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Quantum statistics of nonclassical radiation in dissipative forward four-wave mixing. I. Method of generalized superposition of coherent and chaotic fields

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The Fokker-Planck equation approach to forward four-wave mixing developing from the initial coherent states is adopted to obtain a set of recursive relations for the expectation values of the normally ordered field operators. This set of equations is closed, in accordance with the model of the generalized superposition of coherent and chaotic fields, to provide differential equations for unknown quantities in this model. A numerical solution of these equations enables one to determine the photon number distribution, its factorial moments, squeezing variances, and entropy of radiation under discussion. Fully quantum-mechanical features of radiation such as its sub-Poisson behavior and squeezing of vacuum fluctuations are preserved in this approach. The validity of this approximation is discussed and numerical results are provided.

I. INTRODUCTION

In recent years great progress in generating light exhibiting nonclassical behavior such as photon antibunching, sub-Poisson photon statistics, and squeezing of vacuum fluctuations has been achieved. Experiments have been realized based on resonance-fluorescence light from single atoms[1–4] and from cooperative atoms,[5] Franck-Hertz light,[6] second-harmonic generation,[7,8] four-wave mixing,[9–13] parametric processes,[14–16] semiconductor lasers,[17,18] and light-emitting diodes.[19] Some of these experiments exhibit small sub-Poisson effects[4,6]; substantial sub-Poisson behavior has been observed in Refs. 17, 18, and 19 and the squeezing of vacuum fluctuations achieving tens of percent has been generated.[7–12,15]

As to the theory of nonclassical light much information has been obtained,[20–23] although, due to the quantum properties, various approximations and restrictions must usually be adopted. A unified approach to the photon statistics of light in nonlinear optical processes has been suggested in Ref. 24, which is based on the use of the generalized superposition of coherent and chaotic fields.[23,25,26] Such a superposition describes nonclassical state by means of negative "mean numbers of chaotic photons" and the appropriate photon statistics are expressed in terms of the Laguerre polynomials. This approach has been extended to the three-field optical-parametric process[27] and it will be applied in this paper to the process of four-wave mixing.

The statistical properties of radiation in four-wave mixing have been studied using the short-length, the short-time, or the parametric approximation.[28–33] Generation of squeezed-state light by four-wave mixing has been investigated in a number of papers[34–43] from various points of view.

II. DYNAMICAL PROPERTIES

We consider four-wave mixing described by the effective Hamiltonian

$$\hat H=\hbar\left[\sum_{j=1}^{2}(\omega_j\hat a_j^\dagger\hat a_j+\omega_{j+2}\hat b_j^\dagger\hat b_j)+(g\hat a_1^\dagger\hat a_2^\dagger\hat b_1\hat b_2+\text{H.c.})+\sum_{l}\sum_{j=1}^{2}(\psi_l^{(j)}\hat c_l^{(j)\dagger}\hat c_l^{(j)}+\varphi_l^{(j)}\hat d_l^{(j)\dagger}\hat d_l^{(j)}+(\eta_l^{(j)}\hat c_l^{(j)}\hat a_j^\dagger+\bar\eta_l^{(j)}\hat d_l^{(j)}\hat b_j^\dagger+\text{H.c.}\right]\,, \tag{1}$$

where $\hat a_1,\hat a_2$ are the annihilation operators of signal waves and $\hat b_1,\hat b_2$ are those of pump waves, ω_1,ω_2 are the frequencies of the signal beams and ω_3,ω_4 are the frequencies of the pump beams, g (a complex number) is the coupling constant, $\hat c_l^{(j)},\hat d_l^{(j)}$ are the reservoir annihilation operators, and $\eta_l^{(j)},\bar\eta_l^{(j)}$ are the reservoir coupling constants.

The general treatment of four-wave mixing requires the Hamiltonian to be of the form

$$\hat H_g=\hat H+\hat H_1+\hat H_2\,, \tag{2}$$

where

$$\hat H_1=\hbar(\bar g\hat a_1\hat a_2^\dagger\hat b_1\hat b_2^\dagger+\bar{\bar g}\hat a_1\hat a_2^\dagger\hat b_1^\dagger\hat b_2+\text{H.c.})\,,$$
$$\hat H_2=\hbar[\kappa(\hat a_1^\dagger\hat a_1+\hat a_2^\dagger\hat a_2)(\hat b_1^\dagger\hat b_1+\hat b_2^\dagger\hat b_2)+\bar\kappa(\hat b_1^{\dagger 2}\hat b_1^2+\hat b_1^\dagger\hat b_1\hat b_2^\dagger\hat b_2+\hat b_2^{\dagger 2}\hat b_2^2)]\,, \tag{3}$$

