Dissipative evolution, initial conditions, and information theory

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An information-theoretic approach is used to give a straightforward procedure that allows one to determine whether dissipative behavior can be attained for a given Hamiltonian. This technique stresses the importance of an adequate set of initial conditions in order to obtain dissipative evolution. It is shown that, although dissipative solutions may be compatible with the set of dynamical equations, they may not be realized just because it is impossible to construct the adequate set of initial conditions that would result in dissipative behavior. This is illustrated by reference to the well-known Bateman generalized problem of two interacting harmonic oscillators. We demonstrate the impossibility of obtaining dissipative evolution in linear, time-independent Hamiltonian systems generated with any combination of bilinear products of creation and annihilation operators.

I. INTRODUCTION

The statistical description of temporal evolution has been extensively studied within the information-theory (IT) context beginning with the pioneering work of Jaynes.[1] Extensions of concomitant ideas to some quantum-mechanical problems have been proposed during recent years[2–6] that explicitly exploit the dynamical relevance of the entropy S. Within such a framework, the search of dissipative temporal evolution, starting from a microscopic Hamiltonian remains an open field.[7] The irreversible dissipative behavior characteristic of many physical phenomena contrasts with the reversible intrinsic nature of microscopic physical models.[3,7–13] Dissipation is, in general, the result of interactions between the "observed system" and some other (observed or tacit, i.e., a heat bath) into which energy can flow in an undetermined, uncontrolled (and for this reason irreversible) manner.[7] It is generally believed that no damping occurs at the microscopic level, the observed phenomena being the result of a "collective" many-body effect, in which a given particle interacts with a field arising from the motion of the remaining particles. It is the dissipation of energy, ceded to this field, which would originate the observed damping.[7] Classically, the corresponding many-body situation can be reduced to an effective one-body problem and one then faces the familiar situation in which a particle is subjected to the action of a damping force.

In the quantum case, however, serious difficulties ensue if one wishes to work along similar lines, and two different quantization procedures have been followed: (i) An explicit time-dependent Hamiltonian is introduced, with a canonical momentum which is not the usual (kinetic) one;[12] (ii) A nonlinear Hamiltonian is introduced,[13] which includes a "friction" potential that depends upon the expectation values of canonically conjugate variables. Kanai's treatment has received a considerable amount of attention,[14] but it produces some results that are considered unphysical.[11] Among these difficulties, one should especially single out the fact that the Kanai approach appears to violate the uncertainty principle. The second quantization procedure referred to above, i.e., the nonlinear approach, poses a severe problem of interpretation.[11,14,15] Moreover, solutions can not be superimposed and, until a careful study of their regions of stability is carried out, perturbation theory methods cannot be confidently applied.[12] Consequently, although this way of tackling the problem provides one with rewarding insights, it cannot be denied that it by no means precludes the necessity of investigating other possibilities, along totally different lines.

This is precisely the aim of the present effort in which we are going to analyze an information-theoretic treatment of the quantum-mechanical description of dissipative temporal evolution. A very simple mechanism will yield some original insight into the problem of quantum dissipation. To this end a brief resume of basic IT concepts[16] is given in Sec. II. Our theoretical approach to dissipative temporal evolutions is developed in Sec. III. Section IV is devoted to an illustrative example which generalizes the quantum theory of the damped harmonic oscillator based on the Bateman dual Hamiltonian,[7,17,18] finding in a completely different and very general theoretical context the same difficulties in obtaining a normalized density matrix previously reported in Ref. 19. Finally, in Sec. V, some conclusions are drawn.

II. BASIC IT CONCEPTS

Within the IT context, the statistical operator (or density matrix) $\hat{\rho}$ is constructed[2,3,16] starting from the knowledge of the expectation values of, say, M operators $\hat{O}_j$ ($\hat{O}_0=\hat{I}$=identity operator),

$$\langle \hat{O}_j/\hat{\rho}\rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{O}_j] = o_j, \quad j=0,1,\ldots,M \ . \tag{2.1}$$

The subindex "0" refers to the normalization condition $\mathrm{Tr}\hat{\rho}=1$. The operator $\hat{\rho}$ is expressed, within the IT framework, by

$$\hat{\rho}=\exp\left[-\lambda_0\hat{I}-\sum_{j=1}^{M}\lambda_j\hat{O}_j\right], \tag{2.2}$$

in terms of the $M+1$ Lagrange multipliers λ_i, $i=0,1,\ldots,M$, determined so as to fulfill Eq. (2.1). The density matrix $\hat{\rho}$ maximizes the entropy $S[\hat{\rho}]$ given (in units of the Boltzman constant) by

$$S[\hat{\rho}]=-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})=\lambda_0\hat{I}+\sum_{j=1}^{M}\lambda_j\langle\hat{O}_j/\hat{\rho}\rangle \tag{2.3}$$

subject to the constraints Eq. (2.1). The operator $\hat{\rho}(t)$ obeys[2] the equation of motion

$$i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{H}(t),\hat{\rho}(t)]\ . \tag{2.4}$$

