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Replicators with random interactions: A solvable model

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Mean-field methods of statistical physics are applied to replicator selection with N randomly interacting species and deterministic self-interaction u. We present the outcome of a replica symmetric calculation for the global maximum of the mean fitness which is a Lyapunov function of the system. The replica symmetric solution is stable and no further equilibrium states exist at least for $u > \sqrt{2}$. On the other hand, in the low-u region the dynamics relaxes to states which differ from the replica symmetric result.

Replicator dynamics is an evolutionary strategy well established in different disciplines of biological sciences. It describes the evolution of self-reproducing entities called replicators in various independent models of, e.g., genetics, ecology, prebiotic evolution, and sociobiology.[1–3]

Besides this fundamental importance in theoretical biology, replicator selection has been applied to problem solving in combinatorial optimization[4] and to learning in neural networks.[5]

Replicator selection can be expressed in terms of the following nonlinear system of differential equations

$$\frac{dx_\nu}{dt}=x_\nu(f_\nu-\bar{f}) \quad \text{for } \nu=1,\ldots,N. \tag{1}$$

Here f_ν, the fitness of species ν, is the derivative $f_\nu=\partial f/\partial x_\nu$ of a fitness functional $f(x_1,\ldots,x_N)$.[4] This functional is defined on the unit simplex $\sum_\nu x_\nu=1$, $x_\nu \geq 0$ for $\nu=1,\ldots,N$. x_ν denotes the relative weight of species ν in the population and $\bar{f}$ is the mean fitness $\bar{f}=\sum_\nu x_\nu f_\nu$. It can be easily seen that f is a Lyapunov function of the replicator Eq. (1). Thus solving Eq. (1) provides a search for local maxima of f.

Different biological setups require different choices of f. To our knowledge investigations of replicator dynamics have only been undertaken for finite systems and deterministic interactions between the replicators.[2]

In this Rapid Communication we discuss a system of replicators where the interaction between species is assumed to be random. By taking the limit $N\rightarrow\infty$ we identify typical features of replicator selection for large systems. We refer to the frequently discussed case[2,4] where the fitness f_ν is linear in x_ν.

We consider the fitness functional

$$f=-H=-\tfrac{1}{2}\sum_{\nu\mu} x_\nu c_{\nu\mu} x_\mu , \tag{2}$$

where $c_{\nu\mu}=c_{\mu\nu}$ $(\mu\neq\nu)$ are identically distributed Gaussian random variables with mean zero and variance $1/N$. This defines a fully connected network[6] of replicators. Self-interactions $c_{\nu\nu}$ are introduced, which are not random, but which are all equal to a predetermined control parameter u acting as a cooperation pressure. Large positive values of u favor fixed points in the interior of the simplex $(x_\nu>0,\ \nu=1,\ldots,N)$. Small values of u favor states with a large number of variables x_ν vanishing.

We shall present the results of an analytic computation of the global maximum of the fitness functional $f=\bar{f}/2$ [i.e., the ground state of H in Eq. (2)]. Then, a calculation for the average number of equilibrium points is briefly described. Both investigations are based on mean-field methods of statistical physics. Finally, numerical results are given for comparison.

Interpreting H as energy function we introduce the free energy F of the system and consider the ground state as the limiting case for zero temperatures. F is defined for finite temperatures β^{-1} as

$$F(\beta)=-\beta^{-1}\ln Z \tag{3}$$

with the partition function

$$Z=\mathrm{Tr}(e^{-\beta H}),$$

$\mathrm{Tr}(\cdots)$ is an abbreviation for the integral over the phase space of the system. An extensive free energy is obtained with the normalization $\sum x_\nu=N$.

As usual in the physics of disordered systems the calculation of $F(\beta)$ proceeds via its self-averaging property.[7] Denoting the average over the random interactions c_{ij} by $\langle\cdots\rangle_{\rm av}$ leads to

$$F=-(1/\beta)\langle\ln Z\rangle_{\rm av}. \tag{4}$$

The procedure of calculation is straightforward within a replica symmetric approach.[8] We shall only report the results.

