremes of the modulation ($\Delta a=15$ and -15), the barrier height is reduced and noise-induced transitions can occur from one well to another.

$P(I_1,I_2)$ over the intensity I_2 of mode 2. The analytic expression for the potential is[12]

$$V(I_1)=-\tfrac{1}{4}(\xi^2-1)I_1^2+[\tfrac{1}{2}\bar{a}(\xi-1)-\tfrac{1}{4}\Delta a(\xi+1)]I_1 -\ln[1-\text{erf}(\tfrac{1}{2}\xi I_1-\tfrac{1}{2}\bar{a}+\tfrac{1}{4}\Delta a)] . \quad (7)$$

In Fig. 5 we have plotted this potential for three values of Δa. As Δa is modulated, the potential $V(I_1)$ changes adiabatically in response to the modulation. The potential has two wells, one at I_1-0 and the other approximately at $I_1=a_1$. The potential $V(I_1)$ is the one appropriate for the discussion of stochastic resonance in a two-mode laser. If the asymmetry Δa is modulated over a sufficiently large range, the laser will switch from operation in one direction to the other, demonstrating the corresponding hysteresis cycle. In the experiment of Ref. 5 the amplitude of the modulation of the asymmetry was smaller than the width of the hysteresis loop. The laser does not switch directions under the influence of the modulation alone in this case. Injected noise is necessary to initiate switching from one direction to the other. The detector on which one of the beams is incident responds to the on-off intensity variations. The power spectrum of its output is then obtained and the signal-to-noise ratio plotted as a function of the noise input.

III. NUMERICAL SIMULATIONS

The numerical integration of the stochastic differential equations (5) and (6) is straightforward, and has been described in the literature.[15] The values of the intensity I_1 of mode 1 are stored at each step in time and a fast Fourier transform (FFT) is performed to obtain the power spectrum. A very large number (131 072) of time steps ($\Delta t=0.001$) were taken for a given trajectory. Parameter values were $\bar{a}=60$, $\Delta a=15$, $P=100$, $\lambda=1$, and $\Omega=2\pi/4.096$, while the injected noise strength was varied. Power spectra for $R=1$, 16 and 169 are shown in Figs. 6(a)–6(c). These show the phenomenon of stochastic resonance clearly. The signal to noise ratio is small for $R=1$, reaches a maximum around $R=16$, and then subsides again for $R=169$. References 7 and 9 have discussed in detail the technicalities involved in the definitions of signal-to-noise ratio and fast Fourier transformations. We will not discuss these further here; it is sufficient for our purpose to use the simple definitions of signal-to-noise ratio as is evident from the plots, and note

