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Remarks on stochastic resonance

G. Debnath,* T. Zhou,† and Frank Moss
Department of Physics, University of Missouri at St. Louis, St. Louis, Missouri 63121
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A recent and interesting experimental paper [B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988)] has refocused attention on the problem of stochastic resonance by presenting measurements of the signal-to-noise ratio (SNR) of a noise driven, periodically modulated bistable ring laser. We point out that the theoretical SNR, as defined in this and a previous work, is always infinity, because additive modulation leads to a δ function in the power spectrum of the output. Quantitative information on stochastic resonance is contained in the *strength* of this δ function relative to the noise background. We qualitatively reproduce the SNR data with an analog simulator using a standard quartic bistable potential. In this, as in previous experiments and simulations, a peak in the observed power spectrum is a reflection of the δ function, but the amplitude of the peak is rendered finite (and hence measurable) only because of the finite resolution of the measurement system.

The phenomenon of stochastic resonance was first investigated by Benzi, Sutera, and Vulpiani[1] and later suggested by Nicolis[2] and by Benzi *et al.*[3] as a possible explanation of the observed periodicity in the recurrence of the ice ages. In this model, a pair of stable climate states separated by a barrier is imagined. This bistable system is driven by noise resulting from random fluctuations of the solar constant. Using reasonable climate models, it was demonstrated that such fluctuations could trigger switching events between the two stable states on time scales which are in approximate agreement with the period of the observed recurrences[4,5] (10^5 yr). The switching events would, however, be uncorrelated random occurrences in time. In order to explain the observed periodicity, a modulation of the height of the barrier or of the alternate depths of the potential wells was introduced into the model. The resulting switching events, while still randomly occurring, must now be correlated with the periodic forcing since the switching probability is a strong function of the well depth. In the climate models, the modulation is assumed to result from a weak but periodic variation in the eccentricity of Earth's orbit with period $\cong 10^5$ yr. Digital simulations of the models show finite amplitude peaks in the power spectra located at the modulation frequency.[1,3]

Several years ago, a physical realization of stochastic resonance was demonstrated by Fauve and Heslot[6] using an electronic Schmidt trigger as a bistable system. They modulated periodically the depths of the potential wells, which represent the two stable states of the switch, alternately and simultaneously applied additive white noise. The measured power spectrum of this system displayed a sharp peak at the modulation frequency superimposed on a slowly varying continuum noise background of Lorentzian line shape. These authors defined the signal-to-noise ratio (SNR) as the (measured) amplitude of the peak relative to the (measured) noise background. They observed that the SNR passed through a maximum as the noise intensity D was increased from zero, and that at the maximum the value $D = D_0$ could be associated with the period T of the modulating function by a Kramer's characteristic time (see also Ref. 2), $\tau = T \propto \exp(\Delta u / D_0)$, where ΔU is the unmodulated barrier height.

Very recently, McNamara, Wiesenfeld, and Roy have observed this phenomenon using a bidirectional ring laser as the bistable system.[7] Using an intracavity acousto-optic modulator, they were able to induce changes in the direction of the lasing by controlling the acoustic frequency. Representing the two directions as the stable states separated by a barrier, it was possible to modulate the height of the barrier and to introduce external noise as well by modulating the acoustic frequency. These authors have observed the same phenomenology as those of Ref. 6. In particular, they have observed a sharp peak in the power spectrum of the laser intensity (measured in one direction) superimposed on a broadband noise background spectrum. In order to obtain the SNR, they measured the *amplitude* of the peak and that of the noise background at the modulation frequency. Measurements of the SNR versus the noise intensity in this experiment demonstrated the characteristic maximum, though the Kramer's time was not obtained.