It is well known that if $\hat{\rho}$ is constructed so as to fulfill Eq. (2.4), then S is a constant of the motion.[2-6] Consequently, one should endeavor to find those (relevant) operators entering Eq. (2.2) so as to satisfy Eq. (2.4), in order to guarantee that S is a constant of motion. Using Eqs. (2.2) and (2.4) it is easy to verify that the relevant operators are those that close a partial Lie algebra under commutation with the Hamiltonian $\hat{H}$,

$$[\hat{H},\hat{O}_i]=i\hbar\sum_{j=0}^{q}\hat{O}_jG_{ij}\ , \tag{2.5}$$

where the G_{ij} are the elements (c numbers) of a $q\times q$ matrix $\underline{G}$ (which may depend upon the time if $\hat{H}$ is time dependent). For our present purposes we shall consider the case $M=q$. Equation (2.5) constitutes the central requirement to be fulfilled by the operators entering the density matrix. Moreover, the closure condition (2.5) on the $\hat{O}_j$ leads to the fact that the time-dependent Schrödinger equation can be replaced by a set of coupled equations for the λ_i's,[2-6]

$$\frac{d\lambda_i}{dt}=\sum_{l=0}^{q}G_{il}\lambda_l\ , \tag{2.6}$$

which is easily obtained using Eqs. (2.2), (2.4), and (2.5). The Lagrange multipliers are related to $\langle\hat{O}_r/\hat{\rho}\rangle$ by[1,2]

$$\frac{\partial\lambda_0}{\partial\lambda_j}=-\mathrm{Tr}(\hat{O}_j\hat{\rho})=-\langle\hat{O}_j/\hat{\rho}\rangle\ . \tag{2.7}$$

Since $\hat{\rho}$ is a positive operator, its diagonal and nondiagonal elements satisfy

$$\rho_{ii}\geq 0 \tag{2.8}$$

and

$$\rho_{ii}\rho_{jj}\geq|\rho_{ij}|^2\ . \tag{2.9}$$

Therefore, the mean values are correlated by equations

$$K_{ii}\equiv(\Delta\langle\hat{O}_i\rangle)^2=\langle\hat{O}_i^2\rangle-\langle\hat{O}_i\rangle^2\geq 0 \tag{2.10}$$

and

$$K_{ii}K_{jj}=(\Delta\langle\hat{O}_i\rangle)^2(\Delta\langle\hat{O}_j\rangle)^2 \geq(\tfrac{1}{2}\langle\{\hat{O}_i,\hat{O}_j\}_+\rangle-\langle\hat{O}_i\rangle\langle\hat{O}_j\rangle)^2\equiv K_{ij}^2\ . \tag{2.11}$$

The restrictions imposed to the possible initial mean values by Eqs. (2.10) and (2.11) are generally enhanced by the existence of Casimir operators of the Lie group, which is another interrelation between the initial conditions. Thus they would play a crucial role in the dynamical behavior as they can not be arbitrarily chosen. As shown in Ref. 3, the same Hamiltonian (i.e., the same dynamical problem) can lead to different physical situations if a different set of initial conditions is chosen. The temporal evolution of the expectation values of the operators [Eq. (2.1)] can be obtained by recourse to Ehrenfest's theorem. Assuming that $\hat{O}$ does not depend explicitly upon the time we find

$$\frac{d\langle\hat{O}_i\rangle_t}{dt}=-\sum_{j=0}^{q}\langle\hat{O}_j\rangle_tG_{ji},\quad i=1,2,\ldots,q \tag{2.12}$$

which provides us with a set of coupled linear differential equations that completely determine the time evolution of the expectation values $\langle\hat{O}_j\rangle_t$, provided one knows the corresponding initial values $\langle\hat{O}_j\rangle_{t=0}$,

$$\langle\hat{O}_j/\hat{\rho}\rangle_t=\tilde{\underline{F}}\langle\hat{O}_j/\hat{\rho}\rangle_{t_0}\ , \tag{2.13}$$

where $\tilde{\underline{F}}$ denotes the transposed of a square matrix $\underline{F}$ defined by

$$-\frac{\partial\underline{F}}{\partial t}=\underline{F}\,\underline{G}\ . \tag{2.14}$$

Therefore the entire dynamics of the problem is embedded in the value of the structure factors G_{li}, as long as we confine our interest to the time evolution of the set $\{\hat{O}_i\}$. Using the preceding procedure, we shall now show how dissipative solutions naturally emerge, in principle, as a consequence of the structure of the Hamiltonian and its concomitant algebra, but with the requirement that a proper *set of initial conditions* is at hand.

