Hartree solutions for the self-Yukawian boson sphere

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The ground state of an N-boson assembly interacting through two-body attractive Yukawa forces is analyzed in the Hartree approximation. The solution, which is not universal, can be adequately characterized by a unique parameter $\tilde{\mu}$ proportional to μN^{-1}, where μ^{-1} is the range of the force. It is found that for $\tilde{\mu}$ larger than a critical value, there is no bound solution for the system. Our solutions are shown to rigorously fulfill the virial theorem.

I. INTRODUCTION

An often useful approximation to a self-interacting assembly of N identical particles is the shell model. There it is assumed that the structure is dictated by the (uncorrelated) single-particle stationary states of the average potential created by the assembly. Thus, if one is dealing with bosons, the ground state will correspond to a condensate configuration in which all the particles of the assembly are occupying the lowest-lying orbital of the average potential. The specific structure of this minimum energy orbital will be obtained by a Hartree-like method. Some time ago Ruffini and Bonazzola[1] described the structure of the self-gravitating boson sphere in this fashion. In this paper we extend their analysis to the self-Yukawian case, i.e., to a bound assembly of identical bosons linked together by two-body Yukawa potentials.

Regarding this problem, the Hartree method can be stated either as a one-body self-consistent Schrödinger equation in which the potential energy is a function of the wave-function itself (as presented in Ref. 1), or, in the way we choose here, as a variational problem in the particle density n; obviously both methods must be completely equivalent.

The motivation for our analysis is twofold: (i) On one hand it is a purely formal one of getting a better understanding of the structure of this kind of bound states (note that this is one of the few nontrivial quantum many-body systems that admits an easy closed solution), and (ii) although real assemblies formed by fundamental bosons interacting through short-range forces are not known (so far), sometimes in fermion assemblies it is quite useful to define effective bosonic degrees of freedom; a good example of this is perhaps the old α-particle model for nuclear structure. In this sense our analysis may be useful from the phenomenological point of view too.

Advancing part of our results let us say that the nice universality of the gravitational case is lost, that any problem of this type can be adequately parametrized by a unique parameter $\tilde{\mu}$ proportional to μN^{-1} where μ^{-1} is the range of the force, and that there is a borderline case so that beyond a limit value $\tilde{\mu}$, there is no bound state in the assembly.

The paper is organized as follows: In Sec. II we reanalyze the pure gravitational case from the variational perspective, the general notation is introduced, and a minor mistake in Ref. 1 is corrected; in Sec. III the Yukawa case is analyzed; and in Sec. IV our main conclusions are summarized. For astrophysical motivation in the self-gravitating sphere the rotational velocity curve is obtained.

II. SELF-GRAVITATING BOSON SPHERE

Let us assume N identical particles with mass m interacting through gravitational forces; the Hamiltonian of the system is

$$\hat{H} = \hat{T} + \hat{V} \ , \tag{1a}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 \ , \tag{1b}$$

$$\hat{V} = -Gm^2 \sum_{i>j=1}^{N} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \ . \tag{1c}$$

We denote by $|\psi\rangle$ and E the ground state of the system and its energy, respectively,

$$E = \langle \psi | \hat{H} | \psi \rangle \ . \tag{2}$$

Supposing that the particles are bosons, in the Hartree approximation we have

$$|\psi\rangle = |f\rangle_1 |f\rangle_2 \cdots |f\rangle_N \ , \tag{3a}$$

$$n(\mathbf{r}) = N[f^*(\mathbf{r}) f(\mathbf{r})] \ , \tag{3b}$$

where $n(\mathbf{r})$ is the particle density at the point $\mathbf{r}$, and $f(\mathbf{r})$ the minimum energy $(n=1, l=0)$ single-particle wave function. As $f(\mathbf{r})$ corresponds to a bound state, it is real, i.e., f is simply $\sqrt{n/N}$.

