Electron-positron pair creation in relativistic shocks: Pair plasma in thermodynamic equilibrium

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(Received 19 August 1988; revised manuscript received 28 November 1988)

The thermodynamics of the relativistic shock wave in which electron-positron pairs are produced in the post-shock matter is studied in the temperature range 5×10^9 K $\lesssim T \lesssim 10^{12}$ K, where the baryons are nonrelativistic and electron-positron pairs are the only leptons created. The post-shock matter with pairs in thermodynamic equilibrium are classified according to whether or not the energy density of the radiation (the pairs and the photons) exceeds that of the baryons. The post-shock matter without pairs in thermal equilibrium is also considered in the case where it is composed of a relativistic two-temperature electron-ion plasma. The relevant time scales for such processes as temperature relaxation, pair production, and cooling are examined in order to establish the temperature region where such a plasma can exist. Analytic relations between the pre-shock and post-shock quantities are obtained in certain limits of these three cases. The shocks with and without pairs in the post-shock matter are then compared.

I. INTRODUCTION

Relativity comes into play in hydrodynamic shock phenomena either when the bulk velocity of the matter becomes on the same order of the velocity of light (c) or when the temperature (especially, that of the post-shock matter) becomes comparable to or exceeds the rest-mass energy of the lightest particle species.[1–6] In the latter case, which does not necessarily exclude the former case, particle-pair creation may result. The bulk motion of the hydrogen plasma with a velocity,

$$v \geq \left[\frac{2m_e c^2}{m_p} \right]^{1/2} \simeq 0.033c \ , \tag{1.1}$$

where m_e and m_p are the electron and proton masses, respectively, for example, can be a potential source of electron-positron pair creation, since such a bulk kinetic energy, after the plasma is thermalized, could give rise to a temperature exceeding the electron-rest-mass energy. Velocities of this order may be found in a wide variety of astrophysical phenomena, such as galactic jets. In particular, the typical escape velocity from a compact star and the velocities associated with the apparent superluminal motion are generally on the same order of the velocity of light. In addition, there are a number of phenomena in which pair creation by shock waves may play an important role: heavy ion collisions, accretion disks, early universe, cosmic strings, to list a few.

The production of electron-positron pairs has important implications from the observational point of view. Due to a larger electric-charge–to–rest-mass ratio, electrons and positrons couple to photons more strongly than protons, so that a number of radiative processes, such as bremsstrahlung, Compton processes, and synchrotron radiation, are expected. Furthermore, certain atomic processes involving the positronium and the γ-ray line from the pair annihilation are also possible observational consequences. In addition, the electron-positron pair plasma could dynamically behave differently around the massive object, where gravity plays an important role, since the pair plasma is less gravitationally bound to the system than the baryonic matter.

Although the relativistic shock has been studied in a number of papers,[3–6] the effects of particle-pair production has not been considered seriously so far. The purpose of the present paper is to study the properties of relativistic shocks when particle pairs are created in the post-shock matter. Since we are mainly concerned with the qualitative consequences of such a pair production, we consider the simplest possible cases in which the essential physics is transparent. Therefore, we shall make a number of simplifying assumptions. We first consider the case where the pre-shock matter and the post-shock matter are in thermodynamic equilibrium. The pre-shock matter is assumed to be a nonrelativistic-nondegenerate hydrogen plasma. Then, a few equations of state for the post-shock matter are selected. When the number of pairs far exceeds that of the ions, the pairs and the photons are treated as a blackbody. In comparison to the cases where electron-positron pairs are present, we shall also consider the post-shock matter without pairs.

The outline of the present paper is as follows. In Sec. II, we begin by listing simplifying assumptions and then classify the post-shock matter according to its equation of state. In Sec. III, we treat the post-shock matter in which pairs are present. By using the Rankine-Hugoniot relation and conservation laws, we express the post-shock quantities in terms of the pre-shock quantities. We also derive the self-consistency conditions for the pre-shock quantities so that the temperature of the post-shock matter remains in a range such that electron-positron pairs are the only pairs produced. In Sec. IV, we consider the post-shock matter in which pairs are absent. We take the relativistic two-temperature Maxwellian plasma for the equation of state and do an analysis similar to that in Sec. III. In addition, various relevant time scales are compared and the condition under which such a plasma can exist is discussed. In Sec. V, we summarize and discuss our results. Some of the thermodynamic quantities for a relativistic Maxwellian gas are listed in Appendix A, while calculational details pertaining to Sec. III are given in Appendix B.

II. CLASSIFICATION OF THE POST-SHOCK MATTER

Consider a shock wave in which the pre-shock matter with a baryon number density n_{b1} and an electron number density n_{e1} comes in at a velocity v_1 relative to the shock boundary and the post-shock matter with a baryon number density n_{b2} moves at a velocity v_2, also relative to the shock boundary. The suffixes 1 and 2 are used to denote the pre-shock and post-shock quantities, respectively. We treat the simple cases by assuming the following.

(i) First, we assume that the pre-shock matter and the post-shock matter are in thermodynamic equilibrium with temperatures T_1 and T_2, respectively. Later, we also consider the post-shock matter which is in thermal equilibrium but not in chemical equilibrium.

(ii) We take a nonrelativistic-nondegenerate hydrogen plasma as a pre-shock matter, which we shall also refer to as an electron-ion plasma. It is assumed that the electrons and ions are Maxwellian.

(iii) We consider the temperature range,

$$m_e c^2 \le k_B T_2 \le m_\mu c^2 , \tag{2.1}$$

so that electron-positron pairs are the only pairs which need to be considered. Here, m_μ is the mass of the muon. In certain cases, we relax this constraint (2.1) when pair creation is unimportant.

(iv) The dynamics of pair production is by itself a very important problem. However, it is outside the scope of the present paper. We simply consider the cases where pairs are present or absent in the post-shock matter, and qualitatively discuss the conditions under which such cases are realized.

(v) We assume that the pairs and the photons are confined to the system. Therefore, we are assuming that the mean free path of the electrons, positrons, and photons are much shorter than the characteristic length scale of the system.

(vi) When neutrinos are produced, they may freely escape from the system and act as an energy sink. We shall neglect such an effect. If neutrinos are produced and their mean free path is much smaller than the characteristic length scale of the system, they are trapped. In such a case, the neutrinos may be treated as another blackbody and the generalization is straightforward.

Under these conditions, the ions and the electrons contribute to the energy density of the pre-shock matter as

$$\varepsilon_{i1} = n_{b1} m_p c^2 + \tfrac{3}{2} n_{b1} k_B T_1 , \tag{2.2}$$

$$\varepsilon_{e1} = n_{e1} m_e c^2 + \tfrac{3}{2} n_{e1} k_B T_1 . \tag{2.3}$$

Thus the total energy density of the pre-shock matter is dominated by the ion rest-mass energy density,

$$\varepsilon_1 = \varepsilon_{i1} + \varepsilon_{e1} \simeq n_{b1} m_p c^2 . \tag{2.4}$$

Then, the ion, electron, and total pressures are given, respectively, by

$$p_{i1} = n_{b1} k_B T_1 , \tag{2.5}$$

$$p_{e1} = n_{e1} k_B T_1 , \tag{2.6}$$

$$p_1 = p_{i1} + p_{e1} = 2 n_{b1} k_B T_1 . \tag{2.7}$$

Let us now turn to the post-shock matter. When the pre-shock matter, which is composed of nonrelativistic electrons and ions, is heated by the shock, the resultant post-shock matter may also contain thermally produced electron-positron pairs and photons, in addition to the original electrons and ions. Let us recall that the state of a homogeneous one-component system in thermodynamic equilibrium may be characterized by two thermodynamic variables. In spite of its multicomponent nature of the post-shock matter, it may also be characterized by two thermodynamic variables. The baryon number density n_{b2} and the temperature T_2 are sufficient to specify the state of the system uniquely. The number density of the (blackbody) photons is a function of the temperature and the number densities of the electrons and positrons are functions of the temperature and the baryon number density.[7]

Throughout this paper, we are interested in the case where the electrons and ions in the post-shock matter are either nondegenerate (with a Maxwellian distribution) or partially degenerate (with a Fermi distribution with zero chemical potential). Therefore, let us first establish the degeneracy boundaries in the n_{e2}-T_2 (n_{b2}-T_2) plane. For the nonrelativistic electrons with a number density n_{e2}, the Fermi wave number is given by

$$k_{eF} = (3\pi^2 n_{e2})^{1/3} , \tag{2.8}$$

so that in terms of the electron Fermi temperature,

$$T_{eF} \equiv \frac{\hbar^2 k_{eF}^2}{2 m_e k_B} , \tag{2.9}$$

the degeneracy boundary $T_2 = T_{eF}$ may be expressed as

$$\frac{k_B T_2}{m_e c^2} = \frac{(3\pi^2)^{2/3}}{2} \left[\frac{n_{e2}}{n_0} \right]^{2/3} , \tag{2.10}$$

where $n_0 \equiv (m_e c/\hbar)^3 = 1.74 \times 10^{31}$ cm^{-3}. Similarly, for the nonrelativistic ions with a number density n_{b2}, the Fermi wave number is

$$k_{iF} = (3\pi^2 n_{b2})^{1/3} , \tag{2.11}$$

and the Fermi temperature is

$$T_{iF} \equiv \frac{\hbar^2 k_{iF}^2}{2 m_p k_B} . \tag{2.12}$$

Thus the degeneracy boundary $T_2 = T_{iF}$ may be written as

$$\frac{k_B T_2}{m_e c^2} = \frac{(3\pi^2)^{2/3}}{2} \frac{m_e}{m_p} \left[\frac{n_{b2}}{n_0} \right]^{2/3} . \tag{2.13}$$

The condition for nondegeneracy ($T_2 \gg T_{eF}$ and $T_2 \gg T_{iF}$) may be derived from the condition for the Maxwell-Boltzmann statistics in terms of the chemical potential of a nonrelativistic Maxwellian gas. [See Eq. (A38) in Appendix A.]