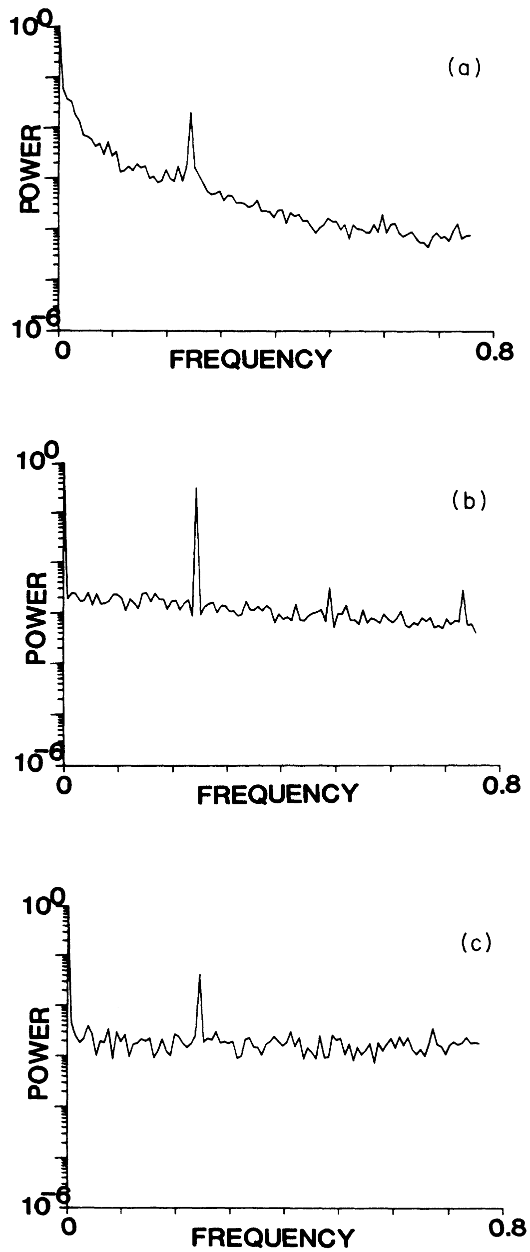

FIG. 6. Power spectra for three different values of the injected noise strength R. (a) $R=1$. (b) $R=16$. (c) $R=169$. There is clear evidence of stochastic resonance, and the second-harmonic peak observed in the experiment of Ref. 5 is clearly seen in the simulations.

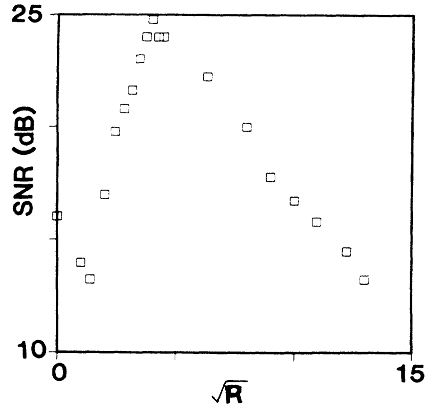

FIG. 7. Signal-to-noise ratio profile as a function of injected noise strength. The stochastic resonance profile is markedly similar to that experimentally observed.

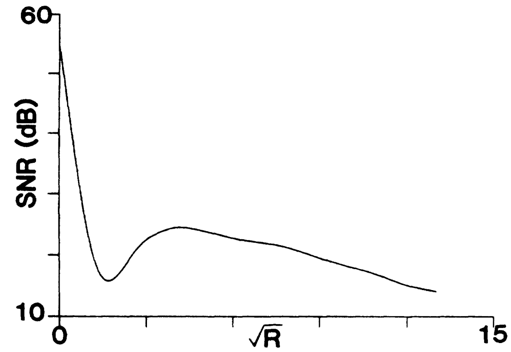

FIG. 9. Stochastic resonance profile from simulations of the laser equations without pump noise. A large peak is seen for small R that was suppressed by the presence of pump noise in Fig. 7. This large initial peak is due to internal motion in a single well of the potential. This is obvious also from the shift of the right-hand-well minimum in Fig. 5.

that signal-to-noise ratios are dependent on the time step and the precise manner in which FFT's are performed (windowing, etc.) These details do not affect the overall behavior of the signal-to-noise ratio as a function of the input noise strength. In Fig. 6(b) it is very clear that signals are present at the second-harmonic and third-harmonic frequences as well. This feature, also present in the experimental observations, has not been previously explained in the literature. It is notable that Eqs. (5) and (6) contain a cubic nonlinearity in the field. When converted to intensity equations, a quadratic term in the intensity is generated, and thus we may expect to see a second-harmonic signal in the power spectrum. The physical situation in the two-mode laser corresponds in a one-dimensional model to the case where the modulation is multiplicative; Debnath, Zhou, and Moss[9] also find peaks at the harmonic frequencies for this system in their analog simulations.

The noise strength R was systematically varied and the signal-to-noise ratio (SNR) obtained from the power spectra was plotted as a function of the square root of R. This plot is shown in Fig. 7. The SNR first decreases to a minimum and then increases sharply by at least 10 dB to reach a maximum. There is then a more gradual decrease in the SNR as R is increased further. Apart from the small initial regime when the SNR decreases, this behavior closely resembles that observed experimentally, which is shown in Fig. 8. We have seen from our simulations that with an appropriate choice of the pump noise strength this initial peak can be suppressed. It was included in Fig. 7 since it serves to clearly point out the role of the pump noise in the observation of stochastic resonance in the ring laser.

It is very pertinent to ask what we would observe if we were in an ideal situation and could eliminate pump noise from our laser completely. This is easy to do in our simulations, though not possible in the experiments. Figure 9 shows the SNR profile versus the square root of R. An enormous sharp peak is visible for very small values of R.