Instead of expressing f as the minimum energy solution of a self-consistent Schrödinger equation,[1]

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{v}\right] f = \varepsilon f \ , \tag{4a}$$

$$\hat{v}(\mathbf{r}_1)=m\phi(\mathbf{r}_1) , \tag{4b}$$

$$\phi(\mathbf{r}_1)=-GNm\int\frac{f^2(\mathbf{r}_2)}{|\mathbf{r}_2-\mathbf{r}_1|}d\mathbf{r}_2 , \tag{4c}$$

we prefer to write it as a variational problem in the particle density n. In (4c) $N-1$ has been taken as N, which is valid for large N. With self-explanatory notation and assuming spherical symmetry, we have

$$E=T+V , \tag{5a}$$

$$\begin{aligned}T&=-\frac{\hbar^2}{2m}\int d\mathbf{r}\, n^{1/2}(\mathbf{r})\nabla^2 n^{1/2}(\mathbf{r})\\&=\int d\mathbf{r}\left[-\frac{\hbar^2}{2m}\left[\frac{n'}{r}-\frac{n'^2}{4n}+\frac{n''}{2}\right]\right] ,\end{aligned} \tag{5b}$$

$$\begin{aligned}V&=\frac{m}{2}\int d\mathbf{r}\, n(\mathbf{r})\phi(\mathbf{r})\\&=-\frac{Gm^2}{2}\int d\mathbf{r}\int d\mathbf{r}'\frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} ,\end{aligned} \tag{5c}$$

so that $E[n]$ must be minimized as a function of $n(r)$ with the constraint

$$\int d\mathbf{r}\, n(\mathbf{r})=N . \tag{6}$$

Thus we face a variational problem

$$\delta\left(E[n]+\lambda\int d\mathbf{r}\, n\right)=0 , \tag{7}$$

λ being the Lagrange multiplier. Putting (5) into (7) we obtain

$$-\lambda=-\frac{\hbar^2}{2m}\left[\frac{n'}{nr}-\frac{n'^2}{4n^2}+\frac{n''}{2n}\right]+m\phi , \tag{8}$$

which is equivalent to the self-consistent Schrödinger equation ($\lambda=-\varepsilon$) written in Eq. (4). In order to circumvent the integro-differential nature of Eq. (8) let us apply the Laplacian operator ∇^2 to it, to obtain the following fourth-order differential equation:

$$\begin{aligned}n''''=&\frac{16\pi m^2Gn^2}{\hbar^2}-\frac{4}{r}n'''+\frac{10n'n''}{nr}-\frac{6n'^3}{n^2r}+\frac{3n'''n'}{n}\\&+\frac{2n''^2}{n}-\frac{7n''n'^2}{n^2}+\frac{3n'^4}{n^3} .\end{aligned} \tag{9}$$

This is reminescent, for example, of the situation one finds when dealing with the Thomas-Fermi (TF) method for a fermion assembly interacting through exponential forces.[2]

Defining dimensionless variables x and $\tilde{n}$ as

$$n=\frac{N\tilde{n}}{4\pi b^3} , \tag{10a}$$

$$r=bx , \tag{10b}$$

$$b=\frac{\hbar^2}{2GMm^2} , \tag{10c}$$

where $M\equiv Nm$ is the total mass, we obtain

$$\begin{aligned}\ddddot{\tilde{n}}=&2\tilde{n}^2-\frac{4}{x}\dddot{\tilde{n}}+\frac{10\dot{\tilde{n}}\ddot{\tilde{n}}}{x\tilde{n}}-\frac{6\dot{\tilde{n}}^3}{x\tilde{n}^2}+\frac{3\dot{\tilde{n}}\dddot{\tilde{n}}}{\tilde{n}}\\&+\frac{2\ddot{\tilde{n}}^2}{\tilde{n}}-\frac{7\dot{\tilde{n}}^2\ddot{\tilde{n}}}{\tilde{n}^2}+\frac{3\dot{\tilde{n}}^4}{\tilde{n}^3} ,\end{aligned} \tag{11}$$

where a dot means differentiation with respect to x. It is convenient to define a dimensionless multiplier $\tilde{\lambda}$ as

$$\lambda=\frac{GMm}{b}\tilde{\lambda} . \tag{12}$$

Then Eq. (8) adopts the form

$$\tilde{\lambda}=\frac{\dot{\tilde{n}}}{x\tilde{n}}+\frac{\ddot{\tilde{n}}}{2\tilde{n}}-\frac{\dot{\tilde{n}}^2}{4\tilde{n}^2}-\tilde{\phi}(x) , \tag{13a}$$

with

$$\tilde{\phi}(x)=-\frac{1}{4\pi}\int dx'\frac{n(x')}{|\mathbf{x}-\mathbf{x}'|} . \tag{13b}$$