For the extremely relativistic electrons and ions with number densities n_{e2} and n_{b2}, the Fermi temperatures are

$$T_{eF} = \hbar c k_{eF} / k_B , \tag{2.14}$$

$$T_{iF} = \hbar c k_{iF} / k_B , \tag{2.15}$$

so that the boundaries $T_2 = T_{eF}$ and $T_2 = T_{iF}$ may be expressed as

$$\frac{k_B T_2}{m_e c^2} = (3\pi^2)^{1/3} \left[\frac{n_{e2}}{n_0} \right]^{1/3} , \tag{2.16}$$

$$\frac{k_B T_2}{m_e c^2} = (3\pi^2)^{1/3} \left[\frac{n_{b2}}{n_0} \right]^{1/3} , \tag{2.17}$$

respectively. In Eqs. (2.16) and (2.17), the electron mass m_e enters as a dummy variable. These boundaries [Eqs. (2.10), (2.13), (2.16), and (2.17)] are shown in Fig. 1. When the temperature is higher (or the number density is lower) than those given in Eqs. (2.16) and (2.17), the extremely relativistic electrons and ions are nondegenerate, respectively. These constraints may also be derived as the condition for the Maxwell-Boltzmann statistics from the expression for the chemical potential of an extremely relativistic Maxwellian gas. [See Eq. (A53) in Appendix A.] One may derive some of the conditions on degeneracy by using an alternative argument. Let us remark that, in addition to the baryon number density n_{b2}, there is another number density which is characteristic to the relativistic electrons. When the electrons are relativistic such that the temperature exceeds the electron-rest-mass energy, $k_B T_2 \gg m_e c^2$, there is only one energy scale $k_B T$, which in turn gives a length scale $\hbar c / k_B T_2$ (i.e., the thermal de Broglie wavelength).[8] Therefore, the quantity $(k_B T_2 / \hbar c)^3$ gives another characteristic number density of the system in addition to the baryon number density n_{b2}. Thus it is natural to classify the post-shock matter into two types, depending upon whether

$$n_{b2} < \left[\frac{k_B T_2}{\hbar c} \right]^3 , \tag{2.18}$$

or

$$n_{b2} > \left[\frac{k_B T_2}{\hbar c} \right]^3 . \tag{2.19}$$

For the electron-ion plasma ($n_{e2} = n_{b2}$), the condition that the electrons to be nondegenerate [the higher temperature and lower electron number density side of (2.16)] is essentially the same as (2.18) in the temperature range (2.1). For ions, (2.18) is essentially the same as the condition that they are nondegenerate [the higher temperature and lower baryon number density side of (2.17)] in the temperature range ($k_B T_2 \gg m_p c^2$). Thus we shall omit the case in which (2.19) holds from our consideration, and only treat the case in which (2.18) holds. In particular, we shall consider only the limiting case when the inequality holds strongly, i.e., when

$$n_{b2} \ll \left[\frac{k_B T_2}{\hbar c} \right]^3 \equiv n_0 \left[\frac{k_B T_2}{m_e c^2} \right]^3 . \tag{2.20}$$

Here, let us examine the condition for thermodynamic equilibrium. First, the post-shock matter must be in thermal equilibrium. This means that the time scale of interest [the characteristic (hydrodynamic) time scale of the system] τ is much larger than the thermal relaxation time scale τ_T. Second, the post-shock matter must be in chemical equilibrium. In the present case, the composition changes by pair creation, so that the time scale for pair creation τ_{pair} must be much shorter than τ. Therefore, thermodynamic equilibrium is reached when

$$\tau \gg \tau_T, \tau_{pair} . \tag{2.21}$$

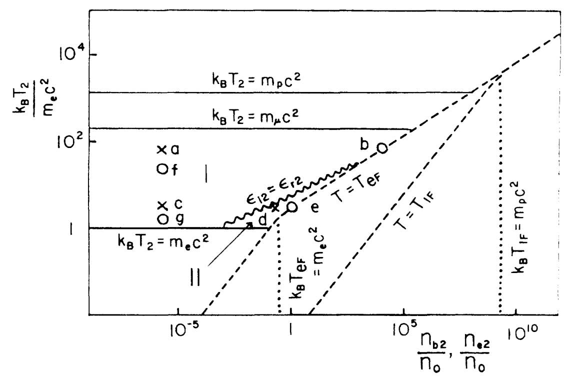

FIG. 1. Classification of the post-shock matter in the n_{b2}-T_2 (and n_{e2}-T_2) plane. The n_{e2} axis is superposed on the n_{b2} axis. The horizontal solid lines correspond to $k_B T_2 = m_e c^2$, $m_\mu c^2$, and $m_p c^2$ from the bottom, respectively. The dashed curves express the degeneracy boundaries for the electrons and the ions. They merge into one line at high temperatures and densities. The dotted vertical lines divide the density region above which the degenerate electrons (the one on the left) and the degenerate ions (the one on the right) become relativistic. The wavy line defines the two regions where the radiation is dominant (on the higher-T_2 and lower-n_{b2} side) or the baryons are dominant (on the lower-T_2 and higher-n_{b2} side) in the energy density of the pair plasma. Regions I and II are bounded by these curves and lines. Region III is defined to be the combined area of region I and region II (the small triangular-shaped area) below the horizontal line $k_B T_2 / m_e c^2 = 100$. The crosses in the figure refer to the sets of values for the baryon number density and the temperature (n_{b2}, T_{i2}), while the circles refer to the sets of values for the electron number density and the temperature (n_{e2}, T_{e2}). These crosses and circles a-g are used as the examples of classification. (See Table II.)

A. The pair plasma

When (2.18) holds and the thermodynamic equilibrium is reached, the electrons and positrons have a Fermi distribution with zero chemical potential and photons have a Planck distribution. Then, their number densities are given by[9,10]

$$n_{e-} \simeq n_{e+} \simeq \frac{3\zeta(3)}{2\pi^2}\left[\frac{k_B T}{\hbar c}\right]^3 , \tag{2.22}$$

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left[\frac{k_B T}{\hbar c}\right]^3 , \tag{2.23}$$

where $\zeta(3)$ is Riemann's ζ function. These number densities are much larger than that of ions:

$$n_{e-} \sim n_{e+} \sim n_\gamma \gg n_{b2} . \tag{2.24}$$

We thus simply call this state a pair plasma. The use of Eq. (2.22) (with n_{e2} replaced by n_{e-}) in Eqs. (2.8) and (2.14) yields

$$T_{eF} \simeq \left[\frac{9\zeta(3)}{2}\right]^{1/3} T = 1.76T . \tag{2.25}$$

One thus sees that the state of the electrons in the n_{e2}-T_2 plane lies close to the degeneracy boundary (2.16) for a given temperature. This is because the electrons have a Fermi distribution with zero chemical potential. Since the ions are nonrelativistic their energy density is given by

$$\varepsilon_{i2} = n_{b2} m_p c^2 + \tfrac{3}{2} n_{b2} k_B T_2 , \tag{2.26}$$

while the combined energy density of the electrons, positrons, and photons (which we shall refer to as the radiation for simplicity) is

$$\varepsilon_{r2} = \xi a T_2^4 \tag{2.27a}$$

$$= \frac{11\pi^2}{60} n_0 m_e c^2 \left[\frac{k_B T_2}{m_e c^2}\right]^4 , \tag{2.27b}$$

where $a \equiv \pi^2 k_B^4 / 15\hbar^3 c^3$, and we only consider electron-positron pairs and photons as a radiation, in which case $\xi = \frac{11}{4}$. In general,

$$\xi = 1 + \tfrac{7}{8} \times (\text{number of spin-}\tfrac{1}{2}\text{ charged lepton species}) + \tfrac{7}{16} \times (\text{number of neutrino species}) . \tag{2.28}$$

The total energy density is

$$\varepsilon_2 = \varepsilon_{i2} + \varepsilon_{r2} . \tag{2.29}$$

Similarly, the ion, radiation, and total pressures are given, respectively, as

$$p_{i2} = n_{b2} k_B T_2 , \tag{2.30}$$

$$p_{r2} = \tfrac{1}{3} \xi a T_2^4 \tag{2.31a}$$

$$= \frac{11\pi^2}{180} n_0 m_e c^2 \left[\frac{k_B T_2}{m_e c^2}\right]^4 , \tag{2.31b}$$

$$p_2 = p_{i2} + p_{r2} . \tag{2.32}$$

With these equations of state, one may compare some of the thermodynamic quantities of the nonrelativistic ions and those of the radiation for a given temperature and a baryon number density. First, from Eqs. (2.30) and (2.31), the inequality

$$p_{i2} \lesseqgtr p_{r2} \tag{2.33}$$

holds when

$$\frac{k_B T_2}{m_e c^2} \gtreqless \left[\frac{180}{11\pi^2}\right]^{1/3} \left[\frac{n_{b2}}{n_0}\right]^{1/3} \tag{2.34a}$$

$$= 1.18 \left[\frac{n_{b2}}{n_0}\right]^{1/3} . \tag{2.34b}$$

Second, from Eqs. (2.26) and (2.27), the inequality

$$\varepsilon_{i2} \lesseqgtr \varepsilon_{r2} \tag{2.35}$$

holds when

$$\frac{k_B T_2}{m_e c^2} \gtreqless \left[\frac{60}{11\pi^2}\frac{m_p}{m_e}\right]^{1/4} \left[\frac{n_{b2}}{n_0}\right]^{1/4} \tag{2.36a}$$

$$= 5.64 \left[\frac{n_{b2}}{n_0}\right]^{1/4} , \tag{2.36b}$$

where only the rest-mass energy density is retained in Eq. (2.26). With the condition (2.20), one sees from Eqs. (2.33)–(2.34) that the pressure of the radiation always dominates over that of the ions ($p_{i2} \ll p_{r2}$). On the other hand, for a given temperature range (2.1) and with the condition (2.20), one finds that the ratio of the energy densities can be either greater or less than unity. Therefore, one can further classify the pair plasma into two (limiting) categories, depending upon whether the radiation or the ions dominate in the energy density

$$\varepsilon_{i2} \ll \varepsilon_{r2};\ \text{i.e.,}\ \frac{k_B T_2}{m_e c^2} \gg \left[\frac{60}{11\pi^2}\frac{m_p}{m_e}\right]^{1/4} \left[\frac{n_{b2}}{n_0}\right]^{1/4} \quad (\text{case I}) , \tag{2.37}$$

or

$$\varepsilon_{i2} \gg \varepsilon_{r2};\ \text{i.e.,}\ \frac{k_B T_2}{m_e c^2} \ll \left[\frac{60}{11\pi^2}\frac{m_p}{m_e}\right]^{1/4} \left[\frac{n_{b2}}{n_0}\right]^{1/4} \quad (\text{case II}) , \tag{2.38}$$

respectively. We have thus defined two cases for the pair plasma.

Let us now compare the heat capacity. From Eq. (2.27) the heat capacity of the radiation is

$$C_{rV} \equiv \left. \frac{\partial E_r}{\partial T_2} \right|_V = V \frac{\partial \varepsilon_{r2}}{\partial T_2} = 4\xi a T_2^3 V \ . \tag{2.39}$$

Thus

$$C_{rV}/V = 4\xi a T_2^3 \tag{2.40a}$$

$$= \frac{11\pi^2}{15} n_0 \left[\frac{k_B T_2}{m_e c^2} \right]^3 k_B \ , \tag{2.40b}$$

where $E_r = \varepsilon_{r2} V$. On the other hand, for the nonrelativistic ions,

$$C_{iV}/V = \tfrac{3}{2} n_{b2} k_B \ . \tag{2.41}$$

Therefore, from Eqs. (2.40) and (2.41) the ratio of the heat capacities is

$$\frac{C_{iV}}{C_{rV}} = \frac{45}{22\pi^2} \left[\frac{n_{b2}}{n_0} \right] \left[\frac{k_B T_2}{m_e c^2} \right]^{-3} . \tag{2.42}$$

Equation (2.42) shows that the pair plasma acts as a huge heat bath at temperatures above the pair-creation threshold $\gtrsim m_e c^2/k_B$. In order to produce the post-shock matter with an energy density $\sim n_0 m_e c^2 \sim 10^{28} m_p c^2$, a large bulk kinetic energy is necessary for the pre-shock matter.

