Characterization of chaotic systems at transition points through dimension spectra

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We study the behavior of chaotic systems at transition points (intermittency and crisis) through their dimension spectra $f(\alpha)$. In the transition regions the finite-statistics $f(\alpha)$ curves display a characteristic doubly peaked structure whose convergence to the asymptotic concave shape occurs for exceedingly large numbers of points. This slowing-down effect is studied for both the Duffing equation and the Hénon map and is used as a guideline in the interpretation of the spectra of NMR-laser experimental data sets.

Sudden qualitative changes in the dynamics of chaotic systems upon variation of a control parameter ("crises",[1] intermittency[2]) appear in many diverse systems. Such events are determined either by collisions between the chaotic attractor and some unstable periodic orbit[3] or by tangent bifurcations. Just above the crisis point, the trajectory is confined for a long time in a small region of phase space containing the unstable periodic orbits generated by successive bifurcations of the original one (this structure mimics the presence of a "small" strange attractor for long times). The remaining part of the dynamics consists of a spatially extended "chaotic transient." Similarly, just below the intermittency threshold, a long laminar (regular) phase alternates with a chaotic burst. The average lifetime of the "transient" states has been studied as a function of the control parameter.[3–5] Here, we investigate the dimension spectrum $f(\alpha)$ (Ref. 6) of systems in the vicinity of such transitions, showing that the convergence of the computed $f(\alpha)$ curves to their asymptotic shape (infinite number of points n) displays a characteristic critical slowing down. This phenomenon is investigated for the NMR laser.[7]

In order to introduce the dimension spectrum $f(\alpha)$, we choose, at random from the natural measure $\mu(\mathbf{y})$ (Ref. 8) on the attractor, a set of m reference points $\mathbf{y}$ and consider balls of size ϵ and mass $P(\epsilon;\mathbf{y})$ centered on each of them. The *pointwise* dimension $\alpha(\mathbf{y})$ (Ref. 8) is then defined as the limit for $\epsilon\to 0$ of $\alpha(\mathbf{y};\epsilon)=\ln P(\epsilon;\mathbf{y})/\ln\epsilon$. The scaling properties of the invariant measure $\mu(\mathbf{x})$ can be characterized either by means of the dimension function $D(q)$, $-\infty<q<+\infty$, or by introducing the scaling function $f(\alpha)$ which describes the spread of values assumed by the pointwise dimension $\alpha(\mathbf{x})$. As for $D(q)$, $f(\alpha)$ can be defined both for fixed-size and fixed-mass methods:[9] here we recall only the latter definition. Let $\hat{P}(\alpha;p)$ be the probability density for a ball of mass p, centered on $\mathbf{y}$, to yield a pointwise dimension $\alpha(\mathbf{y})$ in the interval $[\alpha,\alpha+d\alpha]$. Then $f(\alpha)$ is implicitly defined, for $p\to 0$, by the asymptotic relation

$$\hat{P}(\alpha;p)\sim p^{1-f(\alpha)/\alpha} . \tag{1}$$

The function $f(\alpha)$ is concave and is tangent to the curve $f(\alpha)=\alpha$ at $\alpha=D(1)$. Given a number n of data points, we consider m balls containing a mass p which can be approximated by the fraction k/n ($k,n\to\infty$, $k/n\ll 1$), where k is the number of points in the ball. Therefore, Eq. (1) can be rewritten as[10]

$$f(\alpha)\sim\alpha\left[1-\frac{\ln P(\delta,n,k)}{\ln(k/n)}\right]-1 , \tag{2}$$

where $P(\delta,n,k)$ is the probability density for the size of a ball containing a mass $p=k/n$ to lie in the interval $[\delta,\delta+d\delta]$. The quantity $P(\delta,n,k)$ can be measured numerically, thus obtaining a histogram, on which δ is the independent variable and n and k are parameters. The limit curve is then approximated by plotting $f(\alpha;n,k)$ for large n and various choices of k, keeping k/n small. The normalization of the $f(\alpha)$ curves [which are, in general, vertically and horizontally displaced from their true position, due to the presence of unknown prefactors in the power laws defining α and $f(\alpha)$] is obtained by shifting the curves until they are tangent to the bisector $f(\alpha)=\alpha$ at $\alpha=D(1)$. Thanks to the statistical properties of the method, the large-α tail of $f(\alpha)$ and high-dimensional sets (e.g., embedded experimental signals) can also be analyzed: the convergence criteria for the algorithm are given in Ref. 10, by studying systems of known dimension spectrum. In the study of an experimental signal, a suitable embedding dimension E is selected on the grounds of the convergence of $D(1)$. The m reference points $\mathbf{y}_j$ and the n data points $\mathbf{x}_i$ in the embedding space are then chosen from a random permutation of the original sequence, in order to eliminate possible correlation effects between consecutive iterates.

