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Hamiltonian theory of moving interferometers for light and matter waves

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We present a comprehensive theory for interference in an apparatus whose parts—source, detector, mirrors, beam splitters, and medium—are in general motion. Under the conditions prevailing in all practical cases, geometrical optics is fully adequate to give the phases of the interfering waves; therefore, our treatment is based on the Hamiltonian formalism. We assume that the speeds of all moving parts are small with respect to the velocities of the waves, so that their motions can be viewed as small perturbations to a stationary interferometer. A first-order perturbative expression is given for both the Hamiltonian and the ensuing phase shifts. The results are applied to a wide spectrum of cases, ranging from neutron interference experiments to gyro lasers.

I. INTRODUCTION

The theory of the so-called Sagnac effect, or more generally of the interference of light in moving interferometers, is still based on the fundamental work of Post.[1] In more recent times the subject has grown in importance due to developments in widely different fields. On one side there have been technological applications, mostly related to optical rotation-rate sensors (active and passive gyro lasers); on the other side there is the experimental and theoretical interest aroused by neutron interferometry.

While light and neutron interferometry share the same theoretical bases, between the two systems there are several significant differences, mainly (a) whereas light is an extreme-relativistic physical system, neutrons are extreme nonrelativistic and therefore characterized by completely different dispersion laws; (b) usually the dispersivity of the medium is not an important parameter in the case of light, whereas for neutrons any medium is dispersive; (c) the peculiar nature of the dispersivity of the medium for matter waves is sometimes responsible of cancellations—at first sight mysterious—that wash out the typical effects expected by analogy with the case of light.

The results of Ref. 1 cannot immediately be applied to neutron interferometry mainly because in that paper the dispersivity of the medium is (deliberately) ignored, not to mention the problem of translating its results to comply with the different wavelength-frequency relation. Partly as a consequence of this, the theory of neutron interference experiments has been developed by several authors taking different approaches: optical-path difference in stationary systems,[2] optical analogy,[3] apparent-force terms in the Hamiltonian,[4] Doppler effect on moving surfaces,[5] general relativity,[6] analogy with the Bohm-Aharonov effect,[7] invariance of the phase under Lorentz or Galilean transformations,[8,9] scattering by moving centers.[9] Some of these are rather general and elegant but do not apply to all experiments, while others are simpler and more directly aimed at specific experimental setups.[10] In our opinion this state of affairs does not do full justice to the basic unity of the underlying physics, and we felt that an attempt to a unified treatment, encompassing the whole spectrum of interference phenomena, both for light and neutrons, would be welcome.

We have found that a unified treatment is possible under some hypotheses, widely satisfied in present-day experiments: (a) that a full wave-theoretical analysis is not needed—apart for some special situations we shall discuss in due course—so that the phase in an interference experiment can always be computed as a classical action integral; (b) that the perturbing effect of the interferometer's motion is so small that a first-order perturbative treatment is sufficiently accurate. It should be noted, however, that the latter is not a basic limitation of our approach, which could always be pushed to second order,[11] if the need should arise because of increased experimental accuracy.

The natural tool for the theory is then the Hamiltonian formalism, and the main problem becomes the one of finding the first-order form of the Hamiltonian for an interferometer in general motion. Once this aim has been accomplished, an almost automatic application of the basic equation derived from perturbation theory will enable us to discuss straightforwardly the various experimental situations (e.g., moving interferometer with stationary or comoving medium in it, stationary interferometer with moving medium, etc.) with full generality, i.e., taking into account dispersivity, nonhomogeneity of the medium, etc.

Along the same lines we are also able to compare the results of experiments performed with source and detector at rest and a moving interferometer, with those performed with source and detector comoving with the apparatus. This problem is also discussed in Ref. 1 where, however, due to the special geometry considered (both light beams going around the same loop) the differences are shown to vanish at the first order in the velocity of the apparatus. In the general case these differences are not zero, and we show how they are related to the geometry and calibration of the interferometer: the knowledge of these effects is also relevant for a correct interpretation of the experimental result of any interference experiment.

Finally, we give a concise discussion of the basic equations for the beat frequency in a gyro laser. The theory of the ring laser as well can be presented in several different ways:[12] we show that in its elementary form (the one ignoring effects such as frequency locking) it directly follows from the Hamiltonian formalism, which again provides equations taking into account the possible effects of whatever medium may be present in the light path.

II. THE EQUATIONS OF GEOMETRICAL OPTICS

In order to unify the treatment of light and matter fields we use complex scalar[13] wave fields $\psi(\mathbf{x},t)$ (with positive frequencies) so that the wave equation is of the first order in the time derivative and can generally be written in the form[14]

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t}=H(-i\nabla,\mathbf{x}t)\psi(\mathbf{x},t) \ . \tag{1}$$

In the eikonal (or time-dependent WKB) approximation we put

$$\psi(\mathbf{x},t)=A(\mathbf{x},t)e^{i\Phi(\mathbf{x},t)} \tag{2}$$

with A a slowly varying function of its arguments, and $\Phi(\mathbf{x},t)$ satisfying the equation

$$-\frac{\partial\Phi}{\partial t}=H(\nabla\Phi,\mathbf{x},t) \ . \tag{3}$$

In the above equation the explicit dependence of H on $\mathbf{x},t$ is due to the possible presence of a moving nonhomogeneous medium, while the dependence on $\nabla\Phi$ (i.e., on the wave number) embodies both the behavior of the wave in the vacuum and the possible dispersive nature of the medium. For instance, for light in vacuum $H=c|\nabla\Phi|$.

Geometrical optics is based on Eq. (3), which is well known as the bridge between optics and mechanics, where it is called the Hamilton-Jacobi equation. In optical problems—and specially in interference experiments—the incoming wave is given at all times and in all places preceding the optical apparatus, so that we may choose a surface Σ_0 before the entrance of the interferometer and assume Φ and $\nabla\Phi$ known on Σ_0 for all t. Equation (3) can then be solved as follows.

First let us transform it into the integro-differential equation

$$\Phi(\mathbf{x},t)=\Phi(\mathbf{x}_0,t_0)+\int_\gamma \mathbf{k}\cdot d\mathbf{x}-H(\mathbf{k},\mathbf{x},t)dt \ , \tag{4}$$

where $\mathbf{x}_0\in\Sigma_0$, and t_0 is arbitrary. The vector field $\mathbf{k}(\mathbf{x},t)\equiv\nabla\Phi$ is of course as unknown as Φ. The integration can be performed along any curve γ in the $(\mathbf{x},t)$ space joining $(\mathbf{x}_0,t_0)$ to $(\mathbf{x},t)$. If for γ we choose the trajectory (ray) γ_0 obtained by solving Hamilton's equations

$$\frac{d\mathbf{x}}{dt}=\frac{\partial H}{d\mathbf{k}} \ , \quad \frac{d\mathbf{k}}{dt}=-\frac{\partial H}{\partial\mathbf{x}} \ , \tag{5}$$

then $\mathbf{k}$ is known along any ray, and Eq. (4) gives the required solution of Eq. (3) if the point $(\mathbf{x}_0,t_0)$ is determined as a function of $\mathbf{x}$ and t from the assigned boundary conditions for Φ: $\mathbf{x}_0$ (on Σ_0) and t_0 are to be chosen in such a way that the trajectory leaving $\mathbf{x}_0$ at time t_0 with "momentum" $\mathbf{k}(\mathbf{x}_0,t_0)$ will reach the point $\mathbf{x}$ at time t. This of course is nothing but solving the Hamilton-Jacobi equation via the equivalent Hamiltonian system.