III. DISSIPATIVE TEMPORAL EVOLUTIONS, INITIAL CONDITIONS, AND THE $\underline{G}$ MATRIX

As expressed above, dissipation arises as a consequence of interactions between a given subsystem and the "rest of the universe," often referred to as the "reservoir," whether it is thermal or not.[7] By restriction of our attention to the subsystem of interest, the usual procedure for dealing with irrelevant variables is to eliminate them from the corresponding picture by means of some adequate projection operator. Instead, we will suggest a redefinition[5] of what we are to understand both by the "system" and by the "rest of the universe," assuming that both are parts of a "super system," described by a Hamiltonian.

It was shown in Ref. 3 that, given a Hamiltonian and an (initial) set $(\hat{O}_j^m)$ of operators [i.e., Eq. (2.1)] whose corresponding observables are accessible to experimental manipulation (the superscript "m" stands for "measurable"), a definite prescription for constructing a suitable subspace (the "relevant" one) of the appropriate Hilbert space can be given, based upon the closure[5] procedure of Eq. (2.5). In other words, the set of measurable operators

gives rise to additional operators that can either augment the size of the initial set or, contrary wise, be regarded as "nonmeasurable" ones. For example, in the harmonic oscillator case we can begin with $\{\hat{x},\hat{p},\hat{x}^2,\hat{p}^2/2m\}$ as measurable operators and Eq. (2.5) leads to $\hat{L}=\hat{x}\hat{p}+\hat{p}\hat{x}$ as a "nonmeasurable" operator (the identity operator is always included among the $\hat{O}_j^m$ to ensure that the density matrix is normalized).

Our approach to dissipative temporal evolutions consists of (1) defining our supersystem as described by *all* the measurable and nonmeasurable operators, both defined by Eq. (2.5), and (2) assuming that the latter give rise to the unknown energy flow, while the former describe "our system." The closure relations decouple the "super system" from the "real" external universe.[3] Into this context, the energy and the entropy of the supersystem are time-dependent constants of motion.[2] Instead, the energy and entropy of "our system" vary with time. Dissipation is the result of looking at a part of the supersystem, defined as the system under study.

One finds in Ref. 3 the general form of the solution of Eq. (2.12), which determines the temporal behavior of the mean values for a given Hamiltonian. The situation becomes particularly simple when $\hat{H}$ is independent of time, for which the coefficients G_{ij} of Eq. (2.12) are time independent. The $\langle \hat{O}_i \rangle_t$ are then the solutions of a system of linear differential equations with constant coefficients. Consequently,

$$\langle \hat{O}_j \rangle_t = \sum_{i=1}^{k} \exp(R_i t) \sum_{m=0}^{\gamma} a_{im}^{(j)} t^m , \tag{3.1}$$

where k is the number of different roots R_i of the corresponding secular equations, the $a_{im}^{(j)}$ are constants to be determined by the initial conditions, and $\gamma+1$ is the multiplicity of the R_i. Thus it can be clearly seen that *the eigenvalues of the $\underline{G}$ matrix (R_i) determine the character of the solution* at least in the case of time-independent Hamiltonians. So, a *necessary* condition to obtain dissipative temporal evolutions is the existence of *real eigenvalues* of the $\underline{G}$ matrix. In this case, using the Lie group's theory, we may assure that the $\underline{F}$ matrix, defined via Eq. (2.14), must be related with a noncompact Lie group. Therefore this group must have an infinite volume.[20]

One should here stress the importance of the initial conditions in giving a correct description of the problem at hand. As mentioned in Sec. II, the algebra constructed under commutation with the Hamiltonian determines not only the dynamical aspects of the problem, but also the possible expectation values of the relevant operators. As it was said in Sec. II [see Eqs. (2.10) and (2.11)], this restriction appears because all the relevant operators, measurable or not, must satisfy Eq. (2.1) and the normalization condition imposed on the density matrix. Thus we can not select the mean values of the operators in a completely independent way without violating the normalization condition or the probabilistic character of the density matrix.