We applied our method to the study of the Poincaré section of the Duffing equation $\ddot{x}+0.154\dot{x}-x+4x^3=A\cos\omega t$, with frequency $\omega=1.17$. The integration was performed using 1000 Runge-Kutta steps per period of the external modulation. We first considered amplitudes A of the external forcing slightly larger than the value $A=A_c\approx 0.100476$ at which a period-1 attractor undergoes a crisis. In this situation, the dynamics of the map restricted to the "chaotic period" generates essentially the same spectrum of local dimensions as the logistic map at crisis, i.e., the triangular distribution $f(\alpha)=2\alpha-1$ for $0.5\le\alpha\le 1$, and $f(\alpha)=-\infty$ elsewhere.[9] The iterations in the transient, instead, contribute a component to $f(\alpha)$ which resembles the spectrum corresponding to $A \gg A_c$. A further component, with apparent dimension close to that of phase space, is generated by the re-entries from the transient into the basin of the period since, for finite n, it is not possible to resolve the Cantor structure in small regions around the chaotic period. Its contribution to $f(\alpha)$ depends on the number of iterations necessary to shrink the immediate basin of attraction onto the nearly one-dimensional chaotic period. In Fig. 1 we show the results of a numerical investigation, performed with $m=56\,000$ reference points, $n=925\,139$ data points, nn-order $k=100$, and constructing the histograms over 64 bins. The curves in the figure were obtained for $A=0.1005>A_c$ (dotted line), $A=0.103$ (dashed line), and $A=0.1$ (solid line). The finite-n curves, when A is close to A_c, present a doubly peaked structure, with a spurious tail at large α values. This phenomenology can be explained in the following way. Just above the crisis point very few iterations belong to the transient, and nn distances $\delta(n)$ from reference points in that region are affected by large statistical fluctuations. This gives rise to spuriously large dimensional values (i.e., the decrease rate of the δ's may be smaller than the asymptotic one). Notice that the randomization of the indices of the original data string renders immaterial the order of appearance of both reference and data points. As n is increased, the spurious tail persists over many decades in n, becoming gradually less pronounced. Eventually, a strictly concave structure is recovered and the $f(\alpha)$ curves attain a shape similar to that measured at larger A (solid line in the figure). The line segment visible at left of the tangency point in the solid-line curve reveals the existence of a "true transition," in the framework of the thermodynamical formalism [discontinuity in some derivative of the $f(\alpha)$ spectrum][11,9,12] as expected for a nonhyperbolic system. The segment does not reach the α axis because of a lack of statistics (few reference points with small pointwise dimension α). Measurements performed on the Hénon map showed the presence of a straight-line behavior, with slope in good accordance with the theoretical prediction. The segment, which is already present for a moderately small number of data points, extends gradually with an increasing number of reference points, eventually reaching the horizontal axis.[10] Notice that the "phase-transition," in the case of intermittency, is of infinite order.[13] The minimum number of data points n_0 needed to observe a concave $f(\alpha)$ increases as the control parameter approaches the critical value A_c. Our results agree with the conjecture that $n_0\sim(A-A_c)^{-\gamma}$, where γ is the critical exponent of the transition.[3] Notice that the Lyapunov exponents follow an analogous power-law behavior.[14] Therefore, also the pointwise dimensions [and, consequently, the $f(\alpha)$ curve] should vary smoothly across the transition point.[15] The observed slowing down is not an artifact of the method, but rather an intrinsic feature of transitions between different chaotic states. In fact, the same behavior is found by using fixed-size algorithms which, in addition, exhibit bad convergence properties for large α, independently of the presence of a transition.[10] The same qualitative behavior above discussed for the Duffing system is displayed by the Hénon map with parameter values $a=1.755\,18$ and $b=-0.015$, just above the crisis of a period-3 attractor, which occurs at $a=a_c\approx 1.75\,516$.[1] The numerical analysis (with $m=320\,000$ reference points, nn-order $k=100$ and 64 bins) yields, for small n, a doubly peaked curve which, for $n\approx 300\,000$, gives place to a strictly concave spectrum.