In this Rapid Communication, we point out that, based on physical arguments alone, the power spectra of all such systems regardless of the details of the model, but additively modulated by a single frequency must contain delta functions. Noise in the system does not alter this so long as the noise does not multiply the amplitude of the periodic modulation. By contrast, experimental measurements of the power spectra or digital simulations of models, both obtained from Fourier transforms of time series made at finite resolution, show noninfinite amplitude peaks. The amplitudes so obtained reflect *both* the strength of the singularity, which contains all the information about stochastic resonance, and the finite resolution of the Fourier transform technique used. The latter is an entirely instrumental effect which also determines the observed line shape. Results obtained by measurements of the amplitudes only of the peaks in the power spectra therefore tell only part of the story in the absence of the details of the line shape.[8] While the shape of the stochastic resonance curve is invariant so long as all measurements are made with constant bandwidth, as they were in Refs. 6 and 7, the amplitude of the measured SNR depends on the bandwidth as well as other factors, for example, the frequency stability of the signal generator which produces the modulation. This fact has also been appreciated by Gammaitoni *et al.* who have further pointed out that entirely new insights into the phenomenon can be gained by measurements of the probability density of residence times.[9]

For the purposes of discussion, we consider two models represented by the following Langevin equations:

$$\dot{x} = ax - x^3 + \epsilon\cos\omega t + \xi(t)\,, \quad \epsilon < (4a^2/27)^{1/2}\,, \quad (1)$$

and

$$\dot{x} = a(t)x - x^3 + \xi(t)\,, \quad a(t) = 2(1+\epsilon\cos\omega t)^{1/2}\,, \quad \epsilon < 1\,, \quad (2)$$

which represent an infinitely damped system moving in the standard quartic potential,

$$U(x) = -(a/2)x^2 + \tfrac{1}{4}x^4\,, \quad (3)$$

modulated at the frequency ω and driven by the additive noise $\xi(t)$. Equation (1) represents additive modulation and additive noise. In this case, the modulation raises and lowers the depth of each well alternately on alternate half-cycles. Equation (2) represents the case of multiplicative modulation and additive noise.[10] In this case, the height of the potential barrier $\Delta u = \frac{1}{4}a^2$ is modulated about an average value of 1. Even though the system represented by Eq. (2) does not show stochastic resonance we have included it in order to show that a similarly sharp peak appears in the power spectrum at the modulation frequency. In experimental realizations of systems exhibiting stochastic resonance, such as the bistable ring laser, it is likely that combinations of multiplicative and additive modulation occur. The limits on ϵ ensure that the barrier never vanishes. (In the simulation discussed below, most of the data were taken for $a = 2^{1/2}$ and $\epsilon = 0.4$.)

As Eckmann and Thomas have pointed out, calculations of the statistical properties of such time-modulated bistable systems are by no means trivial,[11] in the first instance because they are nonlinear, and in the second because they are not stationary. Nevertheless, some general remarks can be made. Considering Eq. (1), whatever the Fourier transform of $x(t)$ may be, we note that the modulation term, $\epsilon\cos\omega t$, stands alone and therefore will necessarily contribute a delta function at ω to the transform and hence to the power spectrum. In Eq. (2), or in other versions where the modulation may multiply more highly nonlinear terms, the situation becomes more complicated because $x(t)$ itself is a stochastic function. The Fourier transform can, in principle, be broadened into a continuum which may be repeated at harmonics of the modulation frequency. In the simulation of Eq. (2) described below, however, the measured line shapes seem to be as sharp as those observed in the simulation of Eq. (1). In neither case is the line shape observed to be continuously broadened either by the noise or by the dynamics (see Ref. 10). Theoretical results recently obtained by Jung and Hanggi[12] and by Jung,[13] in an exact analysis of Eq. (1); by Fox,[14] using a perturbation analysis of the associated Fokker-Planck equation; and by McNamara and Wiesenfeld,[15] using a generalized two-state model, all predict delta functions in the power spectrum.

The climate models[1-3] are based on a Langevin equation of the type of Eq. (2). The Schmidtt trigger[6] obeys an equation of the type of Eq. (1). An accurate model for the laser is more difficult for two reasons. First, the laser is a multidimensional system, and even with appropriate adiabatic eliminations probably cannot be adequately represented by any less than two coupled Langevin equations. Second, the details of exactly how the bistable potential is modulated by the acousto-optic modulator (AOM) are not well understood.[16] Nevertheless, the modulation and the noise must enter the experimental systems phenomenologically either as additive or multiplicative terms or some combination of the two. We believe that the generic systems represented by Eqs. (1) and (2) above will thus reproduce the phenomenology observed in the experimental systems.