Therefore we have two special features of our approach: (a) assume the existence of a set of operators $\{\hat{O}_i, i=1,\ldots,q\}$, relevant to the physical problem at hand, that closes a partial Lie algebra under commutation with the Hamiltonian $\hat{H}$ (the g_{ij} elements of the matrix $\underline{G}$ determines the dissipative or conservative behavior of the temporal evolution of the operator's mean values [see Eq. (2.12)], and (b) the existence of a coherent set of initial conditions for *all* those relevant operators. In order to stress the importance of both conditions, in Sec. IV we discuss an example which is characterized by real eigenvalues of the $\underline{G}$ matrix, satisfying only condition (a) for dissipative temporal evolutions, but not condition (b) on the initial mean values.

IV. DISSIPATIVE TEMPORAL EVOLUTIONS IN TWO COUPLED OSCILLATORS MODEL

In this section, we resort to a prescription for the modeling of two interacting systems,[7] in order to apply the formalism described in Sec. III. We shall apply it to a generalization of the well-known Bateman Hamiltonian.[7,17,18] Let $\hat{a}$ and $\hat{a}^\dagger$ be the second-quantization boson operators for one of them and $\hat{b},\hat{b}^\dagger$ the corresponding ones for the second, so that

$$\hat{H}_a=\varepsilon_a(\hat{a}+\hat{a}+\tfrac{1}{2})=\varepsilon_a\hat{T}_a , \tag{4.1a}$$

$$\hat{H}_b=\varepsilon_b(\hat{b}+\hat{b}+\tfrac{1}{2})=\varepsilon_b\hat{T}_b , \tag{4.1b}$$

are the unperturbed Hamiltonians. In order to introduce the interaction we define the quasispin operators

$$\hat{S}_-=\hat{b}\hat{a} , \tag{4.2a}$$

$$\hat{S}_+=\hat{a}^\dagger\hat{b}^\dagger , \tag{4.2b}$$

$$\hat{T}_{ab}=\hat{S}_+ +\hat{S}_- , \tag{4.2c}$$

$$\hat{S}_{ab}=i(\hat{S}_+ -\hat{S}_-) , \tag{4.2d}$$

and cast the total Hamiltonian in the form

$$\hat{H}=\hat{H}_a+\hat{H}_b+v_1\hat{T}_{ab}+v_2\hat{S}_{ab} , \tag{4.3}$$

so that the following partial Lie algebra is closed:

$$[\hat{H},\hat{T}_a]=[\hat{H},\hat{T}_b]=i(v_1\hat{S}_{ab}-v_2\hat{T}_{ab}) , \tag{4.4a}$$

$$[\hat{H},\hat{T}_{ab}]=-i[(\varepsilon_a+\varepsilon_b)\hat{S}_{ab}+2v_2(\hat{T}_a+\hat{T}_b)] , \tag{4.4b}$$

$$[\hat{H},\hat{S}_{ab}]=i[(\varepsilon_a+\varepsilon_b)\hat{T}_{ab}+2v_1(\hat{T}_a+\hat{T}_b)] . \tag{4.4c}$$

From the closure condition Eq. (2.5) we see that the matrix $\underline{G}$ reads ($\hat{O}_0=1$, $\hat{O}_1=\hat{T}_a$, $\hat{O}_2=\hat{T}_b$, $\hat{O}_3=\hat{T}_{ab}$, and $\hat{O}_4=\hat{S}_{ab}$)

$$g_{04}=g_{40}=0,\quad g_{34}=-g_{43}=(\varepsilon_a+\varepsilon_b)/\hbar , \tag{4.5a}$$

$$g_{14}=g_{24}=2v_1/\hbar,\quad g_{41}=g_{42}=2v_1/\hbar , \tag{4.5b}$$

$$g_{13}=g_{23}=-2v_2/\hbar,\quad g_{31}=g_{32}=-2v_2/\hbar , \tag{4.5c}$$

while the remaining elements are zero. As it was demonstrated in Ref. 2–6 the $\underline{G}$ matrix defined by Eq. (2.5) contains the whole dynamics of the system and, in particular, the different roots of the secular equation corresponding to the $\underline{G}$ matrix are the coefficients accompanying the exponential functions that characterize the appropriate solutions (we assume $\hat{H}$ to be time independent). *When*

these roots are real, dissipation can indeed take place if some very special initial conditions are selected. The equations of motion for the relevant operators adopt the appearance

$$\frac{d\langle \hat{T}_a \rangle_t}{dt} = (v_2/\hbar)\langle \hat{T}_{ab} \rangle_t - (v_1/\hbar)\langle \hat{S}_{ab} \rangle_t \ , \tag{4.6a}$$