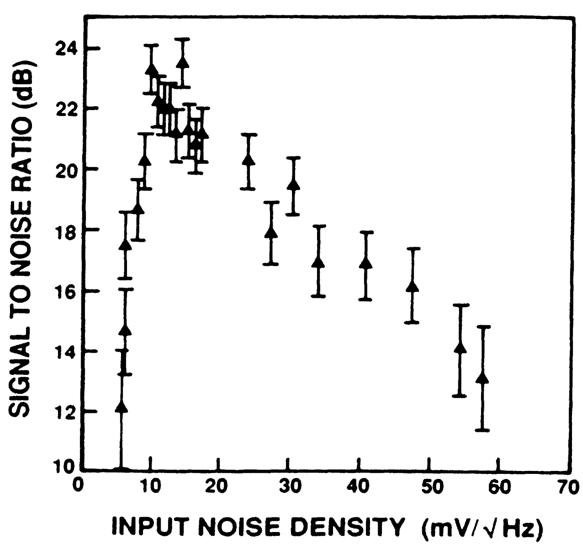

FIG. 8. Experimental observation of stochastic resonance in a bistable ring laser. The data are taken from the experiment in Ref. 5.

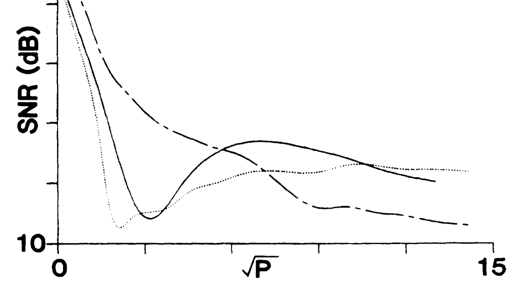

FIG. 10. Signal-to-noise profiles with no injected noise present. The time scale of the pump noise has been varied. (a) ----, $\lambda=1$; (b) · · · ·, $\lambda=100$; (c) ——, white noise.

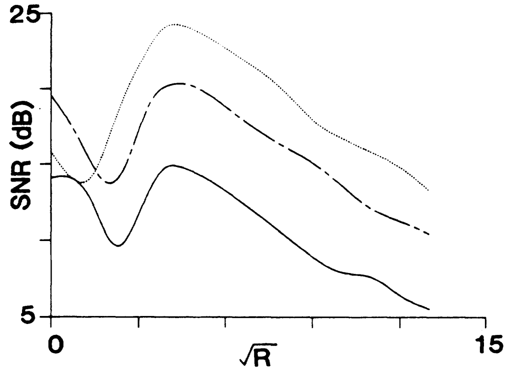

FIG. 11. Effect of reducing the modulation amplitude on the signal-to-noise profile. (a) · · · ·, $\Delta a = 15$; (b) —-—-, $\Delta a = 10$; (c) ——, $\Delta a = 5$.

This initial peak corresponds to motion within a single well that is detected in the power spectrum of the output. In a strictly bistable device, where the on and off states are precisely defined, this sharp initial peak would be absent. The rest of the profile looks very much similar to that of Fig. 7.

What causes the initial sharp peak to be suppressed in the experimental data? The answer to this question is obtained as soon as we examine the behavior of the bistable system when pump noise is present together with the periodic modulation, but the injected noise strength R is set to zero. Figure 10 shows the effect of pump noise as its strength is varied. A glance at Eqs. (5) and (6) reveals that pump noise and the injected noise occur very similarly in them. The only real difference is that the pump noise appears with the same sign in both of the modes, whereas the injected noise appears in an asymmetric fashion. We may then expect the behavior of the SNR profile to be very similar to that of Fig. 9. Indeed, the solid curve in Fig. 10, drawn for the case of white pump noise, does resemble Fig. 9 closely. However, when the pump noise is colored ($\lambda = 100$ and $\lambda = 1$), the SNR profile changes qualitatively. In fact, when the pump noise is strongly colored ($\lambda = 1$), the maximum for intermediate values of P has virtually disappeared. We note that for $P = 100$ on this curve, the signal-to-noise ratio has a value close to 16 dB, which is exactly the initial value of the SNR in Fig. 7. The high initial peak of Fig. 9 is thus seen to be suppressed by the presence of the pump noise. What is indeed remarkable is that even though the presence of the large colored pump noise term suppresses the signal, the injected noise is able to resurrect the SNR to values as large as 25 dB.