and the Hamiltonian $\hat H$ is given in (1); $\bar g,\bar{\bar g},\kappa,\bar\kappa$ are the coupling constants (the constants g are complex numbers, the constants κ are real numbers). This Hamiltonian involves all interactions of photons of single modes. It consists of the terms where two photons of two certain modes are transformed into photons of the two other modes and vice versa (terms of photon pair exchange), of the free field including the photon number conserving interactions, i.e., the energy renormalization terms, and of the terms introducing dissipation or including the reservoir variables. This general case falls beyond the scope of our paper. As a "facet" approach we shall study the nonlinear oscillations in the second part of this paper.

The frequency resonance for the effective process requires $\omega_1+\omega_2=\omega_3+\omega_4$. Two other conditions are $\omega_1+\omega_3=\omega_2+\omega_4$ and $\omega_1+\omega_4=\omega_2+\omega_3$. They are equivalent to $\omega_1-\omega_2=\pm(\omega_3-\omega_4)$, i.e, $\omega_1-\omega_2=\omega_3-\omega_4=0$, the conditions of degeneracy.

We assume that the reservoir spectrum is flat, so that the mean numbers of the reservoir oscillators (phonons) in the mode l are $\langle n_{d_j}\rangle=\langle \hat{c}_l^{(j)\dagger}(0)\hat{c}_l^{(j)}(0)\rangle, j=1,2$, and $\langle n_{d_j}\rangle=\langle \hat{d}_l^{(j)\dagger}(0)\hat{d}_l^{(j)}(0)\rangle, j=3,4$ independently of l and that the reservoir oscillators form a chaotic system.

Using the general procedure proposed in Ref. 44, we arrive at the generalized Fokker-Planck equation for the quasidistribution $\phi_{\mathcal{N}}\equiv\phi_{\mathcal{N}}(\{\alpha_j\},t)$ related to the normal ordering of field operators

$$\frac{\partial\phi_{\mathcal{N}}}{\partial t}=ig\left[\frac{\partial}{\partial\alpha_1}\alpha_2^*\alpha_3\alpha_4+\frac{\partial}{\partial\alpha_2}\alpha_1^*\alpha_3\alpha_4-\frac{\partial}{\partial\alpha_3^*}\alpha_1^*\alpha_2^*\alpha_4-\frac{\partial}{\partial\alpha_4^*}\alpha_1^*\alpha_2^*\alpha_3-\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\alpha_3\alpha_4+\frac{\partial^2}{\partial\alpha_3^*\partial\alpha_4^*}\alpha_1^*\alpha_2^*\right]\phi_{\mathcal{N}}$$
$$+\text{c.c.}+\sum_{j=1}^{4}\left[\left(\frac{\gamma_j}{2}+i\omega_j\right)\frac{\partial}{\partial\alpha_j}+\text{c.c.}+\gamma_j\langle n_{d_j}\rangle\frac{\partial^2}{\partial\alpha_j\partial\alpha_j^*}\right]\phi_{\mathcal{N}}\,. \tag{4}$$

The quasidistribution $\phi_{\mathcal{N}}$ exists, in general, as a generalized function only. By calculating the expectation values of the normally ordered field operators the irregular behavior of $\phi_{\mathcal{N}}$ is smoothed out. From (4) we arrive at the following moment equation:

$$\begin{aligned}
\frac{d}{dt}&\langle m_1m_2m_3m_4n_1n_2n_3n_4\rangle\\
&=-ig[n_1\langle m_1(m_2+1)m_3m_4(n_1-1)n_2(n_3+1)(n_4+1)\rangle+n_2\langle(m_1+1)m_2m_3m_4n_1(n_2-1)(n_3+1)(n_4+1)\rangle\\
&\qquad-m_3\langle(m_1+1)(m_2+1)(m_3-1)m_4n_1n_2n_3(n_4+1)\rangle-m_4\langle(m_1+1)(m_2+1)m_3(m_4-1)n_1n_2(n_3+1)n_4\rangle\\
&\qquad+n_1n_2\langle m_1m_2m_3m_4(n_1-1)(n_2-1)(n_3+1)(n_4+1)\rangle\\
&\qquad-m_3m_4\langle(m_1+1)(m_2+1)(m_3-1)(m_4-1)n_1n_2n_3n_4\rangle]\\
&\quad+ig^*[m_1\langle(m_1-1)m_2(m_3+1)(m_4+1)n_1(n_2+1)n_3n_4\rangle+m_2\langle m_1(m_2-1)(m_3+1)(m_4+1)(n_1+1)n_2n_3n_4\rangle\\
&\qquad-n_3\langle m_1m_2m_3(m_4+1)(n_1+1)(n_2+1)(n_3-1)n_4\rangle-n_4\langle m_1m_2(m_3+1)m_4(n_1+1)(n_2+1)n_3(n_4-1)\rangle\\
&\qquad+m_1m_2\langle(m_1-1)(m_2-1)(m_3+1)(m_4+1)n_1n_2n_3n_4\rangle\\
&\qquad-n_3n_4\langle m_1m_2m_3m_4(n_1+1)(n_2+1)(n_3-1)(n_4-1)\rangle]\\
&\quad-\sum_{j=1}^{4}\left[\frac{\gamma_j}{2}(m_j+n_j)+i\omega_j(m_j-n_j)\right]\langle m_1m_2m_3m_4n_1n_2n_3n_4\rangle\\
&\qquad+\gamma_1\langle n_{d_1}\rangle m_1n_1\langle(m_1-1)m_2m_3m_4(n_1-1)n_2n_3n_4\rangle+\gamma_2\langle n_{d_2}\rangle m_2n_2\langle m_1(m_2-1)m_3m_4n_1(n_2-1)n_3n_4\rangle\\
&\qquad+\gamma_3\langle n_{d_3}\rangle m_3n_3\langle m_1m_2(m_3-1)m_4n_1n_2(n_3-1)n_4\rangle+\gamma_4\langle n_{d_4}\rangle m_4n_4\langle m_1m_2m_3(m_4-1)n_1n_2n_3(n_4-1)\rangle\,,
\end{aligned}\tag{5}$$

where we have denoted

$$\begin{aligned}\langle m_1m_2m_3m_4n_1n_2n_3n_4\rangle&\equiv\langle\alpha_1^{*m_1}\alpha_2^{*m_2}\alpha_3^{*m_3}\alpha_4^{*m_4}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}\rangle_{\mathcal{N}}\\
&=\int\int\int\int\prod_{k=1}^{4}\alpha_k^{*m_k}\alpha_k^{n_k}\phi_{\mathcal{N}}(\{\alpha_j\},t)\prod_{k=1}^{4}d^2\alpha_k\,.\end{aligned}\tag{6}$$

After the substitution

$$\mathcal{A}_j=\alpha_j\exp(i\omega_jt),\quad j=1,2,3,4 \tag{7}$$

reflecting the use of the interaction representation, the generalized Fokker-Planck equation (4), and the moment equations (5) do not contain the ω_j terms. In particular, for (a) $m_l=0$, $n_l=\delta_{jl}$, $j,l=1,2,3,4$; (b) $m_l=\delta_{jl},n_l=\delta_{kl},j,k,l=1,2,3,4$; (c) $m_l=0,n_l=\delta_{jl}+\delta_{kl},j,k,l=$ 1,2,3,4, the moment equations (5) take on the form

$$\frac{d}{dt}\langle\alpha_j\rangle=-ig(\delta_{j1}\langle\alpha_2^*\alpha_3\alpha_4\rangle+\delta_{j2}\langle\alpha_1^*\alpha_3\alpha_4\rangle)-ig^*(\delta_{j3}\langle\alpha_4^*\alpha_1\alpha_2\rangle+\delta_{j4}\langle\alpha_3^*\alpha_1\alpha_2\rangle)-\frac{\gamma_j}{2}\langle\alpha_j\rangle,\quad j=1,2,3,4 \tag{8a}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha_j^*\alpha_k\rangle=&-ig(\delta_{k1}\langle\alpha_2^*\alpha_j^*\alpha_3\alpha_4\rangle+\delta_{k2}\langle\alpha_1^*\alpha_j^*\alpha_3\alpha_4\rangle-\delta_{j3}\langle\alpha_1^*\alpha_2^*\alpha_k\alpha_4\rangle-\delta_{j4}\langle\alpha_1^*\alpha_2^*\alpha_k\alpha_3\rangle)\\
&+ig^*(\delta_{j1}\langle\alpha_3^*\alpha_4^*\alpha_k\alpha_2\rangle+\delta_{j2}\langle\alpha_3^*\alpha_4^*\alpha_k\alpha_1\rangle-\delta_{k3}\langle\alpha_4^*\alpha_j^*\alpha_1\alpha_2\rangle-\delta_{k4}\langle\alpha_3^*\alpha_j^*\alpha_1\alpha_2\rangle)\\
&-\tfrac{1}{2}(\gamma_j+\gamma_k)\langle\alpha_j^*\alpha_k\rangle+\gamma_j\langle n_{d_j}\rangle\delta_{jk},\quad j,k=1,2,3,4\end{aligned}\tag{8b}$$

