Normal-mode diagonalization for two-component topological kinks

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We present a linear-stability analysis for the kink solutions of a two-component nonlinear scalar model in (1+1) dimensions. The study follows the traditional approaches which directly treat the normal-mode problem.

The classical solutions of nonlinear-field theories exhibiting a topological flavor (lumps, kinks, and solitons), with a special attention perhaps to the two-dimensional case, are widely employed in the most varied physical systems. Normally the work concerning these solitary waves has focused on the one-component real-field systems as the $(\lambda\phi^4)_{1+1}$ model or sine-Gordon theory because the explicit analytical solutions are known for both of the aforementioned cases.[1] Nonetheless, treating general situations, the fields to be considered contain more than one component, even if the restriction to one spatial dimension is retained.

Before the phase stability analysis of the kinks in N-component nonlinear models, the fundamental problem is finding the classical solutions. In fact, the conventional methods offer no systematic way of solving general coupled nonlinear-field equations; "trial and error" techniques, therefore, yield at best some solutions.[2] Once we have determined particular topological kinks associated with a two-component nonlinear interaction model, we shall carry out a linear stability analysis following the traditional approach which directly treats the normal-mode fluctuations built over the background provided by the classical solutions. We also notice the existence of stability analysis in terms of the Morse index, which coincides then with the number of independent unstable perturbation modes.[3,4]

We start from a general theory governed by the density Lagrangian

$$L=\tfrac{1}{2}(\partial_\mu\sigma)^2+\tfrac{1}{2}(\partial_\mu\rho)^2-V(\sigma,\rho)\ . \tag{1}$$

In order to impose a model containing topological kinks, we can make, for the $V(\sigma,\rho)$ potential, the choice

$$V(\sigma,\rho)=\tfrac{1}{4}(\sigma^2-1)^2+\tfrac{1}{2}f\rho^2+\tfrac{1}{4}\lambda\rho^4+\tfrac{1}{2}d\rho^2(\sigma^2-1)\ . \tag{2}$$

Thus the potential has two degenerate absolute minima at $(\sigma=\pm1,\ \rho=0)$ if f, d, and λ are all positive, and $\lambda>(d-f)^2$. More parameters could be introduced into this potential while keeping the same polynomial form, but they can always be removed by rescaling the fields and space-time coordinates.

The general classical solutions will satisfy

$$\frac{\partial^2\sigma}{\partial t^2}-\frac{\partial^2\sigma}{\partial x^2}=\sigma-\sigma^3-d\rho^2\sigma\ , \tag{3a}$$

$$\frac{\partial^2\rho}{\partial t^2}-\frac{\partial^2\rho}{\partial x^2}=-f\rho-\lambda\rho^3-d\rho(\sigma^2-1)\ , \tag{3b}$$

which for the static case reduce to

$$\frac{d^2\sigma}{dx^2}=-\sigma+\sigma^3+d\rho^2\sigma\ , \tag{4a}$$

$$\frac{d^2\rho}{dx^2}=f\rho+\lambda\rho^3+d\rho(\sigma^2-1)\ . \tag{4b}$$

Applying the aforementioned "trial and error" method in order to find some topological kinks, the orbit going from one of the minima to the other can be chosen as[2]

$$g(\sigma,\rho)=\rho^n-\alpha(1-\sigma^2)=0\ , \tag{5}$$

with α and n to be determined. Following the conventional steps it is not difficult to see that the orbit written in (5) leads to the classical solutions[2]

$$\sigma_c(x)=\tanh[\sqrt{f}\,(x-x_0)]\ , \tag{6a}$$

$$\rho_c(x)=\left[\frac{(1-2f)}{d}\right]^{1/2}\operatorname{sech}[\sqrt{f}\,(x-x_c)]\ , \tag{6b}$$

if the conditions

$$n=2,\quad \alpha=\frac{(1-2f)}{d},\quad \lambda=\frac{d(d-2f)}{(1-2f)},\quad f<\tfrac{1}{2} \tag{7}$$

are satisfied. Then the orbit represents a semiellipse (see Fig. 1) connecting the two minima of the potential. We shall refer this particular solution as the type-II kink.

Moreover, another classical solution of Eqs. (3a) and (3b) with a topological flavor is easily obtained taking $\rho=0$. Then we get

$$\sigma_c(x)=\tanh\left[\frac{x-x_0}{\sqrt{2}}\right]\ , \tag{8a}$$

$$\rho_c(x)=0\ , \tag{8b}$$

the so-called type-I kink (Fig. 1).

In order to investigate the linear stability of the solitary-wave solutions found so far, we examine small perturbations over the static classical solutions

$$\sigma(x,t)=\sigma_c(x)+h(x,t)\ , \tag{9a}$$

$$\rho(x,t)=\rho_c(x)+g(x,t)\ . \tag{9b}$$

Substituting Eqs. (9a) and (9b) into the time-dependent general equations [(3a) and (3b)] and retaining only terms linear in the small perturbations h and g, we have

$$-\left[\frac{\partial^2 h}{\partial t^2}-\frac{\partial^2 h}{\partial x^2}\right]+(3\sigma_c^2-d\rho_c^2+1)h-2d\rho_c\sigma_c g=0\ , \tag{10a}$$