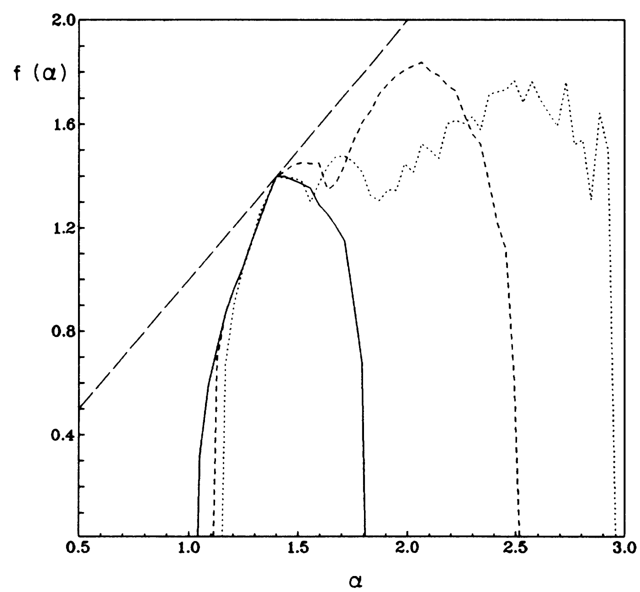

FIG. 1. Dimension spectrum $f(\alpha)$ vs α of the Poincaré map of the forced Duffing system for $A=0.1005$ (dotted line), $A=0.103$ (dashed line), and $A=0.1$ (solid line). In the computation we used $m=56\,000$ reference points, $n=925\,139$ data points, nn-order $k=100$, and we constructed the histograms over 64 bins. The bisector $f(\alpha)=\alpha$ is also indicated (long dashes).

The intermittency transition produces similar effects. We have studied the Duffing equation numerically with $A=0.127\,505\,3$ and $\omega=1.17$, close to the birth of a period-5 attractor by tangent bifurcation, where intermittent behavior is observed. The main difference is that the left tail now extends to $\alpha=0$ and disappears at small n values than for crisis events.

The critical slowing down in the convergence of the $f(\alpha)$ curves to their asymptotic shape should be carefully taken into account in the interpretation of the calculated dimension spectra of experimental signals. The appearance of the above-described phenomena is an indication that the system is close to a transition which, in the case of crises, may not be easily detectable by inspecting two-dimensional projections from the embedding space. On the other hand, if the spectrum $f(\alpha)$ is determined as the Legendre transform of $D(q)$, the two peaks appear to be connected by a line segment. This shape could be erroneously interpreted as due to a "phase transition" in the sense of the above-mentioned thermodynamic formalism for dynamical systems. Of course, spuriously large values of the pointwise dimension α also lead to overestimates of the dimension function for small q. Convergence to the correct $D(q)$ is obtained only for roughly the same amount of data as for $f(\alpha)$.

We have applied the above results to the interpretation of the dimension spectra of the NMR laser[7] with externally modulated cavity Q factor (with modulation frequency $f_{\rm mod}=160$ Hz). The output signal, proportional to the transverse magnetization M_v of the spin system, was sampled about five times per period of the external forcing field and recorded with a 12-bit resolution. In Fig. 2, a two-dimensional projection of the output signal $x(t)$ is shown for a modulation amplitude $Q_{\rm mod}=0.27$, corresponding to a slightly chaotic situation. The $f(\alpha)$ spectrum of this attractor is shown in Fig. 3 (dashed-dotted line), as was obtained using embedding dimension $E=9$, $m=8000$ reference points, nn-order $k=10$, 32 bins, and $n=1\,091\,664$ data points. The convergence with k in the range $10<k<100$ and with E in the range $7<E<21$ was proven to be very satisfactory. At $Q_{\rm mod}\approx 0.275$, the attractor undergoes a crisis. This event appears, in the bidimensional projection x_i-x_{i+1}, as a sudden filling of the inner region of the "loop" seen in Fig. 2. The corresponding $f(\alpha)$ spectrum (shown in Fig. 3, dashed line, for $E=9$, $m=160\,000$, 64 bins, $k=50$, and $n=477\,174$) has the same shape discussed in the preceding paragraphs. The spurious high-dimension tail is even more evident in the spectrum of the attractor just above a second crisis event, taking place at $Q_{\rm mod}\approx 0.32$ (dotted line in Fig. 3). The crisis appears, in the bidimensional projection x_i-x_{i+1}, as a sudden slight expansion of the attractor, without noticeable change of its form. Conversely, at increased distance from the crisis ($Q_{\rm mod}=0.358$, solid line), the same amount of data yields a concave spectrum and 32 000 reference points are sufficient to obtain a smooth histogram. The presence of a line segment in the left part of $f(\alpha)$, indicating, as mentioned above, the occurrence of a true "phase-transition," is evidenced here for the first time in an experimental signal.