$$\frac{d}{dt}\langle\alpha_j\alpha_k\rangle=-ig(\delta_{j1}\langle\alpha_2^*\alpha_k\alpha_3\alpha_4\rangle+\delta_{k1}\langle\alpha_2^*\alpha_j\alpha_3\alpha_4\rangle+\delta_{j2}\langle\alpha_1^*\alpha_k\alpha_3\alpha_4\rangle+\delta_{k2}\langle\alpha_1^*\alpha_j\alpha_3\alpha_4\rangle+(\delta_{j1}\delta_{k2}+\delta_{k1}\delta_{j2})\langle\alpha_3\alpha_4\rangle)$$
$$-ig^*(\delta_{j3}\langle\alpha_4^*\alpha_k\alpha_1\alpha_2\rangle+\delta_{k3}\langle\alpha_4^*\alpha_j\alpha_1\alpha_2\rangle+\delta_{j4}\langle\alpha_3^*\alpha_k\alpha_1\alpha_2\rangle+\delta_{k4}\langle\alpha_3^*\alpha_j\alpha_1\alpha_2\rangle$$
$$+(\delta_{j3}\delta_{k4}+\delta_{k3}\delta_{j4})\langle\alpha_1\alpha_2\rangle)-\tfrac{1}{2}(\gamma_j+\gamma_k)\langle\alpha_j\alpha_k\rangle,\quad j,k=1,2,3,4 \tag{8c}$$

respectively. The system (8) treated as coupled differential equations for the unknown moments of the first and second orders depends on the unknown moments of the third and fourth orders and as such it is not uniquely solvable. To exclude the third- and fourth-order moments and render the system uniquely solvable, we express these moments in terms of the first- and second-order moments using the relations valid for the generalized superposition of coherent fields and quantum noise. The relations chosen may be in fact satisfied only "on average" and they do not imply that the quantum optical system is described by this assumption exactly.

Let us focus on the derivation of the relations mentioned above. The normal quantum characteristic function $C_{\mathcal{N}}(\{\beta_j\},t)$, which is the Fourier transform of the quasidistribution $\phi_{\mathcal{N}}(\{\alpha_j\},t)$, appropriate to the generalized superposition of coherent fields and quantum noise can be written in the form

$$C_{\mathcal{N}}(\{\beta_j\},t)=\exp(-\tfrac{1}{2}\boldsymbol{\beta}^\dagger\mathbf{A}\boldsymbol{\beta}-\boldsymbol{\beta}^\dagger\boldsymbol{\xi})$$
$$=\exp\left[\sum_{j=1}^{4}\{-B_j(t)|\beta_j|^2+\tfrac{1}{2}[C_j^*(t)\beta_j^2+\text{c.c.}]\}+\sum_{j=1}^{3}\sum_{k=j+1}^{4}(-B_{jk}(t)\beta_j^*\beta_k+C_{jk}^*(t)\beta_j\beta_k+\text{c.c.})+\sum_{j=1}^{4}(\beta_j\xi_j^*(t)-\text{c.c.})\right], \tag{9}$$

where

$$\boldsymbol{\beta}=(\beta_1,-\beta_1^*,\beta_2-\beta_2^*,\beta_3,-\beta_3^*,\beta_4,-\beta_4^*)^T,$$
$$\boldsymbol{\xi}=(\xi_1,\xi_1^*,\xi_2,\xi_2^*,\xi_3,\xi_3^*,\xi_4,\xi_4^*)^T,$$