For instance, in the case of light propagation in a nondispersive medium the entire dependence of Φ on $\mathbf{x}$ and t is through $\mathbf{x}_0,t_0$ since in this case the integral in Eq. (4) is zero along a trajectory ($H=ck/n$). For the sake of illustration, let us consider a homogeneous medium, and let $\Sigma(\mathbf{x}_0)=0$ be the equation of the surface Σ_0: then $\mathbf{x}_0$ and t_0 are determined by solving the system of equations

$$\begin{aligned}&\mathbf{x}-\mathbf{x}_0=\frac{c}{n}\frac{\mathbf{k}(\mathbf{x}_0)}{|\mathbf{k}(\mathbf{x}_0)|}(t-t_0) \ , \\ &\Sigma(\mathbf{x}_0)=0 \ .\end{aligned} \tag{6}$$

III. PERTURBATION THEORY

We now assume that $H=H_0+H_1$ where H_1 can be treated as a perturbation. Here and in the following H_0 and H are the Hamiltonians for the stationary and moving interferometer, respectively, i.e., determine the behavior of the wave field in the two cases. We are looking for a perturbative solution of Eq. (3) under given boundary conditions for Φ, starting from the solution $\Phi_0(\mathbf{x},t)$ of the equation

$$-\frac{\partial\Phi_0}{\partial t}=H_0(\nabla\Phi_0,\mathbf{x}) \tag{7}$$

for the stationary interferometer with the same boundary conditions (incoming wave), which is supposedly known. Setting

$$\Phi(\mathbf{x},t)=\Phi_0(\mathbf{x},t)+\Phi_1(\mathbf{x},t) \tag{8}$$

from Eqs. (3) and (7) to first order we get

$$-\frac{\partial\Phi_1}{\partial t}=\mathbf{v}_g(\mathbf{x})\cdot\nabla\Phi_1(\mathbf{x},t)+H_1(\nabla\Phi_0,\mathbf{x},t) \tag{9}$$

where

$$\mathbf{v}_g(\mathbf{x})=\frac{\partial H_0(\mathbf{k}_0,\mathbf{x})}{\partial\mathbf{k}_0}, \quad \mathbf{k}_0=\nabla\Phi_0 \ . \tag{10}$$

Then from the identity

$$\Phi_1(\mathbf{x},t)=\Phi_1(\mathbf{x}_0,t_0)+\int_\gamma\nabla\Phi_1\cdot d\mathbf{x}+\frac{\partial\Phi_1}{\partial t}dt \tag{11}$$

and Eq. (9) we get

$$\begin{aligned}\Phi_1(\mathbf{x},t)=\Phi_1(\mathbf{x}_0,t_0)+\int_\gamma&\nabla\Phi_1\cdot d\mathbf{x}\\ &-[\mathbf{v}_g\cdot\nabla\Phi_1+H_1(\mathbf{k}_0,\mathbf{x},t)]dt \ ,\end{aligned} \tag{12}$$

where the integration path γ—and therefore its starting point $(\mathbf{x}_0, t_0)$—is arbitrary. If, however, we integrate along the unperturbed trajectory γ_0, where

$$d\mathbf{x}=\mathbf{v}_g(\mathbf{x})dt \tag{13}$$

and consequently $\mathbf{x}_0, t_0$ are determined as discussed before, Eq. (12) simplifies to

$$\Phi_1(\mathbf{x},t)=-\int_\gamma H_1(\mathbf{k}_0,\mathbf{x},t)dt \tag{14}$$

(Φ_1 vanishes on Σ_0) and we arrive to

$$\Phi(\mathbf{x},t)=\Phi_0(\mathbf{x},t)-\int_{\gamma_0} H_1(\mathbf{k}_0,\mathbf{x},t)dt \ . \tag{15}$$

IV. THE HAMILTONIAN

The interferometer consists of macroscopic objects such as mirrors, half-reflecting mirrors—i.e., beam splitters—and refracting media. The derivation of the macroscopic equation of wave propagation in a material medium starting from its microscopic structure is due to Foldy[15] and Lax.[16] They obtained the (coherent) wave field $\psi(\mathbf{x},t)$ in the medium as an average of the multiply scattered field $\psi(\mathbf{x},t|\mathbf{x}_1,s_1\cdots\mathbf{x}_N,s_N)$ over the different configurations of the elementary scatterers described by their positions $\mathbf{x}_1$ and parameters s_i (velocity, spin, etc.). The average field $\psi(\mathbf{x},t)$ is shown to satisfy a wave equation

$$i\frac{\partial\psi}{\partial t}=H\psi \tag{16}$$

where for nonrelativistic waves[17] (the case considered in Refs. 15 and 16) H is the sum of the Hamiltonian in empty space (kinetic term) and of a "potential" term $V(\mathbf{k},\mathbf{x})$ given, in terms of the scattering operator $T(\mathbf{k},\mathbf{x},s)$ for a scatterer with given values of the scattering parameters s,[18] by

$$V(\mathbf{k},\mathbf{x})=\int\rho(\mathbf{x}',\mathbf{s})T(\mathbf{k},\mathbf{x}-\mathbf{x}',s)d\mathbf{x}'d\mathbf{s} \ , \tag{17}$$

where $\rho(\mathbf{x},\mathbf{s})$ is the distribution function of the scatterers. In the case of point scatterers[15]

$$T(\mathbf{k},\mathbf{x}-\mathbf{x}',s)=\tilde{T}(\mathbf{k},s)\delta(\mathbf{x}-\mathbf{x}') \tag{18}$$

and Eq. (17) reduces to

$$V(\mathbf{k},\mathbf{x})=\int\rho(\mathbf{x},s)\tilde{T}(\mathbf{k},s)ds \ . \tag{19}$$

Equations (17) or (19) give, in the appropriate cases, the potential of the medium.

From the macroscopic point of view the medium is usually characterized either by its refractive index (for light waves) or by a potential (for matter waves). In both cases the refractive index n can be defined by the following relation between the Hamiltonian in presence of the medium H and the one in empty space H_{ES}:[19]

$$H(\mathbf{k},\mathbf{x})=H_{\mathrm{ES}}(\mathbf{k}/n) \ . \tag{20}$$

For instance, for relativistic particles

$$H(\mathbf{k},\mathbf{x})=[(c\mathbf{k}/n)^2+m^2c^4]^{1/2} \ . \tag{21}$$

Equation (20) is justified by observing that for stationary solutions it leads to the usual relation expressing n as the ratio of the wave numbers in the medium and in the vacuum, or as the ratio of the phase velocities.[20] If Φ is of the form

$$\Phi(\mathbf{x},t)=\vartheta(\mathbf{x})-\omega t \ , \tag{22}$$

the Eq. (3) gives

$$\omega=H(\mathbf{k},\mathbf{x})=H_{\mathrm{ES}}(\mathbf{k}/n) \ , \tag{23}$$

where $\mathbf{k}=\nabla\vartheta$ is the wave vector. When the wave goes from the vacuum into the medium ω stays constant, so that also $H_{\mathrm{ES}}(\mathbf{k}/n)$ is constant. Since H_{ES} depends only on the modulus of $\mathbf{k}$, we find $\mathbf{k}/n=$const.

For dispersive and inhomogeneous media n is a function both of $\mathbf{k}$ and of $\mathbf{x}$. In the case of matter waves Eq. (20) allows to establish the relation between the potential and the refractive index n: if we consider, e.g., a nonrelativistic particle moving in a region of potential $V(\mathbf{k},\mathbf{x})$, i.e., if

$$H(\mathbf{k},\mathbf{x})=\frac{k^2}{2m}+V(\mathbf{k},\mathbf{x}) \ , \tag{24}$$

then from Eq. (20) we get

$$n^2=\frac{k^2}{k^2+2mV}=1-\frac{2mV}{k^2+2mV} \tag{25}$$

or, for stationary waves, the well-known relation

$$n^2=1-V/E \ , \tag{26}$$

where E is the value of H for the given stationary solution.