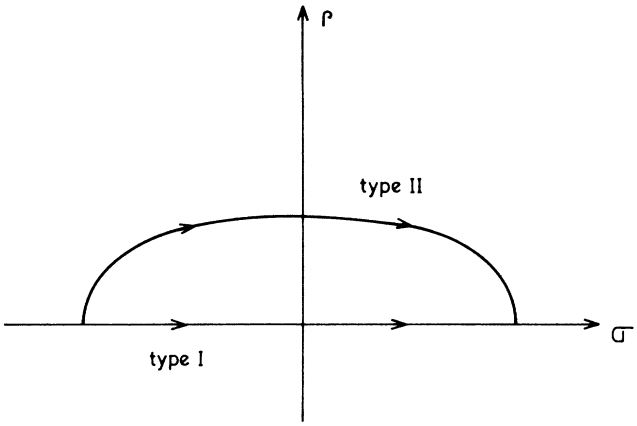

FIG. 1. The two different topological kinks.

$$-\left[\frac{\partial^2 g}{\partial t^2}-\frac{\partial^2 g}{\partial x^2}\right]+[-f-3\lambda\rho_c^2-d(\sigma_c^2-1)]g -2d\rho_c\sigma_c h=0 . \quad (10b)$$

Taking the explicit form

$$h(x,t)=h(x)\exp(-i\omega t) , \quad (11a)$$

$$g(x,t)=g(x)\exp(-i\omega t) , \quad (11b)$$

the coupled pair (10) transforms into

$$-\frac{d^2h}{dx^2}+(3\sigma_c^2+d\rho_c^2-1)h+2d\rho_c\sigma_c g=\omega^2 h , \quad (12a)$$

$$-\frac{d^2g}{dx^2}+[f+3\lambda\rho_c^2+d(\sigma_c^2-1)]g+2d\rho_c\sigma_c h=\omega^2 g . \quad (12b)$$

Now if solutions of Eqs. (12a) and (12b) exist only for $\omega^2>0$, the kink (σ_c,ρ_c) is linearly stable; on the contrary, if $\omega^2<0$, negative eigenvalues appear and then the solitary wave corresponds to the unstable type. To make a detailed analysis of these equations we need the diagonalization of the matrix whose elements are the second derivatives of the potential $V(\sigma,\rho)$ taken in the (σ_c,ρ_c) point, namely,

$$V''(\sigma_c,\rho_c)=\begin{bmatrix}3\sigma_c^2+d\rho_c^2-1 & 2d\rho_c\sigma_c\\ 2d\rho_c\sigma_c & f+3\lambda\rho_c^2+d(\sigma_c^2-1)\end{bmatrix} . \quad (13)$$

After an easy by tiring task, and bearing in mind the conditions written in (5) and (7), we can see that with the choice

$$d=1-\frac{f}{2} \quad (14)$$

the Hessian matrix adopts the diagonal form

$$V''(\sigma_c)=\begin{bmatrix}6f\sigma_c^2-2(1-3f) & 0\\ 0 & 3f\sigma_c^2-2f\end{bmatrix} , \quad (15)$$

while the additional requirement $\lambda>(d-f)^2$ imposes to the only free parameter of the model the condition

$$f<\frac{7-\sqrt{13}}{9} , \quad (16)$$

Nonetheless, we are disposed to make the linear stability analysis according to the normal-mode fluctuations described in (15).

Type-II kink. Taking the classical solutions associated to this case (6), the normal-mode equations are simply

$$\left[-\frac{d^2}{dx^2}+6f\tanh^2(\sqrt{f}x)+2-6f\right]\phi=\Omega^2\phi , \quad (17a)$$

$$\left[-\frac{d^2}{dx^2}+3f\tanh^2(\sqrt{f}x)-2f\right]\eta=\Omega^2\eta . \quad (17b)$$

For the (17a) one there are two eigenfunctions belonging to the discrete spectrum with[5]

$$\Omega_0^2=(2-4f),\quad \Omega_1^2=2-f , \quad (18)$$

both positive within the range marked out by Eq. (16); moreover, we can find the scattering solutions starting from $\Omega^2=2$. Going to the second equation, again only two eigenfunctions constitute the discrete spectrum, namely, those of eigenvalues

$$\Omega_0^2=f\left[1-\frac{(\sqrt{13}-1)^2}{4}\right],\quad \Omega_1^2=f\left[1-\frac{(\sqrt{13}-3)^2}{4}\right] . \quad (19)$$

The continuous part begins at $\Omega^2=f$. Being $\Omega_0^2<0$ the linear instability of this type-II kink is proven.

Type-I kink. Using now the simple topological kink of (8) we directly obtain the uncoupled pair

$$\left[-\frac{d^2}{dx^2}+3\tanh^2\left[\frac{x}{\sqrt{2}}\right]-1\right]\phi=\Omega^2\phi , \quad (20a)$$

$$\left\{-\frac{d^2}{dx^2}+\left[1-\frac{f}{2}\right]\left[\tanh^2\left[\frac{x}{\sqrt{2}}\right]-1\right]\right\}\eta=\Omega^2\eta . \quad (20b)$$

Equation (20a) correspond to the one-component kink of a $(\lambda\phi^4)_{1+1}$ model with $\Omega_0^2=0$ and $\Omega_1^2=\frac{3}{2}$.[1] Therefore, the main interest is focused on the second one, exhibiting an eigenfunction belonging the discrete spectrum with[5]

$$\Omega_0^2=f-\frac{(\sqrt{9-4f}-1)^2}{2} , \quad (21)$$

which is a negative eigenvalue within the validity range outlined for the f constant, a fact also indicating the instability for the type-I kink.

Following the simple approach which directly treats the normal-mode fluctuations over the background provided by the classical solutions, we have performed a linear stability analysis for the two different topological kinks arising in a two-dimensional model.

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