B. The electron-ion plasma

Let us now consider the case in which the post-shock matter does not contain positrons. Such a case is of interest, since it provides a contrasting case for the pair plasma, which we have just discussed in Sec. II A. It should be noted that such a plasma without positrons can exist only temporarily when certain conditions on the time scales are satisfied. This is because when sufficient time has elapsed, pair creation will necessarily occur at temperatures $k_B T/m_e c^2 \gtrsim 1$.

Let us begin with considering some of the relevant time scales. The relaxation time scales of a relativistic Maxwellian plasma due to electron-electron, proton-proton, and electron-proton collision in the temperature range $k_B T/m_e c^2 \gtrsim 1$ have been calculated and compared with various cooling time scales.[11] The cooling of protons occurs mainly due to pion production. At temperatures $k_B T_i/m_e c^2 \gtrsim 100$, the cooling time scale due to pion production (τ_{Ci}) becomes shorter than the proton-proton relaxation time scale (τ_{ii}).[11] Therefore, the protons can maintain a Maxwell distribution in the temperature range

$$1 \lesssim k_B T_i/m_e c^2 \lesssim 100 \ . \tag{2.43}$$

On the other hand, bremsstrahlung is the major mechanism of cooling the electrons. At temperatures $k_B T_e/m_e c^2 \gtrsim 10$, the cooling time scale due to bremsstrahlung (τ_{Ce}) becomes shorter than the electron-electron relaxation time scale (τ_{ee}).[11] Therefore, in order for the electrons to maintain a Maxwell distribution the temperature of the electrons must be

$$1 \lesssim k_B T_e/m_e c^2 \lesssim 10 \ . \tag{2.44}$$

In view of the conditions (2.43) and (2.44), it seems appropriate to consider a two-temperature electron-ion plasma as the post-shock matter. In fact, a two-temperature plasma may be produced quite naturally. Seen in the frame in which the post-shock matter is at rest, the most kinetic energy of the pre-shock matter is carried by the ions. Thus, when the ions and electrons are separately thermalized, one can naturally expect that

$$T_{i2} \gg T_{e2} \ . \tag{2.45}$$

We thus specifically consider a two-temperature Maxwellian electron-ion plasma as post-shock matter. The temperature of the ions is assumed to be higher than that of the electrons [(2.45)] and it is assumed to be in the range (2.43) and (2.44), respectively. When these conditions are satisfied, the plasma can maintain Maxwell distributions for the ions and electrons without being affected by the cooling processes that include pair production. Again, it is to be remarked that such a plasma exists for a certain period in the downstream of the post-shock matter. After a certain time, the electron-ion relaxation processes, pair-creation processes, and cooling processes start to modify the temperatures, distribution, and the composition of the plasma. One can estimate the characteristic time scale within which such a plasma can exist. Using the values for the upper bounds on the temperatures of the ions and electrons, one finds[11] the cooling time scales for both ions and electrons to be on the order of

$$\tau_C \simeq 10^2 \tau_{\rm Th} \ , \tag{2.46}$$

where $\tau_{\rm Th} \equiv 1/n_e \sigma_T c$, and σ_T is the Thomson cross section.

To summarize, the relations among the different time scales are for the ions,

$$\tau_{ii} \ll \tau \ll \tau_{Ci} \ , \tag{2.47}$$

and, for the electrons,

$$\tau_{ee} \ll \tau \ll \tau_{Ce}, \tau_{\rm pair} \ . \tag{2.48}$$

Let us now specify the energy density and the pressure of the plasma under consideration. We shall still assume that the inequality (2.20) holds. However, the electron number density is not given by Eq. (2.22) in the present case. The electrons and ions are taken to be a relativistic and nonrelativistic Maxwellian gases with temperatures T_{e2} and T_{i2}, respectively. Then, the energy density may be expressed as

$$\varepsilon_2 = \varepsilon_{i2} + \varepsilon_{e2} \ , \tag{2.49}$$

where

$$\varepsilon_{i2} = n_{b2} m_p c^2 \left[1 + \frac{3}{2} \frac{k_B T_{i2}}{m_p c^2} + \frac{15}{8} \left[\frac{k_B T_{i2}}{m_p c^2} \right]^2 + \cdots \right] , \tag{2.50}$$

$$\varepsilon_{e2} = 3 n_{b2} k_B T_{e2} \left[1 + \frac{1}{6} \left[\frac{m_e c^2}{k_B T_{e2}} \right]^2 + \cdots \right] . \tag{2.51}$$

TABLE I. Classification of three distinct cases (I, II, and III) of the post-shock matter for given baryon number density (n_{b2}) and temperature (T_2), where $n_0=(m_e c/\hbar)^3$, $t=k_B T_2/m_e c^2$, $\tau_{\rm pair}$ is the characteristic time scale for pair creation; τ is the hydrodynamic time scale; ε_{i2}, ε_{r2}, and ε_{e2} are the energy densities of the ions, the radiation (the electron-positron pairs and the photons), and the electrons, respectively; and p_{i2}, p_{r2}, and p_{e2} are the respective pressures.

Classification	$\frac{n_{b2}}{n_0 t^3}$	$\frac{n_{b2}}{n_0(m_e/m_p)t^4}$	$\frac{\tau_{\rm pair}}{\tau}$	Composition of the post-shock matter	Energy density	Pressure
case I	$\ll 1$	$\ll 1$	$\ll 1$	e^-e^+ plasma	$\varepsilon_{i2}\ll\varepsilon_{r2}$	$p_{i2}\ll p_{r2}$
case II	$\ll 1$	$\gg 1$	$\ll 1$	e^-e^+ plasma	$\varepsilon_{i2}\gg\varepsilon_{r2}$	$p_{i2}\ll p_{r2}$
case III	$\ll 1$	arbitrary	$\gg 1$	e-i plasma[a,b]	$\varepsilon_{i2}\gg\varepsilon_{e2}$	$p_{i2}\gg p_{e2}$

[a]Time scale too short to create pairs.
[b]The electrons and ions are separately thermal equilibrium at temperatures T_{e2} and $T_{i2}(\gg T_{e2})$, respectively.

[See Eqs. (A29) and (A44) in Appendix A.] Since (2.45) holds, one can neglect the contribution of the electrons to the energy density, so that the major contribution comes from the ions. Thus

$$\varepsilon_2\simeq\varepsilon_{i2}=n_{b2}m_p c^2\left[1+\frac{3}{2}\frac{k_B T_{i2}}{m_p c^2}+\cdots\right] . \tag{2.52}$$

Similarly, the pressure is

$$p_2=p_{i2}+p_{e2}\simeq p_{i2}=n_{b2}k_B T_{i2} , \tag{2.53}$$

where

$$p_{i2}=n_{b2}k_B T_{i2} , \tag{2.54}$$

$$p_{e2}=n_{b2}k_B T_{e2} . \tag{2.55}$$

Since the temperature is in the range (2.43), (2.44), and (2.45), it follows from Eqs. (2.50) and (2.51) that

$$\varepsilon_{i2}\gg\varepsilon_{e2} \tag{2.56}$$

and

$$p_{i2}\gg p_{e2} . \tag{2.57}$$

Thus the ions dominate in the energy density and pressure. We shall refer to this post-shock matter as the case III.

C. The classification

We can now summarize the three cases defined so far in terms of the different time scales and the conditions on the baryon number density for a given temperature. In all these three cases, for the electrons to be nondegenerate, the set of parameters (n_{e2},T_2) must lie on the higher-temperature and lower-density side of Eqs. (2.10) and (2.16), while for the ions to be nondegenerate, (n_{b2},T_2) must lie on the higher-temperature and lower-density side of Eqs. (2.13) and (2.17). The conditions on the temperature range and the time scales are (2.20) and (2.21) for cases I and II, and (2.20), (2.43)–(2.45), (2.47), and (2.48) for case III, respectively. In addition, the conditions for the energy density are (2.37) for case I and (2.38) for case II. As we shall see in Sec. III, when one of the above conditions is strongly satisfied, the results are greatly simplified and one can obtain analytic expressions that relate the pre-shock and pre-shock quantities in a transparent form. Thus the above classification enables us to understand each physically representative case. Table I summarizes the characteristic features of the post-shock matter in the above classification scheme. Figure 1 shows the above three cases in the n_{b2}-T_2 (n_{e2}-T_2) plane. To illustrate the classification, several sets of parameters (n_{b2},T_2 and n_{e2},T_2) are shown as circles and crosses a–g in Fig. 1, and their characteristic features are listed in Table II.

III. THERMODYNAMIC QUANTITIES OF THE POST-SHOCK MATTER: THE PAIR PLASMA

In this section we express the physical quantities of the post-shock matter in terms of the pre-shock quantities. We treat two of the three qualitatively distinct cases (case I and case II), which are classified in Sec. II, separately.

TABLE II. Examples of the classification. The sets of parameters n_{b2},T_{i2} and n_{e2},T_{e2}, which are shown as circles and crosses a–g in Fig. 1, are used to illustrate the classification scheme of Table I. The first and second columns list the baryon number density (n_{b2}) and the electron number density (n_{e2}), respectively, to which the circles and the crosses a–g in Fig. 1 correspond. The third column shows the composition which is either the electron-ion plasma (e–i) or the pair plasma (e^-e^+). The fourth and fifth columns show the distribution function [either the Maxwellian (M) or the Fermi distribution with zero chemical potential (F)] of the ions and the electrons, respectively. The last column indicates the classification.

n_{b2}	n_{e2}	Composition	i	e^-	Classification
a	b	e^-e^+	M	F	I
c	e	e^-e^+	M	F	I
d	e	e^-e^+	M	F	II
a	f	e-i	M	M	III
c	g	e-i	M	M	III

There are one kinematical (velocity v) and two thermodynamic quantities (e.g., the baryon number density n_b and temperature T) which specify the pre-shock matter and the post-shock matter, respectively. Therefore, there are six physical quantities altogether which characterize the shock. On the other hand, there are three conservation laws. Thus, only three of the six physical quantities are independent. In certain special cases, which we consider later in this section, the temperature T_1 may be neglected in the energy density ε_1 of the pre-shock matter [as in Eq. (2.4)] as far as $m_p c^2 \gg k_B T_1$. Then, the pre-shock matter is characterized by only one thermodynamic variable n_{b1}. In such a case, there remain only two independent physical quantities after these conservation laws are used. This means that two independent variables, such as n_{b1} and v_1, are sufficient to determine all others (such as n_{b2}, T_2, and v_2).