Therefore, for matter waves, the medium is normally dispersive even if the potential is not velocity dependent; however, the effects of the dispersivity of moving matter will enter only through the difference between the group velocity $\mathbf{v}_g$ and $\mathbf{k}/m$, which is not zero only if the potential V depends on $\mathbf{k}$. Conversely, for light waves Eq. (20) formally allows to define the (macroscopic) potential as $V=H(\mathbf{k},\mathbf{x})-H_{\mathrm{ES}}(\mathbf{k})=ck(1-n)/n$.

Equality—up to a phase—of the incident and reflected wave fields on the mirrors gives rise to a relation between incident and reflected wave vectors $\mathbf{k}(\mathbf{x},t)=\nabla\Phi$ identical to the law of reflection of a particle by a potential barrier (or a potential step of sufficient height); therefore, in geometrical optics the mirrors can be represented by potential terms in the Hamiltonian. Beam splitters have no mechanical analog, and cannot be described in geometrical optics. However, since they are devised to address the wave field in two separated regions of space we can use two different Hamiltonians for the two beams, such that a beam splitter is represented as a mirror for one of them, and as a transparent medium for the other.

A. The Hamiltonian of a moving interferometer

The above discussion enables one—in principle—to construct the Hamiltonian (hereafter designated with H_0) for a given stationary interferometer; our main concern, however, is the Hamiltonian H for the moving interferometer. As in Ref. 1 we shall consider the general case in which the various parts of the interferometer move independently of one another: their velocities are supposed to be very small with respect to the group velocity of the wave field, and their motions are then described by a field of (infinitesimal) displacements $\boldsymbol{\eta}(\mathbf{x},t)$ with respect to the position of the unperturbed (i.e., stationary) interferometer,

$$\mathbf{x}=\mathbf{x}^0+\boldsymbol{\eta}(\mathbf{x},t) \ . \tag{27}$$

The corresponding velocity field $\mathbf{u}(\mathbf{x},t)$, to first order in the $\boldsymbol{\eta}$'s is given by

$$\mathbf{u}(\mathbf{x},t)=\frac{\partial\boldsymbol{\eta}(\mathbf{x},t)}{\partial t} \ . \tag{28}$$

The problem of deriving the Hamiltonian H (in the laboratory frame) for the moving interferometer cannot be given a general solution independently of the microscopic structure of the medium: for instance, if the scatterers which compose the medium would act as long-range force centers, the Hamiltonian at a given point of the moving interferometer would be determined by the displacement field of all the sources and retardation effects could also be relevant. Therefore, having in mind the situations of physical interest, we make the following assumptions: (i) the medium consists of rigid pointlike (possibly nonisotropic) scatterers; (ii) the potential due to the constituent moving scatterers does not depend on their acceleration. Under the above hypotheses the problem can be solved by determining, by means of a Lorentz transformation, the effect of the motion of every single scatterer (in the presence of all other scatterers).

It will be better for our purposes, and also for later reference, to start with a more abstract objective, i.e., to find the transformation properties of the Hamiltonian under a general infinitesimal transformation of coordinates:

$$\begin{aligned}\mathbf{x}'&=\mathbf{x}-\boldsymbol{\xi}(\mathbf{x},t) \ ,\\ t'&=t-\tau(\mathbf{x},t) \ .\end{aligned} \tag{29}$$

Let $\Phi(\mathbf{x},t)$ and $\Phi'(\mathbf{x}',t')$ denote the phase of a wave expressed, respectively, in terms of the old and of the new coordinates,

$$\Phi(\mathbf{x},t)=\Phi'(\mathbf{x}',t') \ . \tag{30}$$

By definition these functions satisfy the equations

$$-\frac{\partial}{\partial t}\Phi(\mathbf{x},t)=H(\nabla\Phi,\mathbf{x},t) \ , \tag{31}$$

$$-\frac{\partial}{\partial t'}\Phi'(\mathbf{x}',t')=H'(\nabla\Phi',\mathbf{x}',t') \ , \tag{32}$$

where H' is the Hamiltonian in the new coordinates.

Differentiating Eq. (30) with respect to t we get

$$\frac{\partial\Phi}{\partial t}=-\nabla\Phi'\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}+\frac{\partial\Phi'}{\partial t'}\left[1-\frac{\partial\tau}{\partial t}\right] , \tag{33}$$

so that

$$H(\nabla\Phi,\mathbf{x},t)=H'(\nabla\Phi',\mathbf{x}',t')\left[1-\frac{\partial\tau}{\partial t}\right]+\nabla\Phi'\cdot\frac{\partial\boldsymbol{\xi}}{\partial t} \ . \tag{34}$$

Differentiating Eq. (30) with respect to x_i

$$\frac{\partial\Phi}{\partial x_i}=\frac{\partial\Phi'}{\partial x_i'}-\frac{\partial\Phi'}{\partial x_j'}\frac{\partial\xi_j}{\partial x_i}-\frac{\partial\Phi'}{\partial t'}\frac{\partial\tau}{\partial x_i} \tag{35}$$

and substituting into Eq. (34), we get after some algebra

$$\begin{aligned}H(\mathbf{k},\mathbf{x},t)=H'(\mathbf{k},\mathbf{x},t)+\mathbf{v}_g\cdot(\mathcal{B}\cdot\mathbf{k}-H\nabla\tau)-H\frac{\partial\tau}{\partial t}\\ -\boldsymbol{\xi}\cdot\nabla H-\tau\frac{\partial H}{\partial t}+\mathbf{k}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t} \ .\end{aligned} \tag{36}$$

In Eq. (36) we have neglected terms of second order in $\boldsymbol{\xi},\tau$ and have introduced the tensor $\mathcal{B}$ defined by

$$B_{ij}=\frac{\partial\xi_j}{\partial x_i} \ . \tag{37}$$

Coming back to physics, in the laboratory reference frame (RF) $\mathcal{F}$ consider, at a given time $\bar{t}$, a macroscopically infinitesimal region centered at the point $\bar{\mathbf{x}}=\mathbf{x}^0+\boldsymbol{\eta}(\bar{\mathbf{x}},\bar{t})=\mathbf{x}^0+\bar{\boldsymbol{\eta}}$. All macroscopic quantities related to the scatterers are constant in that region: in particular their velocity $\bar{\mathbf{u}}=\mathbf{u}(\bar{\mathbf{x}},\bar{t})$ and—in the nonisotropic case—their orientation (with respect to that of the scatterers at the point $\mathbf{x}^0$ of the unperturbed interferometer) determined by the antisymmetric tensor $\mathcal{A}$:

$$A_{ij}=\frac{1}{2}\left[\frac{\partial\eta_i}{\partial x_j}-\frac{\partial\eta_j}{\partial x_i}\right] \tag{38}$$

at the point $\bar{\mathbf{x}},\bar{t}$. These parameters define the RF $\bar{\mathcal{F}}$ "tangent" to the motion of the scatterers in the region.