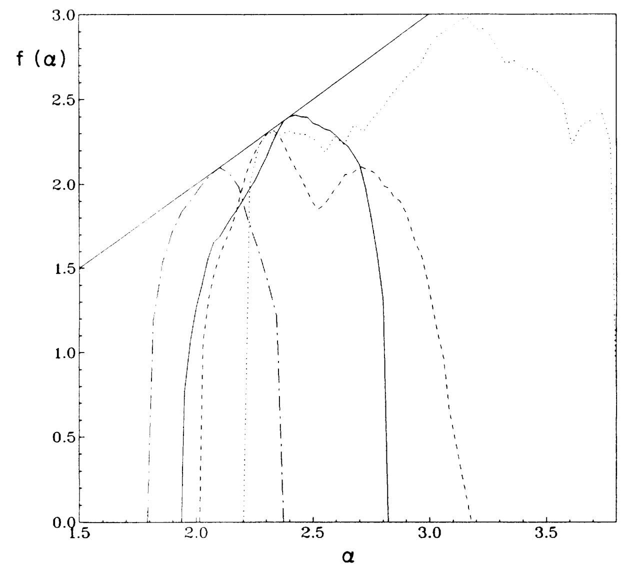

FIG. 3. Dimension spectrum $f(\alpha)$ vs α of the NMR laser attractor of Fig. 2 (dashed-dotted line) and for attractors corresponding to $Q_{\rm mod}=0.275$ (dashed line), $Q_{\rm mod}=0.32$ (dotted line), and $Q_{\rm mod}=0.358$ (solid line). In the computations we used embedding dimension $E=9$.

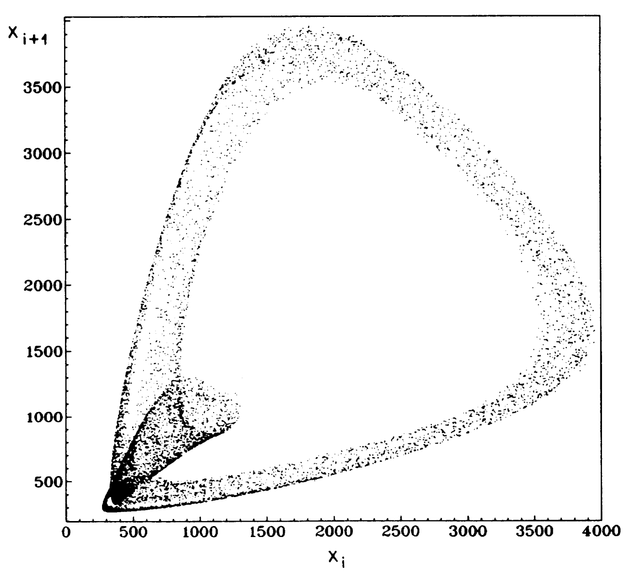

FIG. 2. Two-dimensional projection of the output signal $x(t)$ of the NMR laser, for modulation amplitude $Q_{\rm mod}=0.27$, and frequency $f_{\rm mod}=160$ Hz. The signal was sampled at intervals of 1.25×10^{-3} s. The picture contains 10 000 points.

Recently, a procedure for a hierarchical approximation of the $f(\alpha)$ spectrum has been proposed,[16] based on the localization of all unstable periodic orbits of order n and on the calculation of local Lyapunov dimensions.[16] The method, although appealing, leads in the present case to many difficulties. In fact, close to a transition, the relevant periodic orbits are very long and, therefore, their number is very large. Moreover, they are rather close to one another for a large number of iterates, so that it is not easy to resolve them with good accuracy and the corresponding Lyapunov exponents cannot be computed with meaningful precision. Work is in progress to test applications of both methods to high-dimensional experimental systems.

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