$$\frac{d\langle \hat{T}_b \rangle_t}{dt} = (v_2/\hbar)\langle \hat{T}_{ab} \rangle_t - (v_1/\hbar)\langle \hat{S}_{ab} \rangle_t \ , \tag{4.6b}$$

$$\frac{d\langle \hat{T}_{ab} \rangle_t}{dt} = (2v_2/\hbar)(\langle \hat{T}_a \rangle_t + \langle \hat{T}_b \rangle_t) + [(\varepsilon_a+\varepsilon_b)/\hbar]\langle \hat{S}_{ab} \rangle_t \ , \tag{4.6c}$$

$$\frac{d\langle \hat{S}_{ab} \rangle_t}{dt} = -\{(2v_1/\hbar)(\langle \hat{T}_a \rangle_t + \langle \hat{T}_b \rangle_t) + [(\varepsilon_a+\varepsilon_b)/\hbar]\langle \hat{T}_{ab} \rangle_t\} \ . \tag{4.6d}$$

These can be easily integrated by solving

$$\det(\underline{G}-\lambda \underline{I})=0 \ . \tag{4.7}$$

We obtain that the eigenvalues of the $\underline{G}$ matrix are

$$\lambda^2=[4(v_1^2+v_2^2)-\varepsilon^2]/\hbar^2 \ , \tag{4.8}$$

where

$$\varepsilon=\varepsilon_a+\varepsilon_b \ . \tag{4.9}$$

Therefore the solutions of Eqs. (4.6) are

$$\langle \hat{T}_a \rangle_t = \langle \hat{T}_a \rangle_0 - \frac{1}{\lambda^2\hbar^2}\{2(v_1^2+v_2^2)(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) + \varepsilon(v_1\langle \hat{T}_{ab} \rangle_0 + v_2\langle \hat{S}_{ab} \rangle_0) - [\exp(\lambda t)f(\lambda)+\exp(-\lambda t)g(\lambda)]/2\} \ , \tag{4.10}$$

$$\langle \hat{T}_b \rangle_t = \langle \hat{T}_b \rangle_0 - \frac{1}{\lambda^2\hbar^2}\{2(v_1^2+v_2^2)(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) + \varepsilon(v_1\langle \hat{T}_{ab} \rangle_0 + v_2\langle \hat{S}_{ab} \rangle_0) - [\exp(\lambda t)f(\lambda)+\exp(-\lambda t)g(\lambda)]/2\} \ , \tag{4.11}$$

$$\langle \hat{T}_{ab} \rangle_t = \frac{1}{\lambda^2\hbar^2}\left[2v_1\varepsilon(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) + 4v_1(v_1\langle \hat{T}_{ab} \rangle_0 + v_2\langle \hat{S}_{ab} \rangle_0) + \frac{\exp(\lambda t)[(v_2\lambda\hbar-\varepsilon v_1)f(\lambda)]-\exp(-\lambda t)[(v_2\lambda\hbar+\varepsilon v_1)g(\lambda)]}{2(v_1^2+v_2^2)}\right] \ , \tag{4.12}$$

$$\langle \hat{S}_{ab} \rangle_t = \frac{1}{\lambda^2\hbar^2}\left[2v_2\varepsilon(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) + 4v_2(v_1\langle \hat{T}_{ab} \rangle_0 + v_2\langle \hat{S}_{ab} \rangle_0) - \frac{\exp(\lambda t)[(v_2\lambda\hbar-\varepsilon v_1)f(\lambda)]-\exp(-\lambda t)[(v_2\lambda\hbar+\varepsilon v_1)g(\lambda)]}{2(v_1^2+v_2^2)}\right] \ , \tag{4.13}$$

with

$$f(\lambda)=2(v_1^2+v_2^2)(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) + (v_2\lambda\hbar+\varepsilon v_1)\langle \hat{T}_{ab} \rangle_0 - (v_1\lambda\hbar-\varepsilon v_2)\langle \hat{S}_{ab} \rangle_0 \ , \tag{4.14a}$$

$$g(\lambda)=2(v_1^2+v_2^2)(\langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0) - (v_2\lambda\hbar-\varepsilon v_1)\langle \hat{T}_{ab} \rangle_0 + (v_1\lambda\hbar+\varepsilon v_2)\langle \hat{S}_{ab} \rangle_0 \ . \tag{4.14b}$$