In order to illustrate these remarks, we have built analog simulators of Eqs. (1) and (2) following well developed techniques.[17] The modulation frequency was always set at $f = \omega/2\pi = 500$ Hz, and the modulation amplitude was always $\epsilon = 0.4$. An example measured time series $x(t)$, obtained from the simulator of Eq. (1) is shown in Fig. 1(a). The time scale is delineated by the plot of the 500-Hz modulation shown in Fig. 1(b). The switching events occur randomly in time and, in this example, on a somewhat longer time scale than the modulation period. They are, however, correlated with the modulation as shown by the sharp peak at exactly 500 Hz in the measured power spectrum shown in Fig. 1(c). Figure 1(d) shows a power spectrum obtained from a simulator of Eq. (2). It is very similar except that a small peak appears at the second harmonic of the modulation frequency. These power spectra are qualitatively very similar to the ones published by Fauve and Heslot[6] and by McNamara, Wiesenfeld, and Roy[7] in the sense that very sharp peaks are observed superimposed on a broadband noise background.

The power spectra were measured in the following way: First 2048 points of time-domain data $x(t)$ were digitized with 12 bit accuracy, each point separated from its neighbors by 300 μs. The magnitude of the square of the Fourier transform was then computed and compressed into 1024 points. The final power spectrum was obtained by averaging a number of samples (usually 200) of the individual spectra. In every case, the peak at the modulation frequency was only a few points wide, and the measured amplitude of the peak depended upon which individual point was chosen as "the maximum." Increasing the frequency resolution by increasing the separation time between digitized points resulted in narrower, higher amplitude peaks and the reverse was also true though we explored a range of only a factor of 2 in separation time.

Using the definition adopted in Refs. 6 and 7 (the ratio of the amplitudes of the peaks to the background-noise

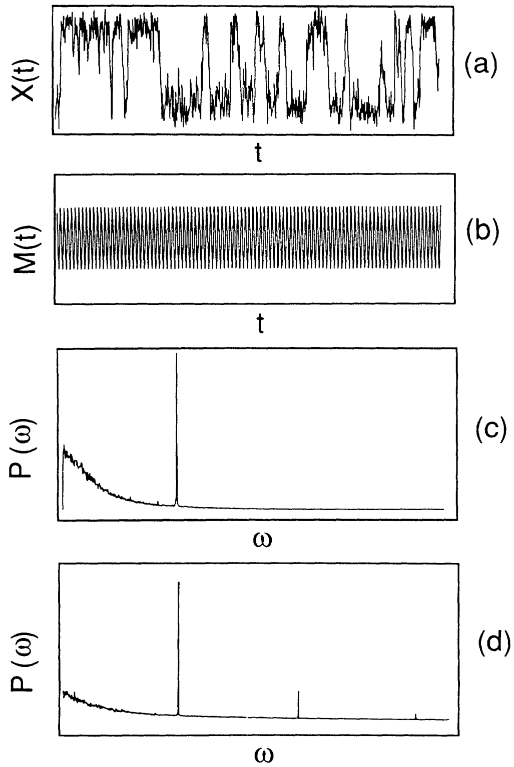

FIG. 1. Example results of the analog simulation. (a) A time series $x(t)$, measured in volts for Eq. (1) with $a=2^{1/2}$, $D=0.20$ v^2/Hz, and $\tau=0.20$. (b) The 500-Hz modulation $M(t)$ measured in volts, which also establishes the time scale for (a) with $\epsilon=0.4$. (c) An example power spectrum which is the result of 200 averages computed from individual time series obtained for the conditions stated in (a) and (b). The shape peak at 500 Hz establishes the frequency scale. (d) A power spectrum obtained for the same conditions as listed in (a) except from the simulator of Eq. (2), and only 100 averages were accumulated. The peaks in (*d*) result from correlated motions of the local minima of the potential.