The baryon number, energy, and momentum fluxes are conserved across the shock. These three conservation laws lead to the following boundary conditions in the frame in which the shock is at rest:

$$n_{b1}\gamma_1 v_1 = n_{b2}\gamma_2 v_2 , \tag{3.1}$$

$$w_1\gamma_1^2 v_1 = w_2\gamma_2^2 v_2 , \tag{3.2}$$

$$w_1\gamma_1^2\beta_1^2 + p_1 = w_2\gamma_2^2\beta_2^2 + p_2 , \tag{3.3}$$

where $\beta = v/c$, $\gamma = 1/(1-\beta^2)^{1/2}$, and $w = \varepsilon + p$ is the enthalpy density. The thermodynamic quantities express the values in the frame in which the fluid element is at rest. Within the energy scale under consideration, the lepton number is also conserved. However, from charge neutrality, this conservation law does not give another independent boundary condition across the shock.

Eliminating the velocities v_1 and v_2 from Eqs. (3.1)–(3.3), one obtains the Rankine-Hugoniot relation,[1-3]

$$\frac{w_1^2}{n_{b1}^2} - \frac{w_2^2}{n_{b2}^2} + (p_2 - p_1)\left[\frac{w_1}{n_{b1}^2} + \frac{w_2}{n_{b2}^2}\right] = 0 . \tag{3.4}$$

The Rankine-Hugoniot relation expresses one of the three conservation laws in terms of the four independent thermodynamic variables, where two kinematical variables have been eliminated by the other two conservation laws. The velocities may also be expressed in terms of the thermodynamic variables

$$\beta_1 \equiv \frac{v_1}{c} = \left[\frac{(p_2-p_1)(\varepsilon_2+p_1)}{(\varepsilon_2-\varepsilon_1)(\varepsilon_1+p_2)}\right]^{1/2} , \tag{3.5}$$

$$\beta_2 \equiv \frac{v_2}{c} = \left[\frac{(p_2-p_1)(\varepsilon_1+p_2)}{(\varepsilon_2-\varepsilon_1)(\varepsilon_2+p_1)}\right]^{1/2} , \tag{3.6}$$

$$\beta_{12} \equiv \frac{v_{12}}{c} = \left[\frac{(p_2-p_1)(\varepsilon_2-\varepsilon_1)}{(\varepsilon_1+p_2)(\varepsilon_2+p_1)}\right]^{1/2} , \tag{3.7}$$

where v_{12} is the relative velocity of the pre-shock matter and the post-shock matter. Here one may use the explicit forms for the thermodynamic quantities ε_1 [Eqs. (2.2)–(2.4)], ε_2 [Eqs. (2.26)–(2.29)], p_1 [Eq. (2.7)], and p_2 [Eqs. (2.30)–(2.32)] to write Eq. (3.5) as

$$\beta_1 = \frac{v_1}{c} \simeq \left[\frac{(1+y-z)\left[1+\frac{1}{3}\frac{n_{b2}}{n_{b1}}x+\frac{1}{2}y+\frac{1}{3}z\right]}{\left[1+\frac{1}{3}\left[\frac{n_{b2}}{n_{b1}}-1\right]x+\frac{1}{2}y-\frac{1}{2}z\right](1+x+y+\frac{3}{2}z)}\right]^{1/2} , \tag{3.5'}$$

where $x \equiv n_{b1}m_p c^2/p_{r2}$, $y \equiv p_{i2}/p_{r2}$, $z \equiv p_1/p_{r2}$, and we have neglected the terms proportional to m_e/m_p.

A. Case I — radiation dominant case

When the energy density and the pressure of the post-shock matter are radiation dominated, $\varepsilon_{i2} \ll \varepsilon_{r2} \simeq \varepsilon_2$ and $p_{i2} \ll p_{r2} \simeq p_2$. Since we consider nonrelativistic ions, $p_{i2} \ll \varepsilon_{i2}$. It thus follows that

$$p_{i2} \ll \varepsilon_{i2} \ll p_{r2} \simeq \varepsilon_{r2} . \tag{3.8}$$

Neglecting the small terms on the orders of $m_e/m_p, p_1/n_{b1}m_p c^2 \simeq p_1/\varepsilon_1$ and $p_{i2}/n_{b1}m_p c^2$, one solves Eq. (3.4) to obtain

$$\frac{n_{b2}}{n_{b1}} \simeq 2\sqrt{3}\left[\frac{p_{r2}}{n_{b1}m_p c^2}\right]^{1/2} + \frac{7}{2} . \tag{3.9}$$

In the following we shall retain only the leading terms in $p_{r2}/n_{b1}m_p c^2$ ($\gg 1$).

The temperature ratio is given by

$$\frac{T_2}{T_1} \simeq 2\left[\frac{45}{\pi^2\xi}\right]^{1/4}\left[\frac{p_{r2}}{n_0 m_e c^2}\right]^{1/4}\left[\frac{p_1}{n_{b1}m_e c^2}\right]^{-1} . \tag{3.10}$$

One finds that for $p_{r2} \to \infty$ the ratio of the post-shock to pre-shock baryon number densities goes as $n_{b2}/n_{b1} \propto p_{r2}^{1/2}$, while the temperature ratio becomes $T_2/T_1 \propto p_{r2}^{1/4}$. This behavior is qualitatively different from the nonrelativistic ideal gas case, in which $n_2/n_1 \to (\Gamma+1)/(\Gamma-1)$, and $T_2/T_1 \propto p_2$ for a strong shock ($p_2 \gg p_1$), where Γ is the adiabatic index.[1] Similarly, from Eqs. (3.5)–(3.7), the velocities are given by

$$\frac{v_1}{c} \simeq 1 - \frac{\varepsilon_1}{\varepsilon_2} , \tag{3.11}$$

$$\frac{v_2}{c} \simeq \frac{1}{3}\left[1+\frac{2\varepsilon_1}{\varepsilon_2}\right] , \tag{3.12}$$

$$\frac{v_{12}}{c} \simeq 1 - \frac{2\varepsilon_1}{\varepsilon_2} , \tag{3.13}$$

which allow one to express ε_2, v_2, and v_{12} in terms of ε_1 and v_1:

$$\varepsilon_2 \simeq 2\gamma_1^2 \varepsilon_1 , \tag{3.14}$$

$$\frac{v_2}{c} \simeq \frac{1}{3}\left[1+\frac{1}{\gamma_1^2}\right] , \tag{3.15}$$

$$\frac{v_{12}}{c} \simeq 1-\frac{1}{\gamma_1^2} , \tag{3.16}$$

where $\gamma_1=(1-v_1^2/c^2)^{-1/2}$. Noting that $p_2 \simeq \frac{1}{3}\varepsilon_2$, $p_2 \simeq p_{r2}$, and $\varepsilon_1 \simeq n_{b1} m_p c^2$, one obtains from Eqs. (3.9) and (3.14)

$$\frac{n_{b2}}{n_{b1}} \simeq 2\sqrt{2}\gamma_1 , \tag{3.17}$$

and from Eq. (3.10),

$$\frac{k_B T_2}{m_e c^2} \simeq \left[\frac{30}{\pi^2 \xi}\frac{m_p}{m_e}\frac{n_{b1}}{n_0}\right]^{1/4} \gamma_1^{1/2} \tag{3.18a}$$

$$\simeq 6.71(n_{b1}/n_0)^{1/4}\gamma_1^{1/2} , \tag{3.18b}$$

where in Eq. (3.18b) $\xi=\frac{11}{4}$ was used. Equations (3.15), (3.17), and (3.18) specify the post-shock quantities in terms of the pre-shock quantities. One again sees that there are no upper limits on the degree of compression [Eq. (3.17)] and to the temperature reached by the post-shock matter [Eq. (3.18)],[12] in contrast to the nonrelativistic ideal gas case. Since the ion rest mass dominates in the energy density of the pre-shock matter, the temperature T_1 does not appear in Eqs. (3.15)–(3.18).

It is easy to show that Eq. (3.14) is a direct consequence of energy flux conservation. In order to illustrate the behavior of the post-shock energy density ε_2 as a function of the Lorentz factor of the pre-shock matter [Eq. (3.14)], let us calculate the bulk ion kinetic energy density of the pre-shock matter seen in the frame in which the post-shock matter is at rest (ε_1'). From Eq. (3.16) one finds

$$\varepsilon_1' = n_{b1}\gamma_{12} m_p \gamma_{12} c^2$$
$$\simeq \gamma_{12}^2 \varepsilon_1 \simeq \tfrac{1}{2}\gamma_1^2 \varepsilon_1 . \tag{3.19}$$

Roughly speaking, this gives the dependence of ε_2 on γ_1. The difference between the numerical factors in Eqs. (3.14) and (3.19) comes from the fact that the energy flux density rather than the energy density is continuous across the shock together with the fact that the work done by the pressure also contributes.

In the extremely relativistic limit ($\gamma_1 \gg 1$), Eq. (3.15) becomes $\beta_2 \simeq \frac{1}{3}$. Physically, this is a general feature which is characteristic of the strong shock with extremely relativistic matter and is independent of the specific form of the equation of state.[1] In fact, when the post-shock matter is extremely relativistic and when the pressure and the energy density of the pre-shock matter are negligible compared to those of the post-shock matter (strong shock), such that

$$p_2 \simeq \tfrac{1}{3}\varepsilon_2 \gg p_1, \varepsilon_1 , \tag{3.20}$$

Eq. (3.6) becomes

$$\beta_2 = \left[\frac{(p_2-p_1)(\varepsilon_1+p_2)}{(\varepsilon_2-\varepsilon_1)(\varepsilon_2+p_1)}\right]^{1/2}$$
$$\simeq \left[\frac{(\frac{1}{3}\varepsilon_2)(\frac{1}{3}\varepsilon_2)}{\varepsilon_2\varepsilon_2}\right]^{1/2} = \frac{1}{3} . \tag{3.21}$$

Similarly, Eq. (3.17) comes essentially from baryon number flux conservation. Here again, one can show that this is a general feature which is characteristic of the strong shock with extremely relativistic matter. In fact, from baryon number flux conservation (3.1) one can write the density ratio as

$$\frac{n_{b2}}{n_{b1}} = \frac{\gamma_1\beta_1}{\gamma_2\beta_2} . \tag{3.22}$$

For the strong shock with extremely relativistic post-shock matter, one may use (3.21) [$\gamma_2 \equiv 1/(1-\beta_2^2)^{1/2} = 3/2\sqrt{2}$] together with $\beta_1 \simeq 1$ in Eq. (3.22) to obtain

$$\frac{n_{b2}}{n_{b1}} \simeq \frac{\gamma_1}{(3/2\sqrt{2})(\frac{1}{3})} = 2\sqrt{2}\gamma_1 . \tag{3.23}$$

It is also straightforward to understand the dependence of the temperature of the post-shock matter on the baryon number density and the velocity of the pre-shock matter in Eq. (3.18a), $k_B T_2 \propto n_{b1}^{1/4}\gamma_1^{1/2}$. The use of the equations of state (2.4) and (2.27) in Eq. (3.14) immediately leads to Eq. (3.18a). Thus Eq. (3.18a) expresses energy flux conservation in terms of the physical quantities associated with the specific equations of state (2.4) and (2.27).