where T is the transposition,

$$\mathbf{A}=\begin{bmatrix} B_1 & C_1 & B_{12} & C_{12} & B_{13} & C_{13} & B_{14} & C_{14}\\ C_1^* & B_1 & C_{12}^* & B_{12}^* & C_{13}^* & B_{13}^* & C_{14}^* & B_{14}^*\\ B_{12}^* & C_{12} & B_2 & C_2 & B_{23} & C_{23} & B_{24} & C_{24}\\ C_{12}^* & B_{12} & C_2^* & B_2 & C_{23}^* & B_{23}^* & C_{24}^* & B_{24}^*\\ B_{13}^* & C_{13} & B_{23}^* & C_{23} & B_3 & C_3 & B_{34} & C_{34}\\ C_{13}^* & B_{13} & C_{23}^* & B_{23} & C_3^* & B_3 & C_{34}^* & B_{34}^*\\ B_{14}^* & C_{14} & B_{24}^* & C_{24} & B_{34}^* & C_{34} & B_4 & C_4\\ C_{14}^* & B_{14} & C_{24}^* & B_{24} & C_{34}^* & B_{34} & C_4^* & B_4 \end{bmatrix} \tag{10}$$

and $B_j(t)$, $C_j(t)$, $B_{jk}(t)$, $C_{jk}(t)$, and $\xi_j(t)$ are unknown functions of time satisfying the initial condition

$$B_j(0)=0,\quad C_j(0)=0,\quad j=1,2,3,4$$
$$B_{jk}(0)=0,\quad C_{jk}(0)=0,\quad j,k=1,2,3,4,\quad j\le k\ ; \tag{11}$$

$\xi_j(0), j=1,2, (j=3,4)$ are the eigenvalues of $\hat{a}_j(0)(\hat{b}_j(0))$ in the coherent state $|\{\xi_j(0)\}\rangle$ and β_j are complex parameters of the characteristic function.

The moments in (8) can be expressed in terms of the coefficients of the normal quantum characteristic function (9). As a consequence of the relation

$$\langle m_1m_2m_3m_4n_1n_2n_3n_4\rangle=\left[\frac{\partial}{\partial\beta_1}\right]^{m_1}\left[-\frac{\partial}{\partial\beta_1^*}\right]^{n_1}\left[\frac{\partial}{\partial\beta_2}\right]^{m_2}\left[-\frac{\partial}{\partial\beta_2^*}\right]^{n_2}\times\left[\frac{\partial}{\partial\beta_3}\right]^{m_3}\left[-\frac{\partial}{\partial\beta_3^*}\right]^{n_3}\left[\frac{\partial}{\partial\beta_4}\right]^{m_4}\times\left[-\frac{\partial}{\partial\beta_4^*}\right]^{n_4}C_{\mathcal{N}}(\{\beta_j\},t)\Big|_{\{\beta_j\}=\{\beta_j^*\}=0}, \tag{12}$$

we get the first- and second-order moments in the form

$$\langle\alpha_j\rangle=\xi_j(t),\quad\langle|\alpha_j|^2\rangle=B_j(t)+|\xi_j(t)|^2,$$
$$\langle\alpha_j^2\rangle=C_j(t)+\xi_j^2(t),\quad j=1,2,3,4$$
$$\left.\begin{aligned}\langle\alpha_j\alpha_k^*\rangle&=B_{jk}(t)+\xi_j(t)\xi_k^*(t)\\ \langle\alpha_j\alpha_k\rangle&=C_{jk}(t)+\xi_j(t)\xi_k(t)\end{aligned}\right\}j,k=1,2,3,4,\quad j<k\ . \tag{13}$$

The third- and fourth-order moments on the right-hand sides in (8) can be rewritten similarly as the moments in (13). In this way we obtain the following system of differential equations for the unknown functions fully determining the statistical properties of the process under study by means of the quantum characteristic function (9):

$$\frac{d}{dt}\xi_1=-ig(\xi_2^*\xi_3\xi_4+B_{24}^*\xi_3+B_{23}^*\xi_4+C_{34}\xi_2^*)-\frac{\gamma_1}{2}\xi_1, \tag{14a}$$

$$\frac{d}{dt}\xi_3=-ig^*(\xi_1\xi_2\xi_4^*+B_{24}\xi_1+B_{14}\xi_2+C_{12}\xi_4^*)-\frac{\gamma_3}{2}\xi_3, \tag{14b}$$

$$\frac{d}{dt}B_1 = -ig(B_{13}^*\bar{B}_{24}^* + B_{14}^*\bar{B}_{23}^* + C_{12}^*\bar{C}_{34}) + \text{c.c.} - \gamma_1 B_1 + \gamma_1 \langle n_{d_1} \rangle \;, \tag{14c}$$