For small x, $\tilde{n}(x)=\sum a_nx^n$, so that the solution of (11) by the Runge-Kutta method requires prior knowledge of a_0, a_1, a_2, and a_3, and the fulfillment of Eq. (6) which in dimensionless units reads

$$1=\int x^2\tilde{n}(x)dx . \tag{14}$$

Note that the solution we seek for fulfilling (11) and (14) is universal (it does not depend on N). Let us see now how among the four coefficients a_0, a_1, a_2, a_3 there are only two that are really independent. From Eq. (11), for small x, we obtain

$$a_3=\left[\frac{a_1}{a_0}\right]\left[\frac{5}{6}a_2-\frac{1}{4}\left[\frac{a_1}{a_0}\right]a_1\right] \tag{15}$$

and analogously from Eq. (13a)

$$a_1=0 , \tag{16a}$$

$$\tilde{\lambda}=\frac{3a_2}{a_0}+\int_0^{\infty}x'\tilde{n}(x')dx' , \tag{16b}$$

so we conclude that

$$a_3=0 . \tag{17}$$

The fact that $a_1=0$ is natural because $\tilde{n}$ must have a maximum at the origin; obviously $a_0>0$. Hence our problem is reduced to finding a_0 and a_2 such that the $\tilde{n}(x)$ obtained from (11) satisfy (14); additionally, $\tilde{n}(x)$ must not have any node ($1s$ state).

It is interesting to emphasize that the function $\tilde{n}(x)=4/x^4$ satisfies (11) and does not have any node, but it does not satisfy (14). Thus this dependence can be used only in the asymptotic regime $x\to\infty$.

The numerical solution which satisfies all the above-mentioned requirements is characterized by

$$\begin{aligned}a_0&=6.912\times10^{-3} ,\\a_2&=-1.7595\times10^{-4} ,\end{aligned} \tag{18}$$

whilst the value of the Lagrange multiplier is

$$\tilde{\lambda}=0.08139\ . \tag{19}$$

$\tilde{n}(x)$ is plotted in Fig. 1.

Although these systems have an infinite size, it is useful to define a radius R which covers 99% of the assembly. In dimensionless units we obtain $R/b=19.8$, i.e.,

$$R=9.9\frac{\hbar^2}{GMm^2}\ . \tag{20}$$

The results for the two components of the energy, and their sum, are

$$T=0.02713\frac{GM^2}{b}=0.05426G^2M^3m^2\hbar^{-2}\ , \tag{21a}$$

$$V=-0.05426\frac{GM^2}{b}=-0.10852G^2M^3m^2\hbar^{-2}\ , \tag{21b}$$

$$E=-0.02713\frac{GM^2}{b}=-0.05426G^2M^3m^2\hbar^{-2}\ , \tag{21c}$$

which proves that the virial theorem ($2T=-V$) is exactly fulfilled. In Ref. 1 it is erroneously assumed that $E=-N\lambda$, for the true total energy; in fact, it is three times smaller.

It is interesting to recall the analogous expressions for the energy terms of the self-gravitating fermion sphere.[3] In the leading semiclassical development, i.e., in TF we have

$$T=0.1499G^2M^{7/3}m^{8/3}\hbar^{-2}\ , \tag{22a}$$

$$V=-0.3003G^2M^{7/3}m^{8/3}\hbar^{-2}\ , \tag{22b}$$

$$E=-0.1499G^2M^{7/3}m^{8/3}\hbar^{-2}\ , \tag{22c}$$

and the TF length parameter b_{TF} is

$$b_{\mathrm{TF}}=0.5577\frac{\hbar^2}{Gm^{8/3}M^{1/3}}\ , \tag{23}$$

i.e., the boson sphere is more compact and more bound than the fermion sphere, and while in the latter there is a definite radius, the density of the bosons decays slowly up to infinity.

In astrophysical literature it is easy to find the rotational velocity curve created by different statistical distributions. This is motivated by the growing concern of that community about the likely existence of dark-matter

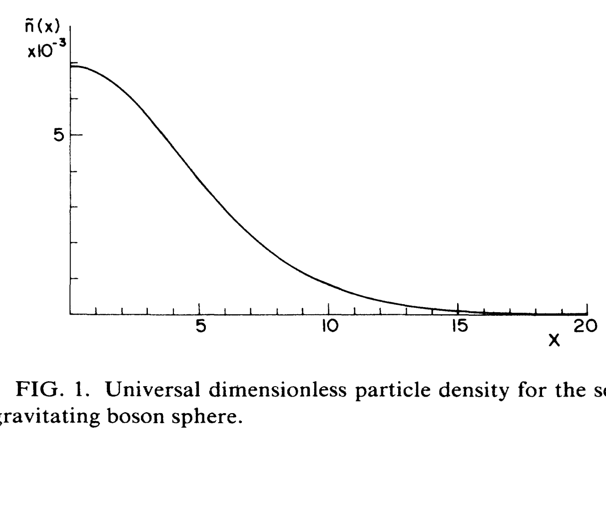

FIG. 1. Universal dimensionless particle density for the self-gravitating boson sphere.