It is to be noted that the pre-shock matter is nonrelativistic and thus the rest-mass energy density of the ions dominates the energy density, so that its state is specified by one thermodynamic variable, the baryon number density n_{b1}, and one kinematical variable, the velocity v_1. In the present case, one finds that above two pre-shock quantities, n_{b1} and v_1, are sufficient to determine one thermodynamic variable T_2 and one kinematical variable v_2 of the post-shock matter. This is because blackbody radiation (electron-positron pairs and photons) dominates the post-shock matter, so that only one parameter, the temperature T_2, specifies the thermodynamic state. Other qualities simply derive from T_2; e.g., the number density is proportional to T_2^3 and the energy density is proportional to T_2^4, etc.

It is also to be noted that the major part of the energy density of the pre-shock matter consists in the bulk kinetic energy of the ions, which is then converted into the electron-positron pairs and the photons. The number densities of the pairs and the photons are determined in such a way that the baryon number, energy, and momentum flux densities are conserved across the shock.

The above results have been obtained under certain simplifying assumptions. Specifically, the conditions (2.1) and (2.37) have been used. It is obvious from Eq. (3.18) that these two conditions constrain the values of the baryon number density and the velocity of the pre-shock matter. Namely, the values of n_{b1} and v_1 must be such that the post-shock matter is heated up sufficiently but not too much [(2.1)] for the equations of state (2.26) and (2.27) to be valid, while radiation is dominant [(2.37)]. We now derive this self-consistency condition.

The use of Eq. (3.18) in the first condition (2.1), $1 < k_B T_2 / m_e c^2 < m_\mu / m_e \simeq 207$, leads to the constraint

$$4.93\times10^{-4}\gamma_1^{-2} < n_{b1}/n_0 < 9.01\times10^{5}\gamma_1^{-2} \ , \qquad (3.24)$$

where $\xi = \frac{11}{4}$ has been used. Since $\varepsilon_1 \simeq n_{b1} m_p c^2$, $\varepsilon_{i2} \simeq n_{b2} m_p c^2$, and $\varepsilon_{r2} \simeq \varepsilon_2$, the second condition $\varepsilon_{i2} \ll \varepsilon_{r2}$ may be written as $\varepsilon_{i2}/\varepsilon_{r2} \simeq (n_{b2}/n_{b1})(\varepsilon_1/\varepsilon_2) \ll 1$. The use of Eqs. (3.14) and (3.17) thus leads to

$$\gamma_1 \gg \sqrt{2} \ . \qquad (3.25)$$

The region in the N_{b1}-γ_1 plane which satisfies (3.24) and (3.25) is shown in Fig. 2.

From Fig. 2, the nature of the pre-shock matter in the present case may be summarized as follows.

(i) The relevant region in the n_{b1}-γ_1 parameter space has a simple shape, which is a consequence of the various simplifying assumptions including the classification scheme for the post-shock matter.

(ii) The velocity of the pre-shock matter relative to the shock boundary (v_1) must be extremely relativistic [(3.25)]. This requirement comes from the condition for the radiation dominance in the energy density of the post-shock matter [(2.37)].

(iii) When the velocity of the pre-shock matter is not so high, the baryon number density of the pre-shock matter must be on the order of $n_0 \equiv (m_e c/\hbar)^3 = 1.74\times10^{31}$ cm^3, which is rather high (e.g., $10^{-5} < n_{b1}/n_0 < 10^4$ for $\gamma_1 \simeq 10$).

(iv) At lower baryon number densities of the pre-shock matter, the required velocities of the pre-shock matter are relatively high (e.g., $10^4 \lesssim \gamma_1 \lesssim 10^8$ for $n_{b1}/n_0 \simeq 10^{-10}$ and $10^8 \lesssim \gamma_1 \lesssim 10^{13}$ for $n_{b1}/n_0 \simeq 10^{-20}$, etc.).

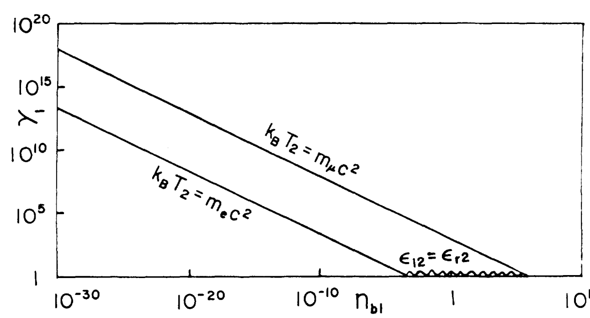

FIG. 2. Parameter region in the n_{b1}-γ_1 plane that satisfies the conditions $m_e c^2 < k_B T_2 < m_\mu c^2$ and $\varepsilon_{i2} \ll \varepsilon_{r2}$. The baryon number density (n_{b1}) and the Lorentz factor ($\gamma_1 \equiv 1/[1-(v_1/c)^2]^{1/2}$) of the pre-shock matter must lie between the solid lines indicated by $k_B T_2 = m_e c^2$ and $k_B T_2 = m_\mu c^2$ for the temperature of the post-shock matter to be in the range $m_e c^2 < k_B T_2 < m_\mu c^2$. Furthermore, the velocity of the pre-shock matter must be such that $\gamma_1 \gg \sqrt{2}$ in order for the energy density of the radiation to exceed that of the baryons in the post-shock matter. The wavy line expresses the condition $\varepsilon_{i2} = \varepsilon_{r2}$.

B. Case II — baryon dominant case

When the energy density of the post-shock matter is baryon dominated and the pressure is radiation dominated, $\varepsilon_{r2} \ll \varepsilon_{i2}$ and $p_{i2} \ll p_{r2}$. Since $p_{r2} \simeq \frac{1}{3}\varepsilon_{r2}$, the condition that the ions be nonrelativistic, $p_{i2} \ll \varepsilon_{i2}$, is satisfied in this case. It thus follows that

$$p_{i2} \ll p_{r2} \simeq \varepsilon_{r2} \ll \varepsilon_{i2} \ . \qquad (3.26)$$

Keeping only the terms to the first order in $p_{r2}/n_{b1} m_p c^2$ and neglecting other small terms in Eq. (3.4), one obtains

$$\frac{n_{b2}}{n_{b1}} \simeq 7 \ . \qquad (3.27)$$

The corrections to Eq. (3.27) are on the orders of p_{i2}/p_{r2}, p_1/p_{r2}, and $p_{r2}/n_{b1} m_p c^2$. Using the equations of state (2.7), (2.30), and (2.31) in Eq. (3.27), one obtains the temperature ratio

$$\frac{T_2}{T_1} \simeq \frac{2}{7} \frac{p_{i2}}{p_1} \ . \qquad (3.28)$$

It is to be noted that these results could be obtained from the nonrelativistic Rankine-Hugoniot relation with the use of appropriate (relativistic) equations of state (cf. Appendix B). It is easy to see from the derivation in Appendix B that the relation (3.27) simply follows if the major contribution to the pressure of the post-shock matter comes from extremely relativistic matter ($\varepsilon_2 \simeq 3p_2$) and if the pressure of the post-shock matter is much larger than that of the pre-shock matter ($p_1 \ll p_2$).

The velocity of the pre-shock matter may be expressed in terms of the thermodynamic quantities as Eq. (3.5′). Noting that $x \gg 1$, $y \ll 1$, and $z \ll 1$ in the present case one immediately obtains

$$\frac{v_1}{c} \simeq \left[\frac{1 + 3\left[\frac{n_{b1}}{n_{b2}}\right]\frac{1}{x}}{\left[\left[1-\frac{n_{b1}}{n_{b2}}\right] + 3\left[\frac{n_{b1}}{n_{b2}}\right]\frac{1}{x}\right](x+1)} \right]^{1/2} \qquad (3.29a)$$

$$\simeq \left[\frac{7}{6x}\right]^{1/2} = \left[\frac{7}{6}\frac{p_{r2}}{n_{b1} m_p c^2}\right]^{1/2} , \qquad (3.29b)$$

where Eq. (3.27) was used to obtain (3.29b). One notices that $v_1/c \ll 1$ since $x \gg 1$. Similarly, one finds that

$$\frac{v_2}{c} \simeq \left[\frac{p_{r2}}{42 n_{b1} m_p c^2}\right]^{1/2} \ll 1 \ . \qquad (3.30)$$

Therefore, the velocities of the pre-shock and post-shock matter are nonrelativistic in the present case. This is the reason why the results (3.27) and (3.28) could have been obtained from the nonrelativistic Rankine-Hugoniot relation together with appropriate equations of state.

Equation (3.27) relates the post-shock baryon number density to the pre-shock baryon number density. One finds that no kinematical parameter enters in this relation. Let us express other post-shock quantities in terms of the pre-shock quantities. From Eqs. (3.29b) and (3.30) one obtains the velocity of the post-shock matter in terms of the pre-shock velocity:

$$v_2 \simeq \tfrac{1}{7} v_1 \ . \tag{3.31}$$

One finds that the velocity of the post-shock matter is solely determined by the velocity of the pre-shock matter and no thermodynamic quantities enter. Equation (3.31) may be obtained more straightforwardly. Noting that the rest-mass energy density of the ions dominate in both the pre-shock and post-shock matter, one obtains from Eqs. (3.5) and (3.6)

$$\frac{\beta_2}{\beta_1} = \frac{\varepsilon_1 + p_2}{\varepsilon_2 + p_1} \tag{3.32a}$$

$$\simeq \frac{\varepsilon_{i1}}{\varepsilon_{i2}} \simeq \frac{n_{b1}}{n_{b2}} \ . \tag{3.32b}$$

The combination of (3.27) and (3.32b) gives (3.31).

It is also easy to see that the velocities of the pre-shock and post-shock matter are nonrelativistic. In fact, using Eq. (3.32b) in the expression for the ratio of the baryon number densities (3.22), which comes from baryon number flux conservation, one obtains

$$\gamma_1 / \gamma_2 \simeq 1 \ . \tag{3.33}$$

Equations (3.31) and (3.33) are compatible only when

$$\beta_1, \beta_2 \ll 1 \ . \tag{3.34}$$

With the use of Eq. (3.30) in Eq. (2.27) one obtains the temperature of the post-shock matter in terms of the velocity of the pre-shock matter:

$$\frac{k_B T_2}{m_e c^2} \simeq \left[\frac{1080}{77\pi^2} \frac{m_p}{m_e} \frac{n_{b1}}{n_0} \right]^{1/4} \left[\frac{v_1}{c} \right]^{1/2} \tag{3.35a}$$

$$\simeq 7.15 (n_{b1}/n_0)^{1/4} (v_1/c)^{1/2} \ . \tag{3.35b}$$

Let us now consider the self-consistency condition. The temperature of the post-shock matter is given by Eq. (3.35). The condition (2.1) for the temperature of the post-shock matter thus constrains the baryon number density and the velocity of the pre-shock matter as

$$3.83 \times 10^{-4} (v_1/c)^{-2} < n_{b1}/n_0 < 7.00 \times 10^5 (v_1/c)^{-2} \ . \tag{3.36}$$

The condition that the pressure of the radiation dominates that of the ions, $p_{r2} \gg p_{i2}$, gives

$$n_{b1}/n_0 \ll 9.79 \times 10^5 (v_1/c)^6 \ , \tag{3.37}$$

where Eqs. (2.30), (2.31), (3.27), and (3.35) have been used. Another condition that the energy density of the ions dominates that of the radiation in the post-shock matter turns out to be

$$v_1/c \ll 1 \ . \tag{3.38}$$

In fact, with the use of Eqs. (2.26), (2.27), (2.31), (3.27), and (3.29b), one finds that this condition is satisfied as far as $v_1/c \ll 1$. The region in the n_{b1}-β_1 plane which satisfies (3.36) and (3.37) is shown in Fig. 3.