Two quite different regimes ensue according to the value of the coupling constant. For

$$(v_1^2+v_2^2)<(\varepsilon/2)^2 \tag{4.15a}$$

the time evolution of $\langle \hat{T}_a \rangle_t$, $\langle \hat{T}_b \rangle_t$, $\langle \hat{T}_{ab} \rangle_t$, and $\langle \hat{S}_{ab} \rangle_t$ adopt a stationary character (oscillating functions). Of more interest to us is the alternative case, namely,

$$(v_1^2+v_2^2)>(\varepsilon/2)^2 \ , \tag{4.15b}$$

for which λ is real [see Eq. (4.8)] and we obtain real exponential functions for the evolution of the mean values [see Eqs. (4.10)–(4.13)]. As usual (see references) we call time-decaying solutions "dissipative ones." The term dissipation should then be regarded within the context of works like the one of Dekker.[7] Notice that the total energy of the system is given by $\langle \hat{H} \rangle$, a constant. The subsystems are represented by $\langle \hat{H}_a \rangle$ and $\langle \hat{H}_b \rangle$, that do evolve with time. Our work differs from previous literature in the fact that, in addition to $\hat{H}_a$ and $\hat{H}_b$, we are including two other operators in order to close a partial Lie algebra. We choose $\hat{H}_a$ as representing that portion of our total system in which an observer is located. The rest is to be considered as a heath bath. The closure above referred to guarantees that our total system is, both dynamically ($\langle \hat{H} \rangle$=const) and thermodynamically (S=const), a closed system.

For the purpose of avoiding divergences for $t \to \infty$ we must impose the condition

$$f(\lambda)=0 \tag{4.16a}$$

or

$$2(v_1^2+v_2^2)(\langle \hat{T}_a \rangle_0+\langle \hat{T}_b \rangle_0)=-(v_2\lambda\hbar+\varepsilon v_1)\langle \hat{T}_{ab} \rangle_0 +(v_1\lambda\hbar-\varepsilon v_2)\langle \hat{S}_{ab} \rangle_0 , \tag{4.16b}$$

and this will result in condition (b) of Sec. III.

In order to explicitly obtain Eq. (4.16b) we need to evaluate the mean values and ask for the significance of this condition. This is done in Sec. IV A. With this, Eqs. (4.10)–(4.13) are seen to describe a damping process. The expectation values of $\hat{T}_a$ and $\hat{T}_b$ decreases as t grows. Thus energy is seen to be "dissipated" by an observer "confined" to any of the corresponding systems.[7] This dissipative behavior would arise as a consequence of a proper choice of ε_a, ε_b, v_1, and v_2, on one hand, and of the initial conditions $\langle \hat{T}_a \rangle_0$, $\langle \hat{T}_b \rangle_0$, $\langle \hat{T}_{ab} \rangle_0$, and $\langle \hat{S}_{ab} \rangle_0$ on the other one. No approximations are needed: all is the result of a straightforward quantum-mechanical manipulation. The basic ingredients needed in order to obtain these results are those of Eqs. (4.4), namely, the closure of a partial Lie algebra under commutation with the total Hamiltonian and Eq. (4.16b).

Equation (2.2) tells us that a properly normalized density matrix is used in evaluating expectation values, representing all possible states of our system. The underlying (partial) Lie algebra relates the different expectation values as described by Eqs. (2.7), (2.10), (2.11), and in this example via the Casimir operator of the group $[(\hat{T}_a+\hat{T}_b)^2-(\hat{T}_{ab}^2+\hat{S}_{ab}^2)]$. The fact that these mean values are not *mutually independent figures* becomes crucial, as shown below. It is of great importance for our present purpose to verify whether Eq. (4.16b) and Eqs. (2.7), (2.10), (2.11), (2.16c), (4.4), and (4.15b) constitute a compatible set. According to Eq. (4.15b) we define

$$\cos(\theta)\equiv\varepsilon/[2(v_1^2+v_2^2)^{1/2}] . \tag{4.17}$$

Therefore Eq. (4.16b) reads

$$\langle \hat{T}_a \rangle_0+\langle \hat{T}_b \rangle_0=[-(v_2\sin\theta-v_1\cos\theta)\langle \hat{T}_{ab} \rangle_0+(v_1\sin\theta+v_2\cos\theta)\langle \hat{S}_{ab} \rangle_0]/(v_1^2+v_2^2)^{1/2}] =(-\alpha\langle \hat{T}_{ab} \rangle_0+\beta\langle \hat{S}_{ab} \rangle_0) , \tag{4.18}$$

with

$$\alpha=\frac{(v_2\sin\theta-v_1\cos\theta)}{(v_1^2+v_2^2)^{1/2}}\equiv\cos\theta'\leq 1 , \tag{4.19a}$$

$$\beta=\frac{(v_1\sin\theta+v_2\cos\theta)}{(v_1^2+v_2^2)^{1/2}}\equiv\sin\theta'\leq 1 , \tag{4.19b}$$

$$\alpha^2+\beta^2=1 . \tag{4.19c}$$

Thus an additional constraint affects the dynamical problem, and this fact can not be ignored.