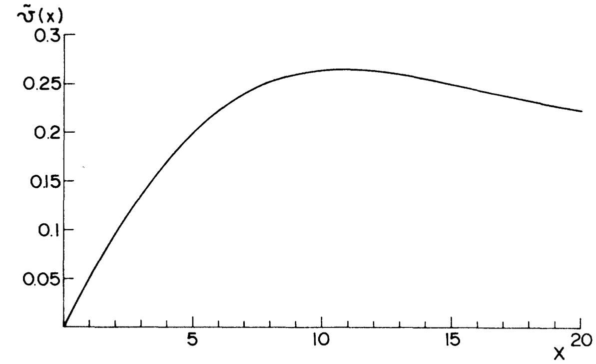

FIG. 2. Universal dimensionless rotational velocity for the self-gravitating boson sphere.

halos[4,5] to explain the observational constancy of the rotational velocities in spiral galaxies, far beyond the central luminous discs. We have not found any reference devoted to studying this curve when the presumed dark matter is a condensate boson sphere, for that reason we present it here too. It is convenient to define a dimensionless velocity $\tilde{v}(x)$ defined as

$$v(r)=\sqrt{GM/b}\,\tilde{v}(x) \tag{24a}$$

with

$$\tilde{v}(x)=\left[\frac{1}{x}\int_0^x\tilde{n}(x')x'^2dx'\right]^{1/2}\ . \tag{24b}$$

This is obtained simply by equating the gravitational attractive force experienced by a test particle at the position x, with the centripetal force when moving along a circle of radius x with velocity x. The resulting v is plotted in Fig. 2.

III. SELF-YUKAWIAN BOSON SPHERE

Let us now analyze an assembly of identical bosons linked by short-range Yukawa forces. In the new Hamiltonian (1c) must be replaced by

$$\hat{V}=-g^2\sum_{i>j=1}^{N}\frac{e^{-\mu|\mathbf{r}_i-\mathbf{r}_j|}}{|\mathbf{r}_i-\mathbf{r}_j|}\ . \tag{25}$$

Following parallel arguments, and identical notation, to Sec. II we find that in the Hartree approximation the total energy can be expressed as

$$\begin{aligned}E=T+V&=-\frac{\hbar^2}{2m}\int d\mathbf{r}\,n^{1/2}(\mathbf{r})\nabla^2n^{1/2}(\mathbf{r})\\&\quad-\frac{g^2}{2}\int d\mathbf{r}\int d\mathbf{r}'\frac{n(\mathbf{r})n(\mathbf{r}')e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\\&=\int d\mathbf{r}\left[-\frac{\hbar^2}{2m}\left[\frac{n'}{r}-\frac{n'^2}{4n}+\frac{n''}{2}\right]\right]\\&\quad+\frac{g}{2}\int d\mathbf{r}\,n(\mathbf{r})\phi(\mathbf{r})\ ,\end{aligned} \tag{26}$$

where we have defined the average potential $\phi(r)$. The variational equation (7) leads, in this case, to

$$-\lambda=-\frac{\hbar^2}{2m}\left[\frac{n'}{nr}-\frac{n'^2}{4n^2}+\frac{n''}{2n}\right]+g\phi \qquad (27)$$

and applying the Laplacian to it, we obtain the following fourth-order differential equation:

$$n''''=\frac{16\pi g^2 mn^2}{\hbar^2}-\frac{4}{r}n'''+\frac{10n'n''}{nr}-\frac{6n'^3}{n^2 r}+\frac{3n'''n'}{n}+\frac{2n''^2}{n}-\frac{7n''n'^2}{n^2}+\frac{3n'^4}{n^3}+\mu^2\left[n''-\frac{n'^2}{2n}+\frac{2n'}{r}-\frac{4m\lambda}{\hbar^2}n\right]. \qquad (28)$$