$$\frac{d}{dt}B_3 = ig(B_{13}^*\bar{B}_{24}^* + B_{23}^*\bar{B}_{14}^* + C_{34}\bar{C}_{12}^*) + \text{c.c.} - \gamma_3 B_3 + \gamma_3 \langle n_{d_3} \rangle \;, \tag{14d}$$

$$\frac{d}{dt}C_1 = -2ig(B_{12}\bar{C}_{34} + C_{13}\bar{B}_{24}^* + C_{14}\bar{B}_{23}^*) - \gamma_1 C_1 \;, \tag{14e}$$

$$\frac{d}{dt}C_3 = -2ig^*(B_{34}\bar{C}_{12} + C_{13}\bar{B}_{24} + C_{23}\bar{B}_{14}) - \gamma_3 C_3 \;, \tag{14f}$$

$$\frac{d}{dt}B_{12} = ig^*(B_{13}\bar{B}_{14} + B_{14}\bar{B}_{13} + C_1\bar{C}_{34}^*) - ig(B_{23}^*\bar{B}_{24}^* + B_{24}^*\bar{B}_{23}^* + C_2^*\bar{C}_{34}) - \tfrac{1}{2}(\gamma_1+\gamma_2)B_{12} \;, \tag{14g}$$

$$\frac{d}{dt}B_{13} = ig[(B_1 - B_3)\bar{B}_{24}^* + B_{12}\bar{B}_{14}^* - B_{34}^*\bar{B}_{23}^* + C_{14}\bar{C}_{12}^* - C_{23}^*\bar{C}_{34}] - \tfrac{1}{2}(\gamma_1+\gamma_3)B_{13} \;, \tag{14h}$$

$$\frac{d}{dt}C_{12} = -ig[(B_1 + B_2 + 1)\bar{C}_{34} + C_{23}\bar{B}_{24}^* + C_{24}\bar{B}_{23}^* + C_{13}\bar{B}_{14}^* + C_{14}\bar{B}_{13}^*] - \tfrac{1}{2}(\gamma_1+\gamma_2)C_{12} \;, \tag{14i}$$

$$\frac{d}{dt}C_{13} = -ig(C_{34}\bar{B}_{23}^* + B_{23}^*\bar{C}_{34} + C_3\bar{B}_{24}^*) - ig^*(C_{12}\bar{B}_{14} + B_{14}\bar{C}_{12} + C_1\bar{B}_{24}) - \tfrac{1}{2}(\gamma_1+\gamma_3)C_{13} \;, \tag{14j}$$

where

$$\bar{B}_{jk} = B_{jk} + \xi_j\xi_k^*, \quad \bar{C}_{jk} = C_{jk} + \xi_j\xi_k \;, \quad j=1,2, \quad k=3,4 \;; \tag{15}$$

the explicit dependence on t being omitted. The system is complete up to the equations yielded by interchanges of indices and the coupling constants g, g^*. The equations for $(d/dt)\xi_2$, $(d/dt)B_2$, $(d/dt)C_2$, $(d/dt)B_{23}$, and $(d/dt)C_{23}$ arise from the equations for $(d/dt)\xi_1$, $(d/dt)B_1$, $(d/dt)C_1$, $(d/dt)B_{13}$, and $(d/dt)C_{13}$ with the interchange of indices 1 and 2, respectively. This symmetry follows from the particular form of the Hamiltonian (1), where the signal modes appear in the symmetrical fashion.

The equations for $(d/dt)\xi_4$, $(d/dt)B_4$, $(d/dt)C_4$, $(d/dt)B_{14}$, and $(d/dt)C_{14}$ result from the equations for $(d/dt)\xi_3$, $(d/dt)B_3$, $(d/dt)C_3$, $(d/dt)B_{13}$, and $(d/dt)C_{13}$ with the aid of the interchange of indices 3 and 4, respectively. This symmetry can be seen from the form of the generalized Fokker-Planck equation (4), where pump modes 3,4 occur symmetrically. Performing the interchanges of indices 1,3, indices 2,4, and constants g, g^* in the equations for $(d/dt)B_{12}$, $(d/dt)C_{12}$, we obtain the equations for $(d/dt)B_{34}$, $(d/dt)C_{34}$, respectively. Also this symmetry reflects the special form of the Hamiltonian (1).