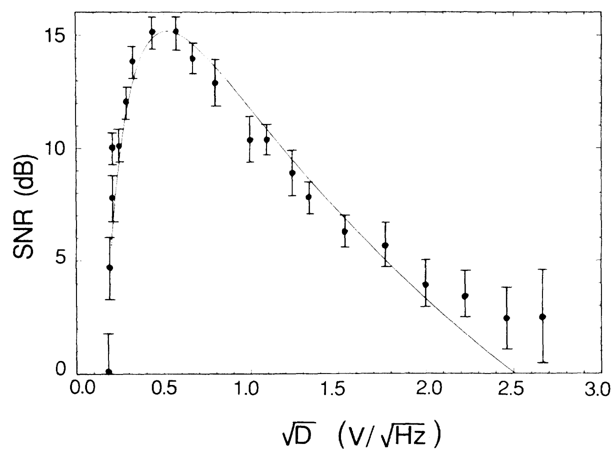

FIG. 2. The measured SNR vs $D^{1/2}$ from the simulator of Eq. (1) with $a=2^{1/2}$ (which implies that $\Delta U=0.5$), $\tau=0.20$, and the modulation as in Fig. 1(b). For a discussion of the error bars, see the text. The curve is Eq. (4) with $c=60.8$ and $\Delta U=0.5$.

power density) we have measured the SNR as a function of noise intensity $D=\tau\langle\xi^2\rangle$, where τ is the noise correlation time,[18] for the simulator of Eq. (1). The data are shown by the experimental points in Fig. 2 and are self-consistent, since all points were measured with the same instrumental resolution. However, it is important to realize that lacking a quantitive measure of the effect of this resolution, i.e., lacking detailed knowledge of the line shape, the vertical scale in Fig. 2 has no quantitative meaning. The error bars shown in Fig. 2 require discussion. They were assigned by making repeatability measurements as well as by testing the results of repositioning the cursors presumably located at the maximum of the peak and near the base of the peak at a place which we hoped would represent the "average" noise power density. The largest amount of scatter by far was incurred with the repeatability measurements. The reason is that the modulation peaks are only a few points wide, so that small variabilities from sample to sample in, for example, the signal generator (modulation) frequency, resulted in large variations in the peak amplitudes as the modulation power density happened to be shared among a few or several "bins" as the Fourier transform was processed. We suggest that the comparably large error bars in the ring-laser experiment[7] might have derived from the same variability.

In their original paper, McNamara, Wiesenfeld, and Roy[7] outlined a theory which has now been written up in detail.[15] We quote here only the result

$$S/N=(c/D^2)\exp(-2\Delta U/D)\,, \qquad (4)$$

where S/N is the power density amplitude ratio and c is some constant. Following the usual definition, the SNR in decibels (db) is given by $\mathrm{SNR}=10\log_{10}(S/N)$. As did the authors of Ref. 7, we have found it necessary to add an offset value $D_0=0.032$ v^2/Hz to all values of D applied to the simulator in order to fit the data with Eq. (4). Presumably, this represents the effect of the internal circuit noise. For $c=60.8$ and $\Delta U=0.5$ the theory is represented by the curve shown in Fig. 2 which is qualitatively comparable to that obtained in the Schmidtt trigger and ring-laser experiments.

We conclude by emphasizing that ideally the power spectra of stochastic resonance systems with additive modulation contain delta functions which are rendered into measurable peaks by experimental systems with finite frequency resolution. Quantitative measurements of the SNR of such systems can be obtained by introducing the details of the instrumentally broadened line shape, or alternatively by integrating the measured power spectra thus transforming the peaks into steps whose amplitudes are independent of the line shapes.

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*Present address: Department of Electrical Engineering, Michigan State University, East Lansing, MI 48823.

†Permanent address: Institute of Semiconductors, Chinese Academy of Sciences, Beijing, People's Republic of China.

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