If $v_2=0$, we obtain a simplified two-body pairing interaction. Such an interaction has been used in the BCS theory of superconductivity and in considering pairing correlations in complex nuclei.[21]

This algebra also includes the Bateman Hamiltonian as a particular case. Bateman's Hamiltonian plays the role of a paradigm in the present literature. It was presented in 1931,[17] and, in the quantal version, has been the subject of much recent and interesting work. (See, for instance, Refs. 7 and 18–20). It is obtained as a special case of our general algebra from Eq. (4.3) if

$$\varepsilon_a=-\varepsilon_b \quad (\varepsilon=0), \quad v_1=0 , \tag{4.20}$$

and we can write also the following Hamiltonian:

$$\hat{H}=\varepsilon_a(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})+iv(\hat{a}^\dagger\hat{b}^\dagger+\hat{b}\hat{a}) . \tag{4.21}$$

The initial condition needed in order to avoid divergences is

$$\langle \hat{T}_{ab} \rangle_0=-(\langle \hat{T}_a \rangle_0+\langle \hat{T}_b \rangle_0) . \tag{4.22}$$

As stated before, this initial condition imposes an *additional restriction not contained explicitly* in the dynamics of the problem.

A. A coherent set of initial conditions

In the following paragraphs we shall demonstrate, based on very general properties, that Eq. (4.16b) or (4.18) can not be fulfilled. Thus, this algebra does not satisfy the requirement (b) of Sec. III in order to obtain temporal dissipative evolutions. The explicit calculation of the pertinent values is performed in the "product" basis,

$$|a_l,b_m\rangle=|a_l\rangle\otimes|b_m\rangle , \tag{4.23}$$

and we write for the matrix elements of the density operator the abbreviation

$$p_{ij}^{lm}=\langle a_i,b_j|\hat{\rho}|a_l,b_m\rangle . \tag{4.24}$$

Our expectation values read

$$\langle \hat{T}_a \rangle=\tfrac{1}{2}+\sum_{l;j=0}^{\infty} l\,p_{lj}^{lj} , \tag{4.25a}$$

$$\langle \hat{T}_b \rangle=\tfrac{1}{2}+\sum_{l;j=0}^{\infty} j\,p_{lj}^{lj} , \tag{4.25b}$$

$$\langle \hat{T}_{ab} \rangle=\sum_{l;j=1}^{\infty} 2(lj)^{1/2}\mathrm{Re}(p_{l-1\,j-1}^{l\,j}) , \tag{4.25c}$$

$$\langle \hat{S}_{ab} \rangle=-\sum_{l;j=1}^{\infty} 2(lj)^{1/2}\mathrm{Im}(p_{l-1\,j-1}^{l\,j}) . \tag{4.25d}$$

If we introduce

$$\mathrm{Re}(p^{l}_{l-1\,j-1}) \equiv |p^{l}_{l-1\,j-1}| \cos(\theta^{l}_{l-1\,j-1}) \,, \tag{4.26a}$$

$$\mathrm{Im}(p^{1}_{l-1\,j-1}) \equiv |p^{1}_{l-1\,j-1}| \sin(\theta^{l}_{l-1\,j-1}) \,, \tag{4.26b}$$

Eq. (4.16b) or (4.18) can be recast as

$$\begin{aligned} \langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0 &= 1 + \sum_{l,j=0}^{\infty} (l+j) p^{lj}_{lj} \\ &= -\cos\theta' \langle \hat{T}_{ab} \rangle_0 + \sin\theta' \langle \hat{S}_{ab} \rangle_0 \\ &= -2 \sum_{l;j=1}^{\infty} (lj)^{1/2} |p^{l}_{l-1\,j-1}| \\ &\quad \times \cos(\theta' - \theta^{l}_{l-1\,j-1}) \,. \end{aligned} \tag{4.27}$$