We then specialize transformation (29) so that $\mathbf{x}',t'$ are the coordinates of $\bar{\mathcal{F}}$, in such a way that $\mathbf{x}'=\mathbf{x}^0$ for $\mathbf{x}=\bar{\mathbf{x}}$, $t=\bar{t}$: this is done by choosing

$$\begin{aligned}\boldsymbol{\xi}&=\bar{\boldsymbol{\eta}}+\bar{\mathcal{A}}\cdot(\mathbf{x}-\bar{\mathbf{x}})+\bar{\mathbf{u}}(t-\bar{t}) \ ,\\ \tau&=\frac{1}{c^2}\bar{\mathbf{u}}\cdot(\mathbf{x}-\bar{\mathbf{x}}) \ ,\end{aligned} \tag{39}$$

i.e.,

$$\begin{aligned}\mathbf{x}'&=\mathbf{x}-\bar{\boldsymbol{\eta}}-\bar{\mathcal{A}}\cdot(\mathbf{x}-\bar{\mathbf{x}})-\bar{\mathbf{u}}(t-\bar{t}) \ ,\\ t'&=t-\frac{1}{c^2}\bar{\mathbf{u}}\cdot(\mathbf{x}-\bar{\mathbf{x}}) \ .\end{aligned} \tag{40}$$

Except for the case of a uniform motion of the whole interferometer, the Hamiltonian $\bar{H}$ in the RF $\bar{\mathcal{F}}$ is different (for $t'=\bar{t}$) from H_0 in the first place because of the motion of all scatterers outside the considered region. However, thanks to the hypothesis that the scatterers are pointlike, these do not affect the properties of the medium in the region, so that $\bar{H}$ will coincide, for $t'=\bar{t}$ and $\mathbf{x}'\simeq\mathbf{x}^0$, with the Hamiltonian of a stationary medium. Since, moreover, the physical properties of the scatterers do not change with motion (rigidity hypothesis), the only difference with respect to H_0—thanks to the principle of relativity—could arise if the motion changed the density of the scatterers, i.e., if $\nabla\cdot\boldsymbol{\eta}\neq 0$:

$$\bar{H}=H_0-(\nabla\cdot\boldsymbol{\eta})V_0 \tag{41}$$

if V_0 is proportional to ρ [Eq. (19)].[21] Moreover, $\bar{H}$ and H_0 differ only by terms of order $\mathbf{u}(\mathbf{x},\bar{t})$ [or $\boldsymbol{\eta}(\mathbf{x},\bar{t})$]; in particular $\bar{H}$ and H_0 differ by second-order terms for $\mathbf{x}=\bar{\mathbf{x}}$, $t=\bar{t}$.

Then Eq. (36) together with Eq. (41), retaining only the first-order terms, give for $H_1=H-H_0$ (the perturbation due to the interferometer's motion):

$$H_1(\mathbf{k},\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)\cdot\left[\mathbf{k}-\frac{\mathbf{v}_g}{c^2}H_0\right]-\boldsymbol{\eta}(\mathbf{x},t)\cdot\frac{\partial H_0}{\partial x} -\mathbf{v}_g\cdot\mathcal{A}\cdot\mathbf{k}-[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0 \ . \tag{42}$$

The term $\mathbf{v}_g\cdot\mathcal{A}\cdot\mathbf{k}$ vanishes if the medium is isotropic, i.e., if $\mathbf{v}_g$ and $\mathbf{k}$ are parallel, while the last term is zero if $\boldsymbol{\eta}(\mathbf{x},t)$ is the displacement field of an incompressible fluid. For a free particle the first term vanishes as well: it follows that $H_1=0$ in the vacuum, for whichever displacement field, as it should be.[22]

For nonrelativistic waves the procedure is similar, and the result is simply obtained by replacing H_0/c^2 with $m/\hbar$ in the second term on the right hand side of Eq. (42).

As already pointed out in the course of the derivation, Eq. (42) is not the general solution of the problem, but should be regarded as the zero-order term of an expansion in powers of the range of the interaction of the scatterers. A simple analysis shows, however, that the expansion parameter is of the order of the ratio between the range of the interaction and the dimension of the interferometer, therefore Eq. (42) is quite acceptable in all practical situations.

B. Microscopic theory

A less formal (but also less general) derivation of Eq. (42) can be given, entirely based on the microscopic theory of the medium: we shall confine our discussion to the case of nonrelativistic matter waves. Let $H_0=H_{\mathrm{ES}}+V_0(\mathbf{k},\mathbf{x})$ where V_0 is given by Eq. (19). The scattering parameters s are taken to be the velocity $\mathbf{s}$ of the scatterer and its orientation, that for simplicity we describe by a unit vector $\boldsymbol{\sigma}$ (other internal parameters are of no concern here). Since we are interested in the coherent scattering $\mathbf{s}$ is taken to be zero (fixed scatterers). Then Eq. (19) takes the form

$$V_0(\mathbf{k},\mathbf{x})=\int\rho_0(\mathbf{x},\boldsymbol{\sigma})\tilde{T}(\mathbf{k},\mathbf{s}=0,\boldsymbol{\sigma})d\boldsymbol{\sigma} \ . \tag{43}$$

Suppose now that the medium is subjected to a (macroscopic) displacement field $\boldsymbol{\eta}(\mathbf{x},t)$ [Eq. (27)]. Then $\rho_0(\mathbf{x},\boldsymbol{\sigma})$ in Eq. (43) is to be replaced by

$$\rho(\mathbf{x},\boldsymbol{\sigma})=\rho_0(\mathbf{x}-\boldsymbol{\eta}(\mathbf{x},t),\mathcal{R}^{-1}(\mathbf{x},t)\cdot\boldsymbol{\sigma})[1-\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)] \ , \tag{44}$$

where $\mathcal{R}(\mathbf{x},t)$ is the rotation matrix given by

$$R_{ij}(\mathbf{x},t)=\delta_{ij}+\frac{1}{2}\left[\frac{\partial\eta_i}{\partial x_j}-\frac{\partial\eta_j}{\partial x_i}\right]=\delta_{ij}+A_{ij}(\mathbf{x},t) \tag{45}$$

and for the potential term we have

$$\begin{aligned}V(\mathbf{k},\mathbf{x},t)&=\int\rho(\mathbf{x},\boldsymbol{\sigma})\tilde{T}(\mathbf{k},\mathbf{u}(\mathbf{x},t),\boldsymbol{\sigma})d\boldsymbol{\sigma}\\&=\int\rho_0(\mathbf{x}-\boldsymbol{\eta}(\mathbf{x},t),\boldsymbol{\sigma})\tilde{T}(\mathbf{k},\mathbf{u},\mathcal{R}\cdot\boldsymbol{\sigma})d\boldsymbol{\sigma}\\&\quad-[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0 \ .\end{aligned} \tag{46}$$

From Galilean and rotational invariance

$$\tilde{T}(\mathbf{k},\mathbf{u},\mathcal{R}\cdot\boldsymbol{\sigma})=\tilde{T}(\mathcal{R}^{-1}\cdot(\mathbf{k}-m\mathbf{u}),\boldsymbol{\sigma}) \tag{47}$$

and (to first order)

$$\begin{aligned}V(\mathbf{k},\mathbf{x},t)&=V_0(\mathbf{k}-\mathcal{A}\cdot\mathbf{k}-m\mathbf{u}(\mathbf{x},t),\mathbf{x}-\boldsymbol{\eta}(\mathbf{x},t))\\&\quad-[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0\\&=V_0(\mathbf{k},\mathbf{x})-\frac{\partial V_0}{\partial\mathbf{k}}\cdot(m\mathbf{u}+\mathcal{A}\cdot\mathbf{k})-\boldsymbol{\eta}\cdot\frac{\partial V_0}{\partial\mathbf{x}}\\&\quad-[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0 \ .\end{aligned} \tag{48}$$

Since

$$\mathbf{v}_g=\frac{\mathbf{k}}{m}+\frac{\partial V_0}{\partial\mathbf{k}} \tag{49}$$

we get

$$H=H_0-\mathbf{u}(\mathbf{x},t)\cdot(\mathbf{k}-m\mathbf{v}_g)-\boldsymbol{\eta}(\mathbf{x},t)\cdot\frac{\partial H_0}{\partial\mathbf{x}} -\mathbf{v}_g\cdot\mathcal{A}\cdot\mathbf{k}-[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0 \ , \tag{50}$$

which coincides with the nonrelativistic version of Eq. (42).

V. EFFECT OF THE MOTION OF THE SOURCE AND DETECTOR

One is often concerned—at least in the case of a rigidly moving interferometer—with experiments performed in the (possibly) noninertial frame of the interferometer, and in this case it is certainly more convenient to work directly with the Hamiltonian in that frame, rather than with the one in the inertial frame [Eqs. (42) or (50)]. This is a particular case of the more general problem in which not only the interferometer is moving (nonrigidly as well) but also the source and the detector are in arbitrary motion. This problem could be treated by taking into account the Doppler effect due to the motion of the source and of the detector, but we prefer to follow a different procedure.