For $N\rightarrow\infty$ the ground-state energy is given by

$$H_0=\lim_{\beta\rightarrow\infty}F(\beta)=Nq(u/2-v), \tag{5}$$

where v and q are order parameters that satisfy

$$u-v=\frac{\sqrt{q}}{\sqrt{2\pi}}\int_{-\Delta}^{\infty}dz\, e^{-z^2/2}(z+\Delta), \tag{6}$$

$$(u-v)^2=\frac{1}{\sqrt{2\pi}}\int_{-\Delta}^{\infty}dz\, e^{-z^2/2}(z+\Delta)^2, \tag{7}$$

and Δ is an abbreviation for

$$\Delta=\sqrt{q}(u-2v). \tag{8}$$

The physical significance of q is

$$q=\langle x^\nu x^\nu\rangle_{\rm av}=\frac{1}{N}\sum_\nu x^\nu x^\nu . \tag{9}$$

By a similar calculation[9] we have obtained the fraction $(1-\alpha_0)$ of the species, which die out $(x_\nu=0)$ in the ground state

$$1-\alpha_0=\int_{-\infty}^{-\Delta}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2} . \tag{10}$$

$1-\alpha_0$ is plotted in Fig. 1. A point of particular interest is $u=\sqrt{2}$ where H_0 reverses sign (see Fig. 2). The system changes its behavior at this value of control parameter u from mainly competitive $(\alpha_0<\frac{1}{2})$ to mainly cooperative $(\alpha_0>\frac{1}{2})$.

Evolution according to the replicator equations may get stuck in metastable states which are different from the ground state. We calculate an upper bound for the number of these relaxed states. From Eq. (1) a necessary condition for a fixed point to be metastable is

$$x_\nu=0 \text{ and } f_\nu<\bar{f} , \tag{11}$$

or

$$x_\nu\neq 0 \text{ and } f_\nu=\bar{f} . \tag{12}$$

Note that the restriction $f_\nu<\bar{f}$ in Eq. (11) characterizes local stability with respect to small changes of state variables which vanish in the relaxed state. But (11) and (12) are not sufficient to define local attractive fixed points. Yet the number of equilibrium points Γ, which satisfy Eqs. (11) and (12), gives an upper bound for the number of metastable states.[10]

To proceed with the calculation of Γ we define Ω as a partition of the index set $I=\{1,\ldots,N\}$ into two sets A and B. A contains indices ν with state variables which vanish: $x_\nu=0$. B comprises indices with state variables which are different from zero. The number of fixed points which correspond to partition Ω is denoted by $\Gamma(\Omega)$. Then we have

$$\Gamma=\sum_\Omega\Gamma(\Omega) , \tag{13}$$

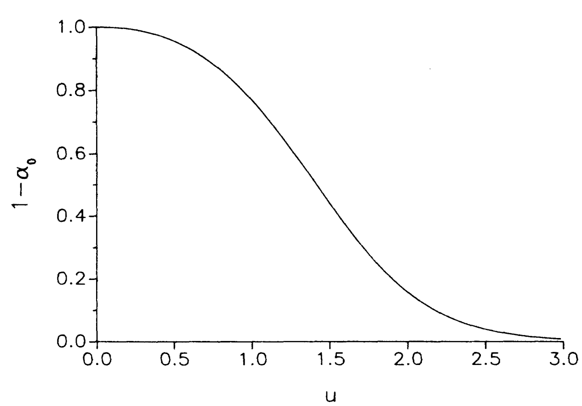

FIG. 1. The fraction of state variables $(1-\alpha_0)$ which vanish in the ground state in dependence upon u.