Let us discuss the difference between the radiation dominant pair plasma (case I) and the baryon dominant pair plasma (case II) as the post-shock matter. First, whether the radiation or the baryons dominate in the energy density determines the velocity of the pre-shock matter to be either extremely relativistic [case I (3.25)] or nonrelativistic [case II (3.38)], respectively. This, then, brings about the entirely different dependence of the post-shock quantities on the pre-shock quantities: cf. (3.17) and and (3.27) for the ratios of the baryon number densities, (3.18) and (3.35) for the temperatures of the post-shock matter, and (3.15) and (3.31) for the velocities of the post-shock matter. Comparing Figs. 2 and 3, one finds the following additional features for the pre-shock matter.

(i) Compared to case I, the region in the (n_{b1}, β_1) plane that fulfills the condition for case II is rather narrow. Numerically, the smallness of this region may be seen from the condition on the velocity of the pre-shock matter, $0.1 \lesssim \beta_1 \ll 1$.

(ii) In addition, the baryon number density must be $10^{-3} \lesssim n_{b1}/n_0 \lesssim 10^5$, which is rather high.

(iii) The triangular-shaped region in Fig. 3 corresponds to a portion of the region which is below the wavy line and above the line $\gamma_1 = 1$ in Fig. 2. Therefore, the

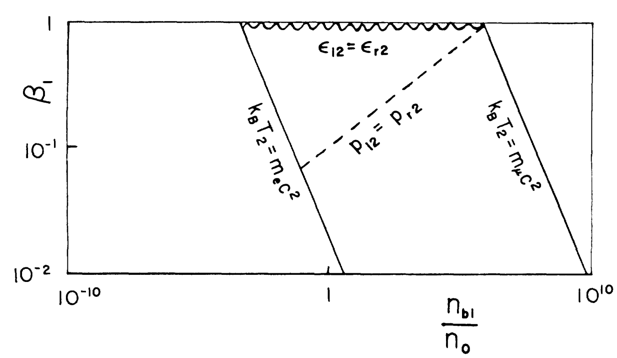

FIG. 3. Parameter region in the n_{b1}-β_1 plane that satisfies the conditions $m_e c^2 < k_B T_2 < m_\mu c^2$, $\varepsilon_{i2} \gg \varepsilon_{r2}$, and $p_{i2} \ll p_{r2}$ [the triangular-shaped region bounded by a wavy line ($\varepsilon_{i2} = \varepsilon_{r2}$), a dashed line ($p_{i2} = p_{r2}$), and a solid line ($k_B T_2 = m_e c^2$)]. The baryon number density (n_{b1}) and the velocity ($v_1 = c\beta_1$) of the pre-shock matter must lie between the solid lines indicated by $k_B T_2 = m_e c^2$ and $k_B T_2 = m_\mu c^2$ in order for the temperature of the post-shock matter to be in the range $m_e c^2 < k_B T_2 < m_\mu c^2$. In addition, the velocity of the pre-shock matter must be such that $\beta_1 \ll 1$ in order for the energy density of the baryons to exceed that of the radiation in the post-shock matter. (The wavy line expresses the condition $\varepsilon_{i2} = \varepsilon_{r2}$.) Furthermore, the pre-shock parameters (n_{b1}, β_1) must lie on the higher-velocity and lower-baryon number-density side of and away from the dashed line, which indicates the condition $p_{i2} = p_{r2}$, in order to satisfy the condition $p_{i2} \ll p_{r2}$.

relevant regions for cases I and II in Figs. 2 and 3 are complimentary. The division is made by the wavy lines in Figs. 2 and 3, which express the condition $\varepsilon_{i2}=\varepsilon_{r2}$.

(iv) The condition $k_B T_2 < m_\mu c^2$ is satisfied as far as $p_{i2} \ll p_{r2}$.

(v) The boundaries $k_B T_2 = m_e c^2$ and $k_B T_2 = m_\mu c^2$ continue from Fig. 2 to Fig. 3.

Finally, let us remark that the relation between the pre-shock quantities and the post-shock quantities may be viewed as a mapping from the parameter space (n_{b1}, γ_1) to another parameter space (n_{b2}, T_2) or (n_{e2}, T_2). In this sense, the triangular-shaped region in Fig. 3 corresponds to another triangular-shaped region in Fig. 1.

IV. THERMODYNAMIC QUANTITIES OF THE POST-SHOCK MATTER: THE RELATIVISTIC TWO-TEMPERATURE PLASMA

In this section, we treat the post-shock matter as a relativistic Maxwellian plasma and obtain the relations between the pre-shock and post-shock quantities, as has been done in Sec. III. We specifically consider a relativistic two-temperature electron-ion plasma in the temperature range (2.43), (2.44), and (2.45), which was classified as case III in Sec. II. In this case, the time scale for pair creation is so large that no appreciable number of positrons is present in thermal equilibrium.[13]

From Eqs. (2.2)–(2.7), the enthalpy density of the pre-shock matter is

$$w_1 \equiv \varepsilon_1 + p_1 \cong n_{b1} m_p c^2 , \tag{4.1}$$

where small terms of orders $k_B T_1 / m_p c^2$ and m_e/m_p have been neglected. (Note that the pre-shock matter is assumed to be nonrelativistic, such that $k_B T_1 \ll m_e c^2$.) On the other hand, the enthalpy density of the post-shock matter is given from Eqs. (2.49)–(2.55) as

$$w_2 \equiv \varepsilon_2 + p_2 \simeq n_{b2} m_p c^2 \left[1 + \frac{5}{2} \frac{k_B T_{i2}}{m_p c^2} \right] . \tag{4.2}$$

Using Eqs. (4.1) and (4.2) together with Eqs. (2.5)–(2.7) and (2.53)–(2.55) in the Rankine-Hugoniot relation (3.4), one obtains the density ratio as

$$\frac{n_{b2}}{n_{b1}} \simeq 4 . \tag{4.3}$$

Next, let us obtain the relations between the velocities and the temperature of the post-shock matter. Using the equations of state (2.2)–(2.7) and (2.49)–(2.55) in the expressions for the velocities (3.5)–(3.7), one immediately obtains

$$\beta_1 \simeq \frac{4}{\sqrt{3}} \left[\frac{k_B T_{i2}}{m_p c^2} \right]^{1/2} , \tag{4.4}$$

$$\beta_2 \simeq \frac{1}{\sqrt{3}} \left[\frac{k_B T_{i2}}{m_p c^2} \right]^{1/2} , \tag{4.5}$$

$$\beta_{12} \simeq \left[\frac{3 k_B T_{i2}}{m_p c^2} \right]^{1/2} . \tag{4.6}$$

One may invert (4.4) to write

$$\frac{k_B T_{i2}}{m_p c^2} \simeq \tfrac{3}{16} \beta_1^2 . \tag{4.7}$$

It is to be noted that Eq. (4.7) is closely related to energy flux conservation. When the pre-shock matter has a nonrelativistic velocity ($\beta_1 \ll 1$, $\gamma_1 \simeq 1$), the bulk kinetic energy of the pre-shock matter seen in the rest frame of the post-shock matter is $\propto \beta_1^2$. On the other hand, the energy density of the post-shock matter is linearly proportional to T_2 at temperatures $k_B T_2 \ll m_p c^2$ [cf. Eqs. (2.49)–(2.51)]. Thus Eq. (4.7) results.

Finally, let us derive the self-consistency condition such that the temperature of the post-shock matter remains in the range (2.43). Using Eq. (4.7) into (2.43), one obtains

$$0.05 \le \beta_1 \le 0.54 . \tag{4.8}$$

One sees that the self-consistency condition is given in terms of only one parameter, the velocity of the pre-shock matter v_1. This contrasts to cases I and II in Sec. III, where the self-consistency conditions are in terms of two parameters, the baryon number density n_{b1} and the velocity of the pre-shock matter v_1 [cf. (3.24), (3.25), (3.36), and (3.37)].

Let us first compare cases I and III by examining how the temperature of the post-shock matter (T_2) depends on the velocity of the pre-shock matter (v_1). In case III, the velocity $\beta_1 \gtrsim 0.05$ is enough to heat the ions to a temperature $k_B T_{i2} \gtrsim m_e c^2$, while the velocity $\beta_1 \lesssim 0.5$ is enough to heat up the post-shock matter to a temperature $k_B T_{i2} \lesssim 100 m_e c^2$. Therefore the required value for the velocity of the pre-shock matter is much smaller than that in case I [cf. (3.25)]. This is because all the bulk kinetic energy of the pre-shock matter is used to heat up the matter. Thus the temperature of the post-shock matter rises more steeply as a function of v_1 in contrast to the pair plasma case (case I), where the bulk kinetic energy of the pre-shock matter is used up in producing pairs.

Note, however, that the self-consistency condition for cases I and III are mutually exclusive [cf. (3.25) and (4.8)]. Thus only one (or none of) these cases is realized for a given set of pre-shock quantities.

Next, let us compare cases II and III. In these cases, there is an overlapping region for the pre-shock quantities [cf. (3.36), (3.37), and (4.8)]. For example, pre-shock matter with $n_{b1}/n_0 = 1$ and $\beta_1 = 0.3$ satisfies (3.36), (3.37), and (4.8) simultaneously. (See Fig. 3.) Such pre-shock matter can result in the post-shock matter of either classification. In case III, the use of the above values in Eq. (4.7) leads to the temperature of the post-shock matter to be $k_B T_{i2}/m_e c^2 \cong 31$. In case II, on the other hand, the above values in Eq. (3.35) give $k_B T_2 / m_e c^2 \cong 4$. One thus sees the difference in the temperature of the post-shock matter. As expected, when pairs are created (case II), the temperature of the post-shock matter does not rise as high as the case without pair creation (case III). This example demonstrates the differences in the post-shock matter between the cases with and without pair creation. One factor that determines which of these two cases is realized is the relation among the different time scales [cf. (2.21) for case II, and (2.47) and (2.48) for case III]. Note that only the conservation laws and thermodynamics cannot uniquely determine the state of the post-shock matter from a given pre-shock condition. This is because the final state depends critically on the various factors, such as the kinetic processes within the shock layer, the size of the shock, the relaxation time scales, the optical thickness, etc.

V. SUMMARY AND DISCUSSION

Let us summarize and discuss the results obtained in the present study. We have assumed the pre-shock matter to be a nonrelativistic electron-ion plasma and considered three types of plasma as the post-shock matter. They are the pair plasmas in thermodynamic equilibrium in which either the radiation or the baryons dominate in the energy density (cases I and II) and a relativistic two-temperature electron-ion plasma. In all three cases, analytic relations were obtained which express the post-shock quantities in terms of the pre-shock quantities.