First, observe that the velocity and displacement fields $\mathbf{u}(\mathbf{x},t)$ and $\boldsymbol{\eta}(\mathbf{x},t)$ need not be restricted to be different from zero only on the moving parts of the interferometer, but can be extended (in an arbitrary way) in the vacuum, as we already noted. Thus for experiments in which the source and/or the detector are in motion in the inertial frame we can choose $\mathbf{u}$ and $\boldsymbol{\eta}$ to correctly represent their motions too.

Then we proceed to describe the experiments by means of the coordinates $\mathbf{x}^0$ defined by Eq. (27) in terms of which the position of the whole device—interferometer, source, and detector—is time independent (the interferometer's coordinate system), i.e.,

$$\mathbf{x}^0=\mathbf{x}-\boldsymbol{\eta}(\mathbf{x},t)\ ,\qquad t^0=t-\frac{1}{c^2}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x} \tag{51}$$

(in the case of a rigid motion these are the coordinates of the interferometer's RF). Let us denote by $H^{\rm NI}$ and $\Phi^{\rm NI}$, respectively, the Hamiltonian and the phase in the new coordinates ("noninertial frame" for brevity), and H^I,Φ^I those in the inertial frame: $H^I\equiv H,\Phi^I\equiv\Phi$ [Eq. (31)]. From Eq. (36) with $\xi(\mathbf{x},t)=\boldsymbol{\eta}(\mathbf{x},t)$ and $\tau(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}/c^2$ we get

$$H^{\rm NI}(\mathbf{k},\mathbf{x},t)=H^I(\mathbf{k},\mathbf{x},t)+\dot G(\mathbf{k},\mathbf{x},t)\ , \tag{52}$$

where

$$G(\mathbf{k},\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}\frac{H_0}{c^2}-\boldsymbol{\eta}(\mathbf{x},t)\cdot\mathbf{k} \tag{53}$$

and $\dot G$ is the total time derivative of G. From Eqs. (42) and (53) we get

$$H^{\rm NI}(\mathbf{k},\mathbf{x},t)=H_0(\mathbf{k},\mathbf{x})+H_2(\mathbf{k},\mathbf{x},t)\ ,$$
$$H_2=H_1+\dot G=\left[\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{x}+\frac{\partial u_i}{\partial x_j}x_iv_{gj}\right]\frac{H_0}{c^2}-\frac{1}{2}\left[\frac{\partial\eta_i}{\partial x_j}+\frac{\partial\eta_j}{\partial x_i}\right]k_iv_{gj}-(\nabla\cdot\boldsymbol{\eta})V_0\ . \tag{54}$$

The presence of a non-null perturbation H_2 may be viewed, apart for the last term, as due to the fictitious forces appearing in the interferometer's noninertial frame.

We shall base on Eq. (54) our discussion of the differences between experiments with stationary source and detector (and moving interferometer) and those with both pieces comoving with the interferometer (or parts of it); in all cases, since the result of a given interference experiment cannot depend on the coordinates used for its description, we can use H_2 as the perturbation instead of H_1, provided of course that $\boldsymbol{\eta}$ and $\mathbf{u}$ are appropriately defined not only on the moving parts of the interferometer, but also on the source and detector.

In an interference experiment what is measured is the phase difference of the two waves reaching point $\mathbf{x}$, the detector, past the interferometer, at time t. If we denote, respectively, with $\Delta\Phi^0$ and $\Delta\Phi$ the phase difference when the apparatus is at rest (in the "inertial frame") and when it is moving, from Eq. (15) we get

$$\Delta\Phi(\mathbf{x},t)=\Delta\Phi^0(\mathbf{x},t)-\int_{\gamma_1-\gamma_2}H_2(\mathbf{k},\mathbf{x},t)dt\ , \tag{55}$$

where, as discussed before, γ_1 and γ_2 are the unperturbed trajectories starting at times t_{01},t_{02} from the appropriate points $\mathbf{x}_{01},\mathbf{x}_{02}$ on the reference surface Σ_0 in such a way that they both reach the desired point $\mathbf{x}$ at time t.

Consider now two experiments in which the motion of the interferometer is the same whereas source and detector, though initially occupying the same positions in both, are at rest (in the inertial frame) in the first and are moving in the second. Denote by $\Phi^1(\mathbf{x},t)$ and $\Phi^2(\mathbf{x},t)$ the phases in the two cases; in both experiments we take for Σ_0 surfaces equally located at (or near) the sources, so that on them $\Phi^1(\mathbf{x},t)=\Phi^0(\mathbf{x},t)$ and, because of the principle of relativity, $\Phi^2(\mathbf{x},t)=\Phi^0(\mathbf{x},t)$.[23] The phase differences $\Delta\Phi^2$ and $\Delta\Phi^1$ are always given by Eq. (55), with the proper choice of H_2 given by Eq. (54). By assumption H_1 is the same in both cases, whereas in the first experiment $\boldsymbol{\eta}=0$, $\mathbf{u}=0$ at the source and detector positions, so that in the difference only the G term is left

$$\Delta\Phi^2(\mathbf{x},t)-\Delta\Phi^1(\mathbf{x},t)=G(\mathbf{k}_2,\mathbf{x},t)-G(\mathbf{k}_1,\mathbf{x},t)+G(\mathbf{k}_{01},\mathbf{x}_{01},t_{01})-G(\mathbf{k}_{02},\mathbf{x}_{02},t_{02})\ . \tag{56}$$

A. Well-aligned and well-calibrated interferometers

Assume now that the two (unperturbed) beams leave the interferometer with the same $\mathbf{k}$: then $\mathbf{k}_1(\mathbf{x},t)=\mathbf{k}_2(\mathbf{x},t)$, so that $\Delta\Phi^0$ is a constant in space, and for stationary solutions in time as well (we shall refer henceforth to this situation as to a "well aligned" interferometer). Then Eq. (56) simplifies to

$$\Delta\Phi^2(\mathbf{x},t)-\Delta\Phi^1(\mathbf{x},t)=G(\mathbf{k}_{01},\mathbf{x}_{01},t_{01})-G(\mathbf{k}_{02},\mathbf{x}_{02},t_{02}) \tag{57}$$

and in this case only the motion of the source is relevant.

A sufficient condition for the vanishing of the right-hand side (rhs) of Eq. (57) is $\mathbf{x}_{01}=\mathbf{x}_{02},t_{01}=t_{02}$. We shall now show that this is equivalent to the experimental requirement that the interferometer is "well calibrated:" by this we mean that the phase difference $\Delta\Phi^0$ (for the interferometer at rest) is insensitive to small variations of the orientation and position of the source, as well as of its frequency.