The interchanges of indices 1,2, and indices 3,4 in the equations for $(d/dt)B_{13}$, $(d/dt)C_{13}$ provide the equations for $(d/dt)B_{24}$, $(d/dt)C_{24}$, respectively. When interchanging the indices, we take account of the properties of the entries of the matrix (10), i.e., the fact that $B_{kj} = B_{jk}^*, C_{kj} = C_{jk}$. The complex conjugate equations also belong to the system (14).

In general, the system of ordinary differential equations (14) is nonlinear and difficult to solve in a closed analytic way. Hence, a numerical solution of this system has been adopted.

In Ref. 45 is has been proved that when using the method of linear quantum correction to classical solutions (applied in Ref. 46) to nonlinear processes described by Hamiltonians of higher than the second order in field operators, the statistics of light arising from this approximation are determined by the generalized superposition of coherent and chaotic fields. Therefore the accuracy of the present approximative method may be compared with that of the method of linear quantum correction, concluding that the generalized superposition of coherent and chaotic fields represents a good approximation provided that

$$\max_j \{|\xi_j(0)|\}|g|t = O(1) \;. \tag{16}$$

III. STATISTICAL PROPERTIES

The numerical solution of system (14), whose continuous dependence on initial values (11) and thus the unicity of the solution in the framework of the validity of used model is supposed, provides fully the photon statistics, squeezing, and entropy related to the normal characteristic function (9). The photon statistics have been calculated for single- and two-mode fields described by the generalized superposition of coherent and chaotic fields in Refs. 23, 47, and 48 and we present necessary formulas only.

A. One-mode cases

Taking account of single modes separately, the normal characteristic function (9) simplifies to the form

$$C_{\mathcal{N}}(\beta_j,t) = \exp\{-B_j(t)|\beta_j|^2 + \tfrac{1}{2}[C_j^*(t)\beta_j^2 + \text{c.c.}] + [\beta_j\xi_j^*(t) - \text{c.c.}]\}, \quad j=1,2,3,4 \;. \tag{17}$$

Supposing

$$K_j(t) = B_j^2(t) - |C_j(t)|^2 \rangle 0 \;, \tag{18}$$

we can determine the corresponding quasidistribution $\phi_{\mathcal{N}}(\alpha_j,t)$ in the form

$$\phi_{\mathcal{N}}(\alpha_j,t)=\frac{1}{\pi[K_j(t)]^{1/2}}\exp\left[-\frac{B_j(t)}{K_j(t)}|\alpha_j-\xi_j(t)|^2+\frac{1}{2K_j(t)}\{C_j^*(t)[\alpha_j-\xi_j(t)]^2+\text{c.c.}\}\right]; \tag{19}$$

in this case a nonclassical behavior of considered fields is excluded.

For the determination of the photon statistics of single modes we can use the standard formulas for the photon number distribution $p(n,t)$ and its factorial moments $\langle W^k\rangle_{\mathcal{N}}$ from Ref. 23 (Sec. 8.5)

$$p(n,t)=(EF)^{-1/2}\left[1-\frac{1}{F}\right]^n\exp\left[-\frac{A_1}{E}-\frac{A_2}{F}\right]$$
$$\times\sum_{k=0}^{n}\frac{1}{\Gamma(k+\frac{1}{2})\Gamma(n-k+\frac{1}{2})}\left[\frac{1+\frac{1}{E}}{1+\frac{1}{F}}\right]^k L_k^{-1/2}\left[-\frac{A_1}{E(E-1)}\right]L_{n-k}^{-1/2}\left[-\frac{A_2}{F(F-1)}\right], \tag{20}$$

$$\langle W^k\rangle_{\mathcal{N}}=k!(F-1)^k\sum_{l=0}^{k}\frac{1}{\Gamma(l+\frac{1}{2})\Gamma(k-l+\frac{1}{2})}\left[\frac{E-1}{F-1}\right]^l L_l^{-1/2}\left[-\frac{A_1}{E-1}\right]L_{k-l}^{-1/2}\left[-\frac{A_2}{F-1}\right], \tag{21}$$

where $L_k^\lambda(x)$ are the Laguerre polynomials, $\Gamma(x)$ is the γ function, the "mean numbers of coherent photons" are given by

$$A_{1,2}=\frac{1}{2}\left[|\xi_j(t)|^2\mp\frac{1}{2|C_j(t)|}[C_j^*(t)\xi_j^2(t)+\text{c.c.}]\right],$$
$$A_{1,2}\equiv A_{1,2(j)}, \tag{22}$$

and the "mean numbers of chaotic photons" representing quantum noise are

$$E=B_j(t)-|C_j(t)|+1,$$
$$F=B_j(t)+|C_j(t)|+1, \tag{23}$$
$$E\equiv E_j, F\equiv F_j.$$