If we pass now to the dimensionless variables defined as

$$\tilde{n}=\frac{4\pi b^3 n}{N}, \qquad (29a)$$

$$r=bx, \qquad (29b)$$

$$b=\frac{\hbar^2}{2mg^2N}, \qquad (29c)$$

$$\lambda=\frac{g^2N}{b}\tilde{\lambda}, \qquad (29d)$$

$$\tilde{\mu}=\mu b, \qquad (29e)$$

Eqs. (28) and (27) read as follows:

$$\ddddot{\tilde{n}}=2\tilde{n}^2-\frac{4}{x}\dddot{\tilde{n}}+\frac{10\dot{\tilde{n}}\ddot{\tilde{n}}}{x\tilde{n}}-\frac{6\dot{\tilde{n}}^3}{x\tilde{n}^2}+\frac{3\dot{\tilde{n}}\dddot{\tilde{n}}}{\tilde{n}}+\frac{2\ddot{\tilde{n}}^2}{\tilde{n}}-\frac{7\dot{\tilde{n}}^2\ddot{\tilde{n}}}{\tilde{n}^2}+\frac{3\dot{\tilde{n}}^4}{\tilde{n}^3}+\tilde{\mu}^2\left[\ddot{\tilde{n}}-\frac{\dot{\tilde{n}}^2}{2\tilde{n}}+\frac{2\dot{\tilde{n}}}{x}-2\tilde{\lambda}\tilde{n}\right], \qquad (30)$$

$$\tilde{\lambda}=\frac{\dot{\tilde{n}}}{x\tilde{n}}-\frac{\dot{\tilde{n}}^2}{4\tilde{n}^2}+\frac{\ddot{\tilde{n}}}{2\tilde{n}}-\tilde{\phi}(x), \qquad (31)$$

where now $\tilde{\phi}(x)$ is defined as

$$\tilde{\phi}(x)=-\frac{1}{4\pi}\int d\mathbf{x}'\frac{n(\mathbf{x}')e^{-\tilde{\mu}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}. \qquad (32)$$

Note that Eq. (30) is no longer universal: the explicit presence of $\tilde{\mu}^2$ implies that each N calls for a specific solution. Furthermore, the appearance of $\tilde{\lambda}$ forces us to work consistently in the sense that the solution $\tilde{n}(x)$ obtained from a $\tilde{\lambda}$ input, must reproduce, when inserted in (31), the same $\tilde{\lambda}$ as output. For small x, $\tilde{n}(x)=\sum a_n x^n$ and to implement the Runge-Kutta method in (30) we need to know a_0, a_1, a_2, and a_3 so that two condition must be fulfilled: (14) and the above-mentioned self-consistency in $\tilde{\lambda}$. These two conditions, plus the appearance of models (1s state), fixes the physical solution for any $\tilde{\mu}$. The reason for this is that two of the a_n are dependent on the other two. This can be seen by analyzing the behavior of (30) and (31) for small x; they imply

$$a_3=\frac{a_1}{a_0}\left[\frac{5}{6}a_2-\frac{1}{4}\left[\frac{a_1}{a_0}\right]a_1+\frac{1}{12}a_0\tilde{\mu}^2\right] \qquad (33)$$

and

$$a_1=0, \qquad (34a)$$

$$\tilde{\lambda}=\frac{3a_2}{a_0}+\int_0^\infty x'\tilde{n}(x')e^{-\tilde{\mu}x'}dx', \qquad (34b)$$

so that $a_3=0$. Thus, again we have $a_0>0$, by definition, and $a_1=0$ and $a_2<0$, because n is a maximum at the origin. In Table I, parametrized as a function of $\tilde{\mu}$, one can observe the value of a_0 and a_2, the resulting Lagrange multiplier $\tilde{\lambda}$, the dimensionless radius R/b where 99% of the community is covered, and the result for the total kinetic energy, total potential energy, and their sum $E=T+V$. To express these quantities it is convenient to define an energy unit as follows:

$$T=\frac{N^2g^2}{b}\tilde{T}, \qquad (35a)$$

$$V=\frac{N^2g^2}{b}\tilde{V}, \qquad (35b)$$

$$E=\frac{N^2g^2}{b}\tilde{E}. \qquad (35c)$$

To check the fulfillment (or not) of the virial theorem by the Hartree solutions just obtained we have to recall that, for bound states linked by Yukawa forces, this theorem states the following:[6]

TABLE I. Some properties of the self-Yukawian boson sphere for several values of the reduced mass parameter $\tilde{\mu}$. a_0 and a_2 are the two independent initial coefficient of the particle density. $\tilde{\lambda}$ is the dimensionless Lagrange multiplier introduced to maintain the total number of particles constant. $X=R/b$ is the distance from the center that covers 99% of the assembly. $\tilde{Y}$ is defined in Eq. (37), and $\tilde{T}$, $\tilde{V}$, and $\tilde{E}$ are the kinetic, potential, and total reduced energies, respectively.