The effect of a small variation in the position and orientation of the source is described by $H_2=\dot G$ [Eq. (54) with $H_1=0$] with $\mathbf{u}=0$ and $\boldsymbol{\eta}=\delta\boldsymbol{\phi}\times\mathbf{x}$ at the source, where $\mathbf{x}$ is from an arbitrary origin. Then from Eq. (55) we get

$$\frac{\partial\Delta\Phi^0}{\partial\delta\boldsymbol{\phi}}=\mathbf{x}_{01}\times\mathbf{k}_{01}-\mathbf{x}_{02}\times\mathbf{k}_{02}\ . \tag{58}$$

The requirement that the rhs of Eq. (58) vanishes entails $\mathbf{k}_{01}=\mathbf{k}_{02}$ (from the arbitrariness of the origin) and $\mathbf{x}_{01}=\mathbf{x}_{02}$.[24] Next, if $\Phi^0(\mathbf{x},t)$ is a stationary solution with frequency ω [Eq. (22)] from Eq. (4) and $\mathbf{x}_{01}=\mathbf{x}_{02}$ we have

$$\Delta\Phi^0=\int_{\gamma_1-\gamma_2}\mathbf{k}\cdot d\mathbf{x}\ . \tag{59}$$

Since in Eq. (59) γ_1,γ_2 are arbitrary, we are allowed to keep them fixed while varying ω and we get

$$\delta(\Delta\Phi^0)=\int_{\gamma_1-\gamma_2}\delta\mathbf{k}\cdot d\mathbf{x}=\int_{\gamma_1-\gamma_2}\delta\mathbf{k}\cdot\mathbf{v}_g dt=\int_{\gamma_1-\gamma_2}\delta\omega\, dt=\delta\omega(t_{02}-t_{01})\ . \tag{60}$$

We have thus found that for a well-calibrated interferometer $\gamma_1-\gamma_2$ is a closed loop: this will be of use in the following. Furthermore, if the interferometer is well aligned too, the rhs of Eq. (56) vanishes, meaning that the motions of the source and detector have no effect on the phase shift.

If this is the case, another important consequence occurs: the interferometer will be sensitive only to accelerations, since when $\mathbf{u}$ is constant $H_2=0$ (and $H_1=-\dot{G}$). This result does not depend on relativistic invariance, as source and detector need not be comoving; a phase shift would appear also in case of uniform motion if the interferometer were not well calibrated.[25] Note at last that all these properties only pertain to the first-order perturbative approximation and would not hold if second-order effects were taken into account.[11]

VI. APPLICATIONS

Since we start considering the case in which source and detector are at rest, we can use Eq. (55) with H_2 replaced by H_1 given by Eq. (42). Then we get

$$\delta\Phi(\mathbf{x},t)=\Delta\Phi(\mathbf{x},t)-\Delta\Phi^0(\mathbf{x},t)$$
$$=-\int_{\gamma_1-\gamma_2}\mathbf{u}(\mathbf{x},t)\cdot\left[\mathbf{k}-\mathbf{v}_g\frac{H_0}{c^2}\right]dt+\int_{\gamma_1-\gamma_2}\boldsymbol{\eta}(\mathbf{x},t)\cdot\frac{\partial H_0}{\partial\mathbf{x}}dt+\int_{\gamma_1-\gamma_2}\mathbf{v}_g\cdot\mathcal{A}\cdot\mathbf{k}\,dt+\int_{\gamma_1-\gamma_2}[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0dt\ . \tag{61}$$

If the medium is isotropic ($\mathbf{v}_g$ and $\mathbf{k}$ parallel) and $\boldsymbol{\eta}$ is the displacement field of an incompressible fluid ($\nabla\cdot\boldsymbol{\eta}=0$) Eq. (61) for stationary solutions of frequency ω takes the form

$$\delta\Phi(\mathbf{x},t)=\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\left[1-\frac{c^2}{v_f v_g}\right]\mathbf{u}(\mathbf{x},t)\cdot d\mathbf{x}-\int_{\gamma_1-\gamma_2}\boldsymbol{\eta}(\mathbf{x},t)\cdot d\mathbf{k}\ , \tag{62}$$

where $v_f=\omega/k$ (or $mc^2/\hbar k$ in the nonrelativistic case) is the phase velocity in the medium.

Equation (61) can be given a different form by transforming the second integral. We have

$$\int_{\gamma_1-\gamma_2}\boldsymbol{\eta}\cdot\frac{\partial H_0}{\partial\mathbf{x}}dt=-\int_{\gamma_1-\gamma_2}\boldsymbol{\eta}\cdot\dot{\mathbf{k}}\,dt=\delta(\boldsymbol{\eta}\cdot\mathbf{k})+\int_{\gamma_1-\gamma_2}\dot{\boldsymbol{\eta}}\cdot\mathbf{k}\,dt\ , \tag{63}$$

where

$$\delta(\boldsymbol{\eta}\cdot\mathbf{k})=\boldsymbol{\eta}(\mathbf{x},t)\cdot(\mathbf{k}_2-\mathbf{k}_1)-[\boldsymbol{\eta}(\mathbf{x}_{02},t_{02})\cdot\mathbf{k}_{02}-\boldsymbol{\eta}(\mathbf{x}_{01},t_{01})\cdot\mathbf{k}_{01}]\ . \tag{64}$$

Then

$$\delta\Phi(\mathbf{x},t)=\delta(\boldsymbol{\eta}\cdot\mathbf{k})+\int_{\gamma_1-\gamma_2}\frac{H_0}{c^2}\mathbf{u}\cdot d\mathbf{x}+\int_{\gamma_1-\gamma_2}(\dot{\boldsymbol{\eta}}-\mathbf{u})\cdot\mathbf{k}\,dt+\int_{\gamma_1-\gamma_2}\mathbf{v}_g\cdot\mathcal{A}\cdot\mathbf{k}\,dt+\int_{\gamma_1-\gamma_2}[\nabla\cdot\boldsymbol{\eta}(\mathbf{x},t)]V_0dt\,. \tag{65}$$

In the applications one is often concerned with situations in which different parts of the interferometer move independently: in these cases $\boldsymbol{\eta}(\mathbf{x},t)$ can be a discontinuous function of the spatial coordinates at the boundaries and, in order to apply Eqs. (61) or (65), one must specify how to treat these discontinuities. The problem is a physical one, since including or excluding the boundaries from the domain of the displacement field corresponds to two physically different situations leading to different results: a classical instance (which we shall discuss later on in this section) are the experiments of Fizeau,[26] where a fluid is moving between fixed boundaries, and of Zeeman,[27] where the boundaries are part of the moving medium.

The mathematical ambiguities arising from a discontinuous $\boldsymbol{\eta}(\mathbf{x},t)$ must therefore be avoided by resorting to a regularization procedure suited to the physical situation considered. Thus, for instance, if different parts of the interferometer—including their boundaries—are in relative motion, we shall assume that these parts are separated by a gap, so that $\boldsymbol{\eta}(\mathbf{x},t)$, which is arbitrary in the vacuum, can be made continuous by extending it within the gap. As a consequence, if the medium consists of rigid parts and possibly contains incompressible fluids the last term in Eqs. (61) and (65) is zero. This is the case we shall consider in the following applications.

In almost all practical situations the gap between the moving parts is actually present and is sufficiently large compared to the wavelength, so that geometrical optics retains its validity. If this is not the case, as for instance in the idealized situation of contiguous moving parts (e.g., a solid in a fluid), we shall regularize $\boldsymbol{\eta}(\mathbf{x},t)$ by assuming the existence of a gap small with respect to the wavelength.[28]

We are now going to apply Eqs. (61) and (65) to the various situations of physical interest.

A. Moving interferometer with a stationary medium

In this case $\mathbf{u}$ and $\boldsymbol{\eta}$ are different from zero only on the reflecting media. The first integral in Eq. (61) gives a vanishing contribution since the integrand remains finite in the infinitesimal time interval where it is not zero, and the last two integrals are absent thanks to the regularization procedure: therefore only the second integral contributes, due to the finite change of $\mathbf{k}$ in the reflection. Thus we have

$$\delta\Phi(\mathbf{x},t)=\sum_{\gamma_2}\boldsymbol{\eta}(\mathbf{x}_{i2},t_{i2})\cdot\Delta\mathbf{k}_{i2}-\sum_{\gamma_1}\boldsymbol{\eta}(\mathbf{x}_{i1},t_{i1})\cdot\Delta\mathbf{k}_{i1}\ , \tag{66}$$

where $\mathbf{x}_{i1,2}$ are the points where the trajectories γ_1 and γ_2 meet the mirrors—at times $t_{i1,2}$—and $\Delta\mathbf{k}_i$ is the variation of $\mathbf{k}$ in the reflection by the ith mirror (at rest).