In order to work with a well defined probability space, p^{lm}_{ij} must fulfill

$$p^{lj}_{lj} \geq 0 \quad \forall i,j \,, \tag{4.28a}$$

$$|p^{lm}_{ij}|^2 \leq p^{ij}_{ij} p^{lm}_{lm} \,, \tag{4.28b}$$

and considering that we are dealing with infinite Hilbert spaces, all temperatures are positive definite quantities ($p^{i\,j}_{i-1\,j} \geq p^{ij}_{ij}$). By taking $\cos(\theta' - \theta^{l}_{l-1\,j-1}) = -1$, one easily finds

$$\begin{aligned} \langle \hat{T}_a \rangle_0 + \langle \hat{T}_b \rangle_0 &= 1 + \sum_{l;j=0}^{\infty} (l+j) p^{lj}_{lj} \\ &> 2 \sum_{l;j=0}^{\infty} [(l+1)(j+1)]^{1/2} p^{lj}_{lj} \\ &> -\cos\theta' \langle \hat{T}_{ab} \rangle_0 + \sin\theta' \langle \hat{S}_{ab} \rangle_0 \,. \end{aligned} \tag{4.29}$$

This means that for the particular case of the generalized quantal Bateman's Hamiltonian [Eq. (4.3)], even though the Hamiltonian dynamics allows one to obtain a dissipative behavior, this can not be attained due to the fact that initial conditions needed to cancel the divergent component cannot be written down. This result is not restricted to the Bateman's Hamiltonian, provided it is obtained as a special case of Eq. (4.3). Instead Eq. (4.3) is the more general linear, time-independent Hamiltonian which can be constructed for bilinear products of creation and annihilation operators. As was said in Sec. III, dissipative evolutions can be attained only for noncompact Lie groups. There exists only one group that can be constructed via bilinears products of creation and annihilation operators satisfying this property. It is the Simplectic group Sp_2, which has bosonic statistics and changes the number of particles.[20] This is just the group to consider in this work and so we can assure that dissipative evolutions can not be obtained with time-independent lineal Hamiltonians generated via any combination of bilinear products of creation and annihilation operators.

V. CONCLUSIONS

Following the previously proposed informational-theoretic approach, a straightforward quantum-mechanical approach has been outlined that allows one to determine whether dissipative behavior can be attained for a given Hamiltonian. The only requirement is that the unperturbed and interacting terms be of such a nature that a partial Lie algebra under commutation with the total Hamiltonian is closed, a fact which allows for the conservation of the entropy of the total system. The second step is to evaluate the eigenvalues of the $\underline{G}$ matrix. If $\underline{G}$ has real eigenvalues we can obtain a dissipative behavior, provided that the algebra's structure allows one to write down a proper set of initial conditions for the relevant operators obtained from Eq. (2.5). The importance of the restrictions imposed by the algebra on the mean values can be clearly seen in the example discussed in Sec. IV.

Neither puzzles nor ambiguities arise in the formalism of Sec. III, as long as a suitable Lie algebra underlies the problem at hand, which is by no means an extraordinary occurrence.[22–25] For the special case we treat here it can be concluded that the particular structure of the Lie algebra leads to an incompatible superposition of the domains of different operators involved in the initial conditions and, for this reason, it is not possible to obtain the cancellation of the divergent component in Eqs. (4.10)–(4.13). It is to be emphasized that the only way to cancel this nonderivable portion of the solution is to obtain an adequate set of initial conditions. In this way the crucial role of the initial conditions is stressed. It is important to notice that difficulties in finding normalized solutions for the dual Bateman Hamiltonian were previously reported.[19] In this respect, we can say that the non-normalizability of the eigenfunctions is the manifestation of a deeper problem, as it is the nondissipative character of the Hamiltonian, as we demonstrate here from a very general formalism. This linear, time-independent Hamiltonian is the more general one which can be constructed with bilinear products of creation and annihilation operators leading to a noncompact Lie algebra.[21] If a scheme is considered in which attention is focused upon a given subsystem of the total system, then, for that subsystem, dissipative behaviors are sought by recourse to Bateman's Hamiltonian.

One may tentatively conclude that quantum dynamics *per se* does not seem to preclude the possibility of describing dissipative behavior. The fact that we cannot derive it from the equations of motion can be attributed *exclusively* to the failure in selecting adequate initial conditions. The clue can be guessed from Eq. (2.13) where, on the one hand, the $\underline{F}$ matrix determines that the Lie group has an associated infinite volume, while on the other hand, the involved initial conditions [Eqs. (4.16)] must be selected to diminish this volume in order to determine a dissipative behavior. So, the Lie algebra associated with the physical problem is crucial to determine the dynamical evolution of the system (conservative or dissipative) but also the addition of a metric to this algebra (i.e., a normalized density matrix with positive eigenvalues), in order to evaluate normalized mean values [see Eq. (2.2)], implies the appearance of extra conditions that determine

whether such temporal evolutions can be physically realized or not. It is necessary to stress that this conclusion is reached because we have insisted upon conserving not only the total energy but also the total entropy, so that the total system is both dynamically and thermodynamically closed. This "double closure" distinguishes the present work from previous ones.

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