Particle-pair creation adds another degree of freedom to how the bulk kinetic energy of the pre-shock matter is converted. If the dynamical time scales are such that particle pair creation is allowed [(2.21)], the resultant pair plasma contains the maximum number of electron-positron pairs, which leads to the minimum temperature rise for a given energy input from the pre-shock matter. In the case of radiation-dominated pair plasma (case I), in particular, the bulk kinetic energy of the pre-shock matter, which is proportional to γ_1^2 in the rest frame of the post-shock matter, is spent in producing pairs, whose energy density is proportional to T_2^4. Thus the temperature of the post-shock matter in such a case rises as $\gamma_1^{1/2}$. Therefore, the radiation-dominated pair plasma case sets a lower limit on the temperature that can be reached by the post-shock matter for a given set of pre-shock parameters (n_{b1},γ_1). In contrast, if the dynamical time scales are such that no pairs are created [(2.48)], the composition will not change due to the shock (i.e., the matter remains as an electron-ion plasma, case III), in which case the matter is maximally compressed and heated for a given energy input from the pre-shock matter. Therefore, the post-shock quantities, such as the temperature T_2, depends sensitively on the composition. In particular, the pre-shock matter that satisfies (3.36), (3.37), and (4.8) simultaneously can produce post-shock matter of either classification II or III. It is demonstrated that the post-shock conditions are appreciably different, depending on whether pairs are created or not. The conditions for these cases to be realized may be expressed in terms of the time scales as (2.21) for case II, and as (2.47) and (2.48) for case III. (Note that these are only necessary conditions.) Thus, it is possible that case III is first realized then followed by case II for the post-shock matter from a certain pre-shock condition. The present analysis, which is based on the conservation laws and thermodynamics, is insufficient for uniquely determining the final state from a given pre-shock condition. In this sense, cases II and III set two limits on the post-shock condition for a given pre-shock condition.

When the post-shock matter is composed of a radiation-dominated pair plasma (case I), the relations among the pre-shock and post-shock quantities take especially simple forms. For example, the velocity of the post-shock matter is $\simeq c/3$ [Eq. (3.15)], and the ratio of the baryon number densities (n_{b2}/n_{b1}) is a sole function of the velocity of the pre-shock matter [Eq. (3.17)]. This is because the post-shock matter is extremely relativistic and may be described by a simple equation of state $p_2 \simeq \varepsilon_2/3$.

For the radiation-dominated pair plasma (case I), the characteristic number density of the pair is $\sim n_0(k_B T_2/m_e c^2)^3$. Therefore, as seen from Fig. 2, the baryon number density of the pre-shock matter must be on the order of $n_0[\equiv (m_e c/\hbar)^3 = 1.74\times 10^{31}\ \mathrm{cm}^{-3}]$ with the pre-shock velocity $\gamma_1 > 1$ for such post-shock matter to be realized. Only when $\gamma_1 \gg 1$ may the baryon number density of the pre-shock matter be low, $n_{b1} \ll n_0$. Therefore, one finds a rather extreme condition, either $\gamma_1 \gg 1$ or $n_{b1} \simeq n_0$, on the pre-shock quantities for this case to be realized.

We have neglected a number of important problems, such as the dynamics and the mechanism of electron-positron pair creation, photon creation, and thermal equilibration, and the effects of magnetic fields and the optical thickness. These are the subjects of future study.

ACKNOWLEDGMENTS

I wish to thank David Seibert for extensively commenting on the original version of the manuscript. I also thank Jonathan Katz for encouragement.

APPENDIX A: THERMODYNAMIC PROPERTIES OF AN IDEAL RELATIVISTIC MAXWELL-BOLTZMANN GAS

In this appendix some of the thermodynamic properties of an ideal relativistic Maxwell-Boltzmann gas are summarized.[14] First, general expressions for the thermodynamic quantities are given. Next, they are reduced in two limiting cases—the nonrelativistic limit and the extremely relativistic limit.

Consider N particles of mass m and the internal degrees of freedom g in thermal equilibrium at a temperature T in a volume V. The single particle energy spectrum is assumed to be

$$\varepsilon_p = c(m^2c^2+p^2)^{1/2}\ , \tag{A1}$$

where p is the momentum. Then, the Gibbs sum is

$$Z_1 = gV\int \frac{d^3p}{(2\pi\hbar)^3} e^{-c(m^2c^2+p^2)^{1/2}/k_B T} \tag{A2a}$$

$$= \frac{gV}{2\pi^2}\left[\frac{mc}{\hbar}\right]^3 \frac{K_2(a)}{a}\ , \tag{A2b}$$

where $a \equiv mc^2/k_BT$ and $K_\nu(a)$ is the modified Bessel function of the second kind (the Macdonald function) of order ν (cf. 9.6.23 of Ref. 15). The partition function of the system is

$$Z_N = \frac{1}{N!} Z_1^N \tag{A3}$$

which gives the Helmholtz free energy as

$$F = -k_BT \ln Z_N \tag{A4a}$$

$$= -Nk_BT \ln \left[\frac{ge}{2\pi^2 n} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a} \right], \tag{A4b}$$

where $n \equiv N/V$. The entropy becomes

$$S \equiv -\left.\frac{\partial F}{\partial T}\right|_{V,N} \tag{A5a}$$

$$= Nk_B \left\{ 1 - \frac{aK_2'(a)}{K_2(a)} + \ln \left[\frac{ge}{2\pi^2 n} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a} \right] \right\}. \tag{A5b}$$

The energy is

$$E = F + TS \tag{A6a}$$

$$= Nk_BT \left[1 - \frac{aK_2'(a)}{K_2(a)} \right]. \tag{A6b}$$

To rewrite (A6b) as

$$E/N = mc^2 \frac{3K_3(a) + K_1(a)}{4K_2(a)}, \tag{A6c}$$

one may use the recurrence formula for the modified Bessel function (cf. 9.6.26 of Ref. 15),

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -(2\nu/z)K_\nu(z), \tag{A7a}$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K_\nu'(z). \tag{A7b}$$

The pressure is

$$p = -\left.\frac{\partial F}{\partial V}\right|_{T,N} = nk_BT, \tag{A8}$$

which is the equation of state for an ideal gas.

The Gibbs free energy is

$$\Phi(T,p,N) = F + pV \tag{A9a}$$

$$= -Nk_BT \ln \left[\frac{g}{2\pi^2} \frac{k_BT}{p} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a} \right]. \tag{A9b}$$

Since the chemical potential is related to the Gibbs free energy as

$$\Phi = \mu N, \tag{A10}$$

one obtains the chemical potential from (A9) as

$$\mu = -k_BT \ln \left[\frac{g}{2\pi^2 n} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a} \right]. \tag{A11}$$

From (A11) the condition for the Maxwell-Boltzmann statistics becomes

$$\frac{1}{n} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a} \gg 1. \tag{A12}$$

The enthalpy is

$$W(S,p,N) = E + pV \tag{A13a}$$

$$= Nk_BT \left[2 - \frac{aK_2'(a)}{K_2(a)} \right], \tag{A13b}$$

where the quantity a must be expressed as $a = nmc^2/p$. The thermodynamic potential is

$$\Omega(T,V,\mu) = -pV \tag{A14a}$$

$$= -\frac{gV}{2\pi^2} k_BT e^{\mu/k_BT} \left[\frac{mc}{\hbar} \right]^3 \frac{K_2(a)}{a}. \tag{A14b}$$

The heat capacity at constant volume is

$$C_V \equiv \left.\frac{\partial E}{\partial T}\right|_V \tag{A15a}$$

$$= Nk_B \left\{ 1 + a^2 \left[\frac{K_2''(a)}{K_2(a)} - \left[\frac{K_2'(a)}{K_2(a)} \right]^2 \right] \right\}, \tag{A15b}$$

while the heat capacity at constant pressure is

$$C_p \equiv \left.\frac{\partial W}{\partial T}\right|_p \tag{A16a}$$

$$= Nk_B \left\{ 2 + a^2 \left[\frac{K_2''(a)}{K_2(a)} - \left[\frac{K_2'(a)}{K_2(a)} \right]^2 \right] \right\}. \tag{A16b}$$

The combination of Eqs. (A15) and (A16) gives Meyer's relation

$$C_p - C_V = Nk_B. \tag{A17}$$

The adiabatic index is

$$\Gamma \equiv \left.\frac{\partial \ln p}{\partial \ln n}\right|_S \tag{A18a}$$

$$= 1 + \frac{1}{1 + a^2 \left[\frac{K_2''(a)}{K_2(a)} - \left[\frac{K_2'(a)}{K_2(a)} \right]^2 \right]}, \tag{A18b}$$

which may also be given as

$$\Gamma = C_p/C_V \tag{A19}$$

for an ideal gas. The adiabatic compressibility is

$$\kappa_S = \frac{1}{p\Gamma} \tag{A20a}$$

$$= \frac{1}{nk_BT} \left[1 - \frac{1}{2 + a^2 \left[\frac{K_2''(a)}{K_2(a)} - \left[\frac{K_2'(a)}{K_2(a)} \right]^2 \right]} \right], \tag{A20b}$$

while the isothermal compressibility is

$$\kappa_T = 1/p = \frac{1}{nk_BT}. \tag{A21}$$

The squared velocity of sound is given as

$$\left[\frac{u}{c}\right]^2 \equiv \frac{\partial p}{\partial \varepsilon}\bigg|_S = \frac{\Gamma p}{w} = \frac{1}{w\kappa_S} \tag{A22a}$$

$$= \left[1+\frac{1}{1+a^2\left[\frac{K_2''(a)}{K_2(a)}-\left[\frac{K_2'(a)}{K_2(a)}\right]^2\right]}\right] \Bigg/ \left[2-a\frac{K_2'(a)}{K_2(a)}\right], \tag{A22b}$$

where $\varepsilon \equiv E/V$. The thermal expansion coefficient is

$$\alpha \equiv \frac{1}{V}\frac{\partial V}{\partial T}\bigg|_p = \frac{1}{T} . \tag{A23}$$

1. Nonrelativistic limit ($a \equiv mc^2/k_B T \gg 1$)

In the nonrelativistic limit, one uses the asymptotic form of the modified Bessel function (cf. 9.7.2 of Ref. 15)

$$K_2(z) \sim \left[\frac{\pi}{2z}\right]^{1/2} e^{-z}\left[1+\frac{15}{8z}+\frac{105}{128z^2}+\cdots\right], \quad z \gg 1 . \tag{A24}$$

It follows from (A24) that (cf. 9.7.4 of Ref. 15)

$$K_2'(z) \sim -\left[\frac{\pi}{2}\right]^{1/2} e^{-z} z^{-1/2}\left[1+\frac{19}{8z}+\frac{465}{128z^2}+\cdots\right], \tag{A25}$$

$$\frac{zK_2'(z)}{K_2(z)} = -z\left[1+\frac{1}{2z}+\frac{15}{8z^2}+\cdots\right] . \tag{A26}$$