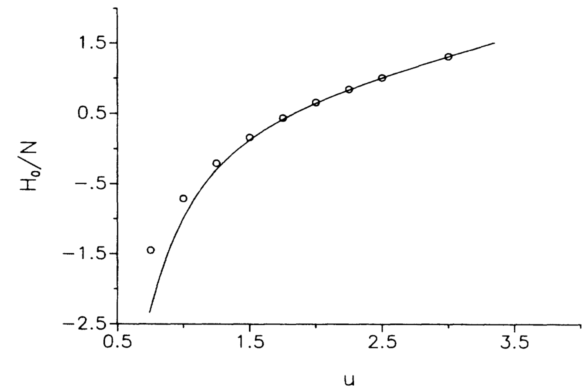

FIG. 2. The ground-state energy H_0/N in dependence upon u. Energies of relaxed states per particle are represented by circles.

where the summation refers to all partitions Ω. For simplicity of presentation we consider the partition with $A=\{1,\ldots,P\}$ and $B=\{1+P,\ldots,N\}$ only. Fixed points which belong to this partition satisfy

$$x_\nu=0 \text{ for } \nu=1,\ldots,P ,$$

and

$$\sum_{\mu=P+1} c_{\nu\mu}x_\mu=-\bar{f} \text{ for } \nu=P+1,\ldots,N .$$

These equations determine $x_1,\ldots,x_{P+1},\bar{f}$ uniquely if we take into account the normalization of state variables. Employing this unique solution $\Gamma(\Omega)$ can simply be written as

$$\Gamma(\Omega)=\prod_{\nu=1}^{P}\Theta(\bar{f}-f_\nu)\prod_{\mu=P+1}^{N}\Theta(x_\mu) . \tag{14}$$

$\Theta(x)$ is the Heaviside function and f_ν is given by

$$f_\nu=-\sum_{\mu=P+1}^{N}c_{\nu\mu}x_\mu .$$

The configuration average $\bar{\Gamma}$ of Γ is calculated via Eqs. (13) and (14) by mean-field methods[11]

$$\bar{\Gamma}=\langle\Gamma\rangle_{\rm av}=\sum_\Omega\langle\Gamma(\Omega)\rangle_{\rm av} , \tag{15}$$

where the summation over Ω leads to an integral over $\alpha=(N-P)/N$ with a combinatorial weight factor.

Detailed presentation of the calculation will be given elsewhere. Generally, the number of equilibrium points increases exponentially with the system size N

$$\bar{\Gamma}\sim e^{\lambda N} . \tag{16}$$

However, we find $\lambda=0$ for $u>\sqrt{2}$. Decreasing u below $\sqrt{2}$ results in increasing λ, e.g., $\lambda=0.02$ at $u=1$, whereas it exceeds 0.1 for small values of u. Thus, λ still remains very small if $u>1$.

The saddle-point integration over $\alpha=(N-P)/N$, which appears instead of the summation over Ω in Eq. (15), leads to a saddle-point value $\bar{\alpha}$ in the limit $N\to\infty$. It turns out that $\bar{\alpha}$ is identical with the corresponding ground-state value α_0 given by Eq. (10) if $u>\sqrt{2}$. Since

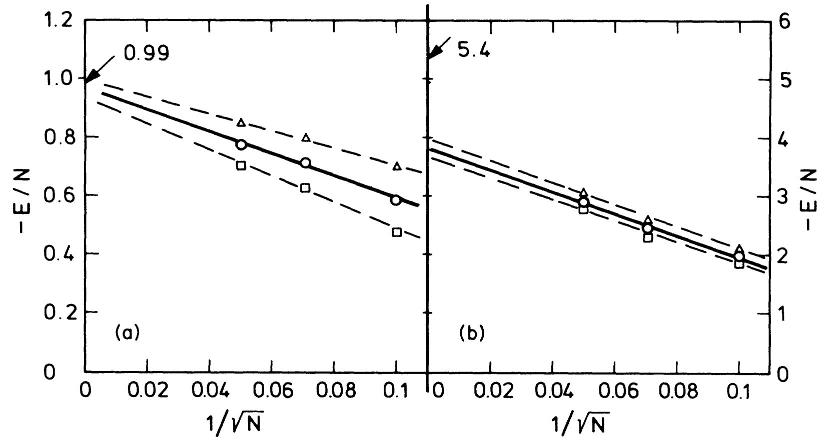

FIG. 3. Finite-size corrections to the average energies E of relaxed states. $-E/N$ is plotted for $N=100$, 200, and 400 against $1/\sqrt{N}$ (open circles) for two values of u: (a) $u=1$, (b) $u=0.5$. The arrows represent the mean-field results $-H_0/N$ and the dashed lines join points which are located one standard deviation apart from the average.