Equation (66) has been used in Ref. 11 for a uniformly accelerated and for a rotating interferometer.

The properties of the medium enter Eq. (66) only implicitly—mainly through the times $t_{i1,2}$ which are related to the propagation time. In special cases, however, Eq. (66) can be given a different form: consider, for instance, a rigidly moving, well-aligned and calibrated interferometer, filled with a stationary, homogeneous and isotropic (dispersive) medium. Since

$$\delta\Phi(\mathbf{x},t)=\int_{\gamma_1-\gamma_2}\boldsymbol{\eta}(\mathbf{x},t)\cdot\frac{\partial H_0}{\partial \mathbf{x}}dt \tag{67}$$

and $\partial H_0/\partial\mathbf{x}=0$ outside the mirrors[29] we are allowed to arbitrarily extend $\boldsymbol{\eta}$ everywhere.

Take for $\boldsymbol{\eta}$ the expression of a rigid motion, i.e.,

$$\boldsymbol{\eta}(\mathbf{x},t)=\mathbf{x}-\mathbf{x}^0=\mathbf{x}+\boldsymbol{\xi}(t)-\mathcal{R}^{-1}(t)\cdot\mathbf{x}=\boldsymbol{\xi}(t)+\mathcal{A}(t)\cdot\mathbf{x}\ , \tag{68}$$

where $\mathcal{R}(t)$ and $\mathcal{A}(t)$ are related as in Eq. (45). From Eq. (67), after partial integration, we get

$$\delta\Phi=\int_{\gamma_1-\gamma_2}\mathbf{k}\cdot\dot{\boldsymbol{\eta}}(\mathbf{x},t)dt\ . \tag{69}$$

Now

$$\dot{\boldsymbol{\eta}}=\dot{\boldsymbol{\xi}}(t)+\dot{\mathcal{A}}(t)\cdot\mathbf{x}+\mathcal{A}\cdot\dot{\mathbf{x}} \tag{70}$$

and for an isotropic medium $\mathbf{v}_g$ and $\mathbf{k}$ are parallel, so that

$$\dot{\boldsymbol{\eta}}\cdot\mathbf{k}=\mathbf{u}\cdot\mathbf{k}=\frac{k}{v_g}\mathbf{u}\cdot\mathbf{v}_g\ , \tag{71}$$

where

$$\mathbf{u}(\mathbf{x},t)=\dot{\boldsymbol{\xi}}(t)+\dot{\mathcal{A}}(t)\cdot\mathbf{x}\ . \tag{72}$$

Thus we arrive to

$$\delta\Phi=\int_{\gamma_1-\gamma_2}\mathbf{k}\cdot\mathbf{u}\,dt=\int_{\gamma_1-\gamma_2}\frac{k}{v_g}\mathbf{u}\cdot d\mathbf{x}\ . \tag{73}$$

For a nondispersive medium Eq. (73) reduces to the corresponding equation of Ref. 1.

From the above derivation it follows that Eq. (73) remains true also if the homogeneous medium does not completely fill the interferometer, provided its boundaries are tangent to the displacement field $\boldsymbol{\eta}$ defined inside the interferometer by Eq. (68).

B. Moving interferometer with comoving medium

In this case the interferometer and the medium are rigidly moving, and we take for $\boldsymbol{\eta}$—also in the regions not filled with the refracting medium—the expression given by Eq. (68). The last integral in Eq. (65) is absent, the second and third one give no contribution because of Eq. (70) and we are left with

$$\delta\Phi(\mathbf{x},t)=\delta(\boldsymbol{\eta}\cdot\mathbf{k})+\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\mathbf{u}\cdot d\mathbf{x}\ , \tag{74}$$

where $\mathbf{u}(\mathbf{x},t)$ is given by Eq. (72).

If $\delta(\boldsymbol{\eta}\cdot\mathbf{k})=0$ and $\mathbf{u}$ has no explicit t dependence i.e., in the case of a uniform rotation, we get from Eq. (74) the known result that the fringe shift does not depend on the properties of the comoving medium.

Particularly relevant is the use of the interferometer as a rotation-rate sensor, e.g., in inertial navigation systems. For this application the apparatus should be, as much as possible, sensitive only to the rotational motion: for this to be the case, it must in the first place be well aligned and calibrated. Then Eq. (74) reduces to

$$\begin{aligned}\delta\Phi(\mathbf{x},t)&=\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\mathbf{u}\cdot d\mathbf{x}\\&=\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\dot{\boldsymbol{\xi}}\cdot\dot{\mathbf{x}}\,dt-\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\mathbf{x}\cdot\dot{\mathcal{A}}(t)\cdot d\mathbf{x}\\&=-\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\ddot{\boldsymbol{\xi}}\cdot\mathbf{x}\,dt-\frac{\omega}{c^2}\int_{\gamma_1-\gamma_2}\mathbf{x}\cdot\dot{\mathcal{A}}(t)\cdot d\mathbf{x}\ .\end{aligned} \tag{75}$$

In order that translational accelerations do not cause unwanted phase shifts, the first integral should give a vanishing or negligible contribution. This actually happens for a constant $\ddot{\boldsymbol{\xi}}$ if the two beam paths γ_1 and γ_2 are the same loop traversed in opposite directions,[1] so that the only contribution comes from the time variation of the translational acceleration. In this case, denoting by $\mathbf{A}$ the area enclosed by the loop and by $\boldsymbol{\Omega}$ the angular velocity we get

$$\delta\Phi(t)=4\frac{\omega}{c^2}\boldsymbol{\Omega}(t)\cdot\mathbf{A} \tag{76}$$

leaving aside terms of the order $\dot{\boldsymbol{\Omega}}\tau$ and $\dddot{\boldsymbol{\xi}}\tau$, where $\tau=t-t_0$ is the traversal time. In all practical cases these terms are safely negligible.

Equation (75) differs from Eq. (73) by the substitution of the factor k/v_g—which depends on the properties of the medium—with ω/c^2. For nonrelativistic matter waves ω/c^2 in Eqs. (74)–(76) is replaced by $m/\hbar$ and there is no difference with the case of a stationary medium unless $\mathbf{v}_g\neq\mathbf{k}/m$, i.e., unless the potential in the medium is velocity dependent [Eq. (49)].

C. Moving medium inside a stationary interferometer

There are two physically different situations which historically go back to the optical experiments of Fizeau[26] and Zeeman:[27] the former is realized by a fluid moving within fixed boundaries, or by a homogeneous medium whose boundaries move tangentially (e.g., a rotating disk); in the latter instead the boundaries are part of the moving medium. For matter waves both cases have been the subject of recent experiments with neutrons.[8,30] A detailed theoretical discussion of these experiments has been given by Horne *et al.*[9] and Bonse and Rumpf:[8] here we take up the discussion again both for completeness and because our Hamiltonian approach is formally different and, perhaps, simpler.

The only difference between the two cases comes from the second integral in Eq. (61), which is absent if all the discontinuities—the boundaries and the mirrors—are stationary or tangential to the displacement field $\boldsymbol{\eta}$. In this case the entire effect is given by the difference between the canonical momentum $\mathbf{k}$ and the kinetical momentum $\mathbf{v}_g H_0/c^2$ ($m\mathbf{v}_g$ in the nonrelativistic case): clearly this difference in not zero if the (stationary) medium is described by a velocity-dependent potential,[30] otherwise the cancellation we referred to in the Introduction results.