$\tilde{\mu}$	$10^3 a_0$	$10^4 a_2$	$\tilde{\lambda}$	X	$\tilde{T}$	$\tilde{V}$	$\tilde{Y}$	$\tilde{E}$
0.000	6.9120	−1.7595	0.08139	19.8	0.027 13	−0.054 26	0.000 00	−0.027 13
0.001	6.9110	−1.7591	0.08039	20.0	0.027 13	−0.053 76	0.000 49	−0.026 63
0.010	6.8150	−1.7198	0.07168	20.4	0.026 86	−0.049 27	0.004 44	−0.022 41
0.050	5.0550	−1.0601	0.03850	22.8	0.021 74	−0.030 12	0.013 35	−0.008 38
0.080	2.8780	−0.4251	0.01869	28.0	0.014 62	−0.016 65	0.012 59	−0.002 03
0.090	2.0380	−0.2428	0.01276	33.0	0.011 47	−0.012 12	0.010 83	−0.000 64
0.095	1.5800	−0.1606	0.00978	36.0	0.009 60	−0.009 69	0.009 50	−0.000 09

$$2\langle \hat{T} \rangle = -\langle \hat{V} \rangle + Y , \quad (36a)$$

being

$$Y = g^2\mu \left\langle \sum_{i>j=1}^{N} e^{-\mu|\mathbf{r}_i - \mathbf{r}_j|} \right\rangle . \quad (36b)$$

Y can also be expressed in the above units

$$Y = \frac{N^2 g^2}{b} \tilde{Y} . \quad (37)$$

In Table I one can see that the virial theorem is scrupulously verified in all the range of existence of bound states.

IV. CONCLUSIONS

Our variational reanalysis of the self-gravitational boson sphere has led again to the basic original results of Ruffini and Bonazzola.[1] Our discrepancy, however (a factor of 3), in the total energy of the assembly arises because they compute it simply as $N\varepsilon$ where ε is the energy eigenvalue of the Schrödinger problem for N particles. Physically this is incorrect because in the process of building up the assembly of N particles the total binding energy stored can be expressed as the following sum:

$$E = -\sum_{n=1}^{N} \lambda_n = -\sum_{n=1}^{N} \frac{GM_n m}{b_n} \tilde{\lambda} = -\frac{2G^2 m^5 \tilde{\lambda}}{\hbar^2} \sum_{n=1}^{N} n^2 = -\frac{2G^2 m^5 \tilde{\lambda}}{\hbar^2} \frac{N(N+1)(2N+1)}{6} \quad (38)$$

($-\lambda_n$ denotes the Schrödinger eigenvalue for a problem with n particles), and in the limit of large N

$$E = -\frac{N\lambda}{3} , \quad (39)$$

which is our result. This result correctly verifies the virial theorem.

With respect to the velocity curve plotted in Fig. 2, we wish to emphasize that as expected it does not fall as steeply as in fermion case (because n decays slowly up to infinity). In this sense it would be worth analyzing the case of a boson halo around a central luminous mass to study if it can be phenomenologically acceptable when compared with astrophysical observations.

With respect to the self-Yukawian boson sphere our main results are as follows.

(i) The universality ruling when $\mu=0$ (gravitational case) is lost.

(ii) The different solutions are characterized by the unique dimensionless parameter $\tilde{\mu} = 2^{-1}\mu\hbar^2 m^{-1} g^{-2} N^{-1}$, with existing bound solutions in the narrow range of

$$0 \le \tilde{\mu} \le 0.095 . \quad (40)$$

(iii) Throughout this range the virial theorem is verified.

Finally, let us say that unlike the TF description of the equivalent fermion case[7], the boson assembly extends smoothly to infinity.

ACKNOWLEDGMENTS

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