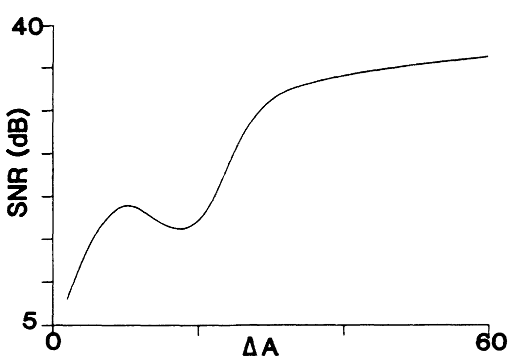

FIG. 12. Signal-to-noise ratio as a function of modulation amplitude when there is no injected noise present.

When the amplitude of the modulation is decreased, stochastic resonance is still observed, though the enhancement becomes smaller in magnitude. Figure 11 shows the SNR profiles for three different values of Δa. The peaks of all three curves lie at about the same value of the input noise strength.

It is also interesting to study the SNR response of the bistable system as a function of the amplitude of the modulation when there is no injected noise. This is shown in Fig. 12. A definite kink appears in this curve at just about the amplitude that is sufficient to observe the complete hysteresis loop. We conjecture that this peculiar behavior is representative of bistable systems with hysteresis that are modulated by a periodic signal in the presence of noise.

IV. DISCUSSION

We have presented an analysis of stochastic resonance effects recently observed in a bistable ring laser[5] based on the equations for a two-mode laser. Our work is also closely related to the recent study by Gage and Mandel[11] of hysteresis effects in a ring dye laser. We have clarified the connection between the observation of hysteresis and the phenomenon of stochastic resonance. In the experiments of Gage and Mandel, the modulation amplitude Δa was held fixed. $\Delta a(t)$ was varied between -20 and $+20$ (note the difference of a factor of 2 in our definition of Δa with regard to the definition in Ref. 11), the only noise sources being pump noise and spontaneous-emission noise. Figures 2–5 explain clearly what was seen in their experiment. The modulation amplitude $\Delta a = 20$ is large enough to observe hysteresis even in the absence of the noise sources, for an average pump parameter of about 60 or smaller. When the pump parameter is much smaller than 60, i.e., about 25, the area of the hysteresis loop is very small, since the barrier height is low, and the spontaneous noise present is sufficient to cause frequent hopping over the barrier. It would be interesting to check whether the area of the loop changes with modulation frequency in this regime of operation. From Fig. 5 ($\bar{a} = 60$, maximum $\Delta a = 15$) we can see that if $\bar{a}$ is increased much further, the barrier height would not be sufficiently depleted by the modulation to cause a hysteresis loop to appear in the absence of noise. It is in this regime, where the width of the hysteresis loop (without noise) is greater than the amplitude of the periodic modulation, that stochastic resonance occurs.

The experiments of Gage and Mandel and those of Ref. 5 are performed in different regimes of experimental parameters; in the former, the modulation amplitude of the periodic signal is greater than the width of the hysteresis loop (without noise), while in the latter, the reverse is the case. Stochastic resonance makes visible the effect of a periodic modulation of amplitude smaller than the width of the hysteresis loop, which would normally be invisible in the output of the bistable system. It is this effect that has been studied in Ref. 5.

The main features of the experiment of Ref. 5, including the observation of a peak at the second-harmonic frequency, are explained by our simulations. The role of pump noise is clearly revealed and the remarkable fact observed that the injected noise in the asymmetry is able to resurrect the SNR to high values even though it is initially (in the absence of the injected noise) suppressed by the pump noise. The effect of the colored nature of pump noise was investigated. The time scale of the pump noise is seen to have a definite effect on the response of the bestable system, changing the SNR profile from nonmonotonic to monotonic when the noise is highly colored. These simulations show that it is certainly possible to explain the phenomenon of stochastic resonance on the basis of realistic equations that describe the laser system. They also indicate that the phenomenon of stochastic resonance exists in a wide variety of systems; the one-dimensional models that have so far been used in the literature do represent (qualitatively) a widespread behavior characteristic of a number of systems. The fundamental issue of whether stochastic resonance can be utilized to obtain signal-to-noise ratios in the output that exceed the input signal-to-noise ratio has yet to be addressed. But even in the limited form that stochastic resonance has been demonstrated so far (where it enhances the response of a bistable system to a periodic signal), there are, doubtless, many applications to be discovered.

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