the number of equilibrium states in this case is $\bar{\Gamma}\sim 1$ [Eq. (16)], the ground state obtained by the replica symmetric calculation is in fact the true ground state for u above $\sqrt{2}$.

We have sketched a calculation of the average $\bar{\Gamma}=\langle\Gamma\rangle_{\rm av}$. Due to $\ln\Gamma$ being an extensive quantity, $\exp\langle\ln\Gamma\rangle_{\rm av}$ represents the most probable value of Γ for a large system and actually should have been evaluated. However, this is much more complicated. According to the Peierls inequality, we have

$$\langle\Gamma\rangle_{\rm av}\geq\exp\langle\ln\Gamma\rangle_{\rm av}\,.$$

Thus our calculation of $\langle\Gamma\rangle_{\rm av}$ gives an exact upper bound for the number of equilibrium points and metastable states in a large system.

The analytic results described above imply that metastable states do not exist in the infinite system for $u>\sqrt{2}$ and suggest that they are of minor importance for the dynamics of the system for u above approximately one. This is confirmed by the numerical results.

In Fig. 2 we have plotted the ground-state energy H_0/N as a function of u calculated from Eqs. (5) to (8). Results obtained from numerical solutions of the replicator equations are included in Fig. 2. The circles correspond to the mean over 100 samples of different system configurations with different initial conditions. The selected system size is $N=200$. For high values of u, very good agreement with the mean-field results for the ground-state energy is obtained. With decreasing u deviations between theoretical and numerical results appear. These are due to finite-size effects. The application of simple finite-size scaling obviously is sufficient to restore agreement between theory and numerical results for u values larger than approximately 1 [see Fig. 3(a)]. We conclude that the replicator dynamics typically relaxes to the ground state of the system for $u>1$. Although metastable states were found in our numerical calculation they did not modify the average results for large N values in this range of u.

The situation is more complex for parameter values u significantly smaller than one. Deviations between mean-field and extrapolated numerical results appear with decreasing u [Fig. 3(b)]. In this region the applied extrapolation procedure to $N\rightarrow\infty$ may not be sufficient to identify the thermodynamic limit for very small values of u. The reason is that only a small fraction of state variables x_ν remains different from zero in the relaxed state. Thus the effective system size decreases with decreasing u.

Up to now it is not yet understood whether the states found solving the replicator Eqs. (1) for small values of control parameter u are metastable states or whether we typically find the ground state. We leave this for further studies together with investigating the stability of the replica symmetric result.

Let us summarize our findings. We have studied replicator selection in dependence upon the self-interaction u. The system behavior is mainly cooperative ($\alpha_0>0.5$) for $u>\sqrt{2}$, where the replica symmetric theory is exact. Metastable states typically do not exist. This is at variance with the results obtained for the region $u<\sqrt{2}$. Here, the system behaves mainly competitive ($\alpha_0<0.5$). The average number of equilibrium points increases exponentially with N and it is unknown whether the replica symmetric ground state looses its stability if u goes down below a certain limit.

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[10]In the context of game theory a fix point which satisfies (11) and (12) is usually referred to as game theoretic equilibrium, see, e.g., Ref. 2, p. 162.

[11]A calculation for the average number of equilibrium configurations of the Sherrington-Kirkpatrick spin glass is available, see S. F. Edwards and F. Tanaka, J. Phys. F **10**, 2769 (1980); see further A. J. Bray and M. A. Moore, J. Phys. C **13**, L469 (1980); and also P. Baldi and S. S. Venkatesh, Phys. Rev. Lett **58**, 913 (1987).