In order to explicitly show where the difference between the two cases comes from, consider a plane slab of an isotropic refracting medium, orthogonal to the wave vector $\mathbf{k}$ and moving with constant velocity $\mathbf{u}$ parallel to $\mathbf{k}$. The expression for the displacement field is $\boldsymbol{\eta}=\mathbf{u}t$ in the medium—including or excluding the boundaries according to whether (a) they are moving with the medium of (b) they are stationary—and $\boldsymbol{\eta}=0$ elsewhere. In both cases, wherever $\boldsymbol{\eta}$ is independent of the coordinates H_1 is the time derivative of $-G$ [Eq. (53)] and from Eq. (15) we get for the phase shift due to the motion of the medium

$$\Phi-\Phi^0=uL\left[\frac{\omega}{c^2}-\frac{k}{v_g}\right], \tag{77}$$

where L is the slab thickness, v_g is the group velocity in the medium (L/v_g is the traversal time), and k is the wave vector just outside the boundaries—i.e., in the vacuum—in case (a), inside the medium in case (b).

From Eq. (77) the known expressions in terms of the refractive index for light waves,[1] or in terms of the potential for matter waves[8,9] (ω/c^2 replaced by $m/\hbar$), can be easily derived. For light waves

$$\begin{aligned}&\text{(a)}\ \Phi-\Phi^0=uL\frac{\omega}{c^2}\left[1-n-\omega\frac{dn}{d\omega}\right],\\ &\text{(b)}\ \Phi-\Phi^0=uL\frac{\omega}{c^2}\left[1-n^2-n\omega\frac{dn}{d\omega}\right].\end{aligned} \tag{78}$$

For matter waves (to the first order in V):

$$\begin{aligned}&\text{(a)}\ \Phi-\Phi^0=uL\frac{m}{\hbar}\left[\frac{\partial V}{\partial E}-\frac{V}{2E}\right],\\ &\text{(b)}\ \Phi-\Phi^0=uL\frac{m}{\hbar}\frac{\partial V}{\partial E}.\end{aligned} \tag{79}$$

Taking the difference between expressions (a) and (b) we get the phase shift for an experiment with moving boundaries and stationary medium, which would not be zero because of the Doppler effect in the refraction from the moving boundaries.

In the case of stationary boundaries the expression of the phase shift for an interferometer in arbitrary motion is obtained from Eq. (61) simply by dropping the second integral; assuming $\mathbf{v}_g$ parallel to $\mathbf{k}$ we get

$$\delta\Phi=\int_{\gamma_1-\gamma_2}\left[\frac{\omega}{c^2}-\frac{k}{v_g}\right]\mathbf{u}(\mathbf{x},t)\cdot d\mathbf{x}, \tag{80}$$

which is the difference between the rhs of Eqs. (74) and (73), as expected. On the other hand, for a rigid motion of the medium together with its boundaries, $\delta\Phi$ cannot be given a general simple expression, but is always the difference between Eqs. (74) and (66), where in the latter the exact geometry of the interferometer is to be taken into account.

D. The ring laser

As a last application we show how the formalism we used so far can also be employed to derive the basic equations of the active ring-laser gyro. The active ring-laser gyro is essentially a ring resonator with an active laser medium in it; two eigenmodes, corresponding to the propagation of light in opposite directions along the same loop γ_0, are degenerate with frequency ω_0 when the ring laser is at rest in an inertial frame. When the device is subjected to accelerations, in the noninertial frame of the device the degeneracy is lifted by H_2 [Eq. (54)], and what is measured is the beat frequency $\omega_{\text{beat}}=\omega_+-\omega_-$ between the two counterpropagating beams.

Since in the noninertial frame we are dealing with stationary solutions, Eq. (8) takes the form

$$\Phi_\pm(\mathbf{x},t)=\vartheta_{0\pm}(\mathbf{x})-\omega_0 t+\vartheta_{1\pm}(\mathbf{x})-\delta\omega_\pm t \tag{81}$$

and from Eq. (9), with H_2 as the perturbation, we get

$$\delta\omega_\pm=\mathbf{v}_g(\mathbf{x})\cdot\nabla\vartheta_{1\pm}(\mathbf{x})+H_2(\nabla\vartheta_0,\mathbf{x},t). \tag{82}$$

Integrating both members of Eq. (82) along the unperturbed trajectory γ_0 and taking into account that by assumption γ_0 forms a loop, we get

$$\tau\delta\omega_\pm=\pm\int_{\gamma_0}H_2(\nabla\vartheta_0,\mathbf{x},t)dt, \tag{83}$$

where $\tau=\int_{\gamma_0}ds/v_g$ is the traversal time. Then

$$\omega_{\text{beat}}=\delta\Phi/\tau, \tag{84}$$

where $\delta\Phi$ can be obtained, for the various experimental situations, from the already derived equations by dropping the boundary terms $\delta(\boldsymbol{\eta}\cdot\mathbf{k})$.

The description from the point of view of the inertial frame, i.e., in terms of H_1, is quite different even if leading to the same result. Taking for instance a circular resonator rotating about its center and with no refracting medium in it, inside the cavity the frequencies of the two counterpropagating beams are still degenerate at the value ω_0, i.e., are unaffected by the motion: in fact, since the boundaries are moving tangentially there is no Doppler effect (in our formalism H_1 is zero). It is only after the two beams are brought to propagate in the same direction towards the detector by reflection on comoving mirrors, that their Doppler-shifted frequencies will differ by the amount given by Eq. (84).

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[17]The generalization to relativisitic waves is possible, although not straightforward since in this case the way the potential enters the Hamiltonian depends on its transformation properties (scalar, vector, etc.)

[18]The transition amplitude is given by the matrix elements of $T(\mathbf{k},\mathbf{x},s)$ between plane waves.

[19]Since we are ignoring polarization, we need only a scalar refractive index (rather than a tensor).

[20]Notice that according to the adopted definition n is exclusively a property of the medium, i.e., $n=1$ for the vacuum. The alternative definition $H=kc/n$ is not the natural one for matter waves and generally not adopted.

[21]The exact form of the term multiplying $\nabla\cdot\boldsymbol{\eta}$ in Eq. (41) depends on the way the potential enters H_0.

[22]The appearance of $\mathbf{v}_g$ in Eq. (42) does not imply that our treatment only holds for wave packets of small size. Actually our derivation is independent of the shape of the incoming wave. This point is more extensively discussed in Ref. 11.

[23]Since H_2 does not vanish outside the interferometer Φ^2 will not equal Φ^0 on the arbitrary surface, but only at the source.

[24]The occurence that $\mathbf{x}_{01}-\mathbf{x}_{02}$ is parallel to $\mathbf{k}$ can be ruled out by the fact that here Σ_0 is arbitrary: once it is proved that $\mathbf{x}_{01}=\mathbf{x}_{02}$ on a suitable Σ_0 this will be true for any other reference surface.

[25]It is not always desirable to work with a well-calibrated interferometer: for instance, in the famous Kennedy-Thorndike experiment the effect of the supposed motion with respect to the ether would be proportional to $t_{01}-t_{02}$.

[26]A. H. L. Fizeau, C. R. Acad. Sci. B **33**, 349(1851); Ann. Chem. Phys. **57**, 385 (1859).

[27]P. Zeeman, Amst. Versl. **23**, 245(1914); **24**, 18(1915).

[28]In this case, however, geometrical optics might fail to give the correct result, and a wave-theoretical treatment in the region of the gap could be necessary. An instance is given later (Ref. 29).

[29]In the case of a small-gap regularization procedure, $\Delta\mathbf{k}$ in Eq. (66) is not correctly given by geometrical optics and is easily shown to coincide with the variation of $\mathbf{k}$ in the medium (rather than in the gap). Therefore, formally the result is the same as if $\boldsymbol{\eta}$ had been extended to include the boundary of the medium contiguous to each mirror.

[30]M. Arif, M. Kaiser, R. Clothier, S. A. Werner, R. Berliner, W. A. Hamilton, A. Cimmino, and A. G. Klein, Physica **151B**, 63 (1988); M. Arif, M. Kaiser, R. Clothier, S. A. Werner, W. A. Hamilton, A. Cimmino, and A. G. Klein (to be published).