Using Eqs. (A24)–(A26) in the general expressions, one obtains the following formulas in the nonrelativistic limit:

$$F = Nmc^2\left\{1-\frac{15}{8}\left[\frac{k_B T}{mc^2}\right]^2+\cdots -\frac{k_B T}{mc^2}\ln\left[\frac{ge}{n}\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{3/2}\right]\right\}, \tag{A27}$$

$$S = Nk_B\left\{\frac{3}{2}+\frac{15}{4}\left[\frac{k_B T}{mc^2}\right]+\cdots +\ln\left[\frac{ge}{n}\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{3/2}\right]\right\}, \tag{A28}$$

$$E = Nmc^2\left[1+\frac{3}{2}\frac{k_B T}{mc^2}+\frac{15}{8}\left[\frac{k_B T}{mc^2}\right]^2+\cdots\right], \tag{A29}$$

$$\Phi = Nmc^2\left\{1-\frac{15}{8}\left[\frac{k_B T}{mc^2}\right]^2+\cdots -\frac{k_B T}{mc^2}\ln\left[\frac{g}{n}\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{3/2}\right]\right\}, \tag{A30}$$

$$\mu = mc^2\left\{1-\frac{15}{8}\left[\frac{k_B T}{mc^2}\right]^2+\cdots -\frac{k_B T}{mc^2}\ln\left[\frac{g}{n}\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{3/2}\right]\right\}, \tag{A31}$$

$$W = Nmc^2\left[1+\frac{5}{2}\frac{k_B T}{mc^2}+\frac{15}{8}\left[\frac{k_B T}{mc^2}\right]^2+\cdots\right], \tag{A32}$$

$$C_V = \tfrac{3}{2}Nk_B\left[1+\frac{5}{2}\frac{k_B T}{mc^2}+\cdots\right], \tag{A33}$$

$$C_p = \tfrac{5}{2}Nk_B\left[1+\frac{3}{2}\frac{k_B T}{mc^2}+\cdots\right], \tag{A34}$$

$$\Gamma = \frac{5}{3}\left[1-\frac{k_B T}{mc^2}+\cdots\right], \tag{A35}$$

$$\kappa_S = \frac{3}{5}\frac{1}{nk_B T}\left[1-\frac{k_B T}{mc^2}+\cdots\right], \tag{A36}$$

$$\left[\frac{u}{c}\right]^2 = \frac{5}{3}\frac{k_B T}{m}\left[1-\frac{7}{2}\frac{k_B T}{mc^2}+\cdots\right] . \tag{A37}$$

From (A31) the condition for the Maxwell-Boltzmann statistics is

$$\frac{1}{n}\left[\frac{mk_B T}{2\pi\hbar^2}\right]^{3/2} \gg 1 . \tag{A38}$$

Extremely relativistic limit ($a \equiv mc^2/k_B T \ll 1$)

In the extremely relativistic limit, one can use the ascending series expansion of the modified Bessel function (cf. 9.6.11 of Ref. 15)

$$K_2(z) \equiv \frac{2}{z^2}-\frac{1}{2}-\frac{z^2}{8}\ln\frac{z}{2}+\left[\frac{3}{32}-\frac{\gamma_E}{8}\right]z^2 +O\left[z^4, z^4\ln\frac{z}{2}\right], z \ll 1 \tag{A39}$$

where $\gamma_E = 0.5772$ is Euler's constant. From (A39) it then follows that

$$K_2'(z) = -\frac{4}{z^3}-\frac{z}{4}\ln\frac{z}{2}+\left[\frac{1}{16}-\frac{\gamma_E}{4}\right]z+O(z^3, z^3\ln z) , \tag{A40}$$

$$\frac{zK_2'(z)}{K_2(z)} = -2\left[1+\frac{z^2}{4}+O(z^4)\right] . \tag{A41}$$

Using (A39)–(A41) in the general expressions, one obtains the following formulas in the extremely relativistic limit:

$$F=-Nk_BT\left\{\ln\left[\frac{ge}{\pi^2 n}\left[\frac{k_BT}{\hbar c}\right]^3\right]-\frac{1}{4}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right\}, \tag{A42}$$

$$S=3Nk_B\left\{1+\frac{1}{12}\left[\frac{mc^2}{k_BT}\right]^2+\cdots+\frac{1}{3}\ln\left[\frac{ge}{\pi^2 n}\left[\frac{k_BT}{\hbar c}\right]^3\right]\right\}, \tag{A43}$$

$$E=3Nk_BT\left[1+\frac{1}{6}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A44}$$

$$\Phi=-Nk_BT\left\{\ln\left[\frac{g}{\pi^2 n}\left[\frac{k_BT}{\hbar c}\right]^3\right]-\frac{1}{4}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right\}, \tag{A45}$$

$$\mu=-k_BT\left\{\ln\left[\frac{g}{\pi^2 n}\left[\frac{k_BT}{\hbar c}\right]^3\right]-\frac{1}{4}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right\}, \tag{A46}$$

$$W=4Nk_BT\left[1+\frac{1}{8}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A47}$$

$$C_V=3Nk_B\left[1-\frac{1}{6}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A48}$$

$$C_p=4Nk_B\left[1-\frac{1}{8}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A49}$$

$$\Gamma=\frac{4}{3}\left[1+\frac{1}{24}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A50}$$

$$\kappa_S=\frac{3}{4}\frac{1}{nk_BT}\left[1-\frac{1}{24}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right], \tag{A51}$$

$$\left[\frac{u}{c}\right]^2=\frac{1}{3}\left[1-\frac{1}{12}\left[\frac{mc^2}{k_BT}\right]^2+\cdots\right]. \tag{A52}$$

From (A46), the condition for the Maxwell-Boltzmann statistics is

$$\frac{1}{n}\left[\frac{k_BT}{\hbar c}\right]^3\gg 1 . \tag{A53}$$

APPENDIX B: DERIVATION OF THE BARYON NUMBER-DENSITY RATIO FROM THE NONRELATIVISTIC RANKINE-HUGONIOT RELATION (CASE II)

In this appendix, we derive the ratio of the baryon number density of the post-shock matter to that of the pre-shock matter, Eq. (3.27), for case II (the baryon dominated pair plasma) from the nonrelativistic Rankine-Hugoniot relation.

The nonrelativistic form of the Rankine-Hugoniot relation is[1]

$$e_1-e_2+\tfrac{1}{2}(\tilde V_1-\tilde V_2)(p_1+p_2)=0 , \tag{B1}$$

where e is the energy per unit mass, and $\tilde V(\equiv 1/\rho)$ is the specific volume with ρ the mass density. Here, the energy does not include the rest-mass energy. The energy density $\tilde\varepsilon$, which also does not include the rest-mass energy density, is related to e by $\tilde\varepsilon=\rho e$.

From (B1), the ratio of the mass densities may be written as

$$\frac{\rho_2}{\rho_1}=\frac{\tilde\varepsilon_2+\frac{1}{2}(p_1+p_2)}{\tilde\varepsilon_1+\frac{1}{2}(p_1+p_2)} . \tag{B2}$$

In the present case, the pre-shock matter is nonrelativistic, so that

$$\tilde\varepsilon_1=\tilde\varepsilon_{i1}+\tilde\varepsilon_{e1}=3n_{b1}k_BT_{1,} \tag{B3}$$

$$p_1=p_{i1}+p_{e1}=2n_{b1}k_BT_2 , \tag{B4}$$

$$\tilde\varepsilon_1=\tfrac{3}{2}p_1 . \tag{B5}$$

On the other hand, the post-shock matter is composed of the nonrelativistic ions and the radiation, so that

$$\tilde\varepsilon_{i2}=\tfrac{3}{2}n_{b2}k_BT_2 , \tag{B6}$$

$$\tilde\varepsilon_{r2}=3p_{r2} , \tag{B7}$$

$$p_{i2}=n_{b2}k_BT_2 . \tag{B8}$$

Thus

$$\tilde\varepsilon_2\equiv\tilde\varepsilon_{i2}+\tilde\varepsilon_{r2}=\tfrac{3}{2}p_{i2}+3p_{r2} . \tag{B9}$$

Inserting (B5), (B9), and

$$p_2\equiv p_{i2}+p_{r2} \tag{B10}$$

into (B2), one obtains

$$\frac{\rho_2}{\rho_1}=\frac{\frac{1}{2}p_1+2p_{i2}+\frac{7}{2}p_{r2}}{2p_1+\frac{1}{2}p_{i2}+\frac{1}{2}p_{r2}} . \tag{B11}$$

When the pressure of the radiation (p_{r2}) is much larger than those of the ions in the post-shock matter (p_{i2}) and the pre-shock matter (p_1),

$$p_1,p_{i2}\ll p_{r2} , \tag{B12}$$

(B11) becomes

$$\frac{\rho_2}{\rho_1}\simeq 7 , \tag{B13}$$

which is the same as Eq. (3.27).

[1]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics,* 2nd ed. (Pergamon, Oxford, 1987).

[2]Ya. B. Zel'dovich and Yu. P. Raiser, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena* (Academic, New York, 1967).

[3]A. H. Taub, Phys. Rev. **74**, 328 (1948).

[4]W. Israel, Proc. R. Soc. London, Ser. A **259**, 129 (1960).

[5]M. H. Johnson and C. F. McKee, Phys. Rev. D **3**, 858 (1971).

[6]K. S. Thorne, Astrophys. J. **179**, 897 (1973).

[7]L. D. Landau and E. M. Lifshitz, *Statistical Physics,* 3rd ed. (Pergamon, Oxford, 1980), Pt. 1.

[8]When the electrons are nonrelativistic, such that $k_B T_2 < m_e c^2$, the Compton wavelength $\hbar/m_e c$ gives an additional length scale. However, in the temperature range of our interest, $m_e c^2 < k_B T_2 < m_\mu c^2$, this matters only when $m_e c^2 \lesssim k_B T_2$, in which case the two characteristic number densities $(m_e c/\hbar)^3$ and $(k_B T_2/\hbar c)^3$ are on the same order. Therefore, only the characteristic number density $n_0(k_B T_2/m_e c^2)^3$ needs to be considered in this temperature range.

[9]The nearly equal sign in (2.22) reflects the charge neutrality condition $n_{b2}+n_{e+}=n_{e-}$.

[10]We use the notation n_{e2} for the electron number density of the post-shock matter (especially, that for the electron-ion plasma), while n_{e-} is also used when referring to the pair plasma. The corresponding positron number density is n_{e+}.

[11]See S. Stepney, Mon. Not. R. Astron. Soc. **202**, 467 (1983), and the references therein.

[12]Since we are interested in the temperature range (2.1), the nominal upper limit on the temperature is $m_\mu c^2/k_B$. When the condition (2.1) is relaxed and the pair creation of other particles is taken into account, one still expects a behavior similar to Eq. (3.18).

[13]It is also assumed that the processes involving photons may be neglected.

[14]See, e.g., H.-Y. Chiu, *Stellar Physics* (Blaisdell, Waltham, MA, 1968).

[15]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).