For $E=1$, $F=1$, and $A_1=A_2=|\xi_j(t)|^2/2$ we obtain from (20) and (21)

$$p(n,t)=\frac{1}{n!}|\xi_j(t)|^{2n}\exp[-|\xi_j(t)|^2], \tag{24}$$

$$\langle W^k\rangle_{\mathcal{N}}=|\xi_j(t)|^{2k}, \tag{25}$$

i.e., the Poisson distribution and its factorial moments. The deviation from the Poisson statistics depends on the sign of the quantity $E-1$ ($F-1$ is always positive). If it is positive, the uncertainty (bunching effects of photons) is higher than that for the Poisson photon statistics. If it is negative, this uncertainty is reduced below that for the coherent state and the antibunching of photons and the sub-Poisson statistics may occur as discussed in Ref. 23.

For the mean number of photons we obtain from (21)

$$\langle W(t)\rangle_{\mathcal{N}}=|\xi_j(t)|^2+B_j(t) \tag{26}$$

and for the variance of the integrated intensity

$$\langle(\Delta W)^2\rangle_{\mathcal{N}}=\langle W^2\rangle_{\mathcal{N}}-\langle W\rangle_{\mathcal{N}}^2 \tag{27}$$

it holds that

$$\langle[\Delta W(t)]^2\rangle_{\mathcal{N}}=B_j^2(t)+|C_j(t)|^2+2B_j(t)|\xi_j(t)|^2+2\,\text{Re}[C_j^*(t)\xi_j^2(t)]. \tag{28}$$

For $\langle[\Delta W(t)]^2\rangle_{\mathcal{N}}<0$ the antibunching of photons occurs.[25] In this case the photon number distribution $p(n,t)$ is narrower than the Poisson distribution corresponding to the coherent state and we have the sub-Poisson radiation. Testing the second reduced factorial moment

$$\frac{\langle W^2\rangle_{\mathcal{N}}}{\langle W\rangle_{\mathcal{N}}^2}-1, \tag{29}$$

its negative value represents a sufficient condition for observing the sub-Poisson behavior of radiation.

The squeezing properties of one-mode fields can be deduced from the simple expressions

$$\langle(\Delta_{\hat{P}_j}^{\hat{Q}_j})^2\rangle=1+2[B_j(t)\pm\text{Re}C_j(t)],\quad j=1,2,3,4 \tag{30}$$

where the operators $\hat{Q}_j$ and $\hat{P}_j$ are defined in terms of the photon annihilation and creation operators $\hat{A}_j=\hat{a}_j\exp(i\omega_j t)$, $\hat{A}_j^\dagger=\hat{a}_j^\dagger\exp(-i\omega_j t)$ as follows:

$$\hat{Q}_j=\hat{A}_j+\hat{A}_j^\dagger,\quad \hat{P}_j=-i(\hat{A}_j-\hat{A}_j^\dagger),\quad [\hat{Q}_j,\hat{P}_{j'}]=2i\delta_{jj'}. \tag{31}$$

According to the standard definition of squeezing, this phenomenon can be observed in the jth mode if

$$\mu_j=\min\{\langle(\Delta\hat{Q}_j)^2\rangle,\langle(\Delta\hat{P}_j)^2\rangle\}<1. \tag{32}$$

The definition of principal squeezing[49] requires the fulfilling of the condition

$$\lambda_j=\min\{\langle(u_1\Delta\hat{Q}_j+u_2\Delta\hat{P}_j)^2\rangle;u_1^2+u_2^2=1\}<1. \tag{33}$$

As $\lambda_j \le \mu_j$, the definition of principal squeezing would be more tolerant and squeezing would occur more frequently. Assuming

$$\langle \Delta\hat{Q}_j \Delta\hat{P}_j \rangle = -i(\langle \Delta\hat{A}_j^{2}\rangle - \langle \Delta\hat{A}_j^{\dagger 2}\rangle) = 2\,\mathrm{Im}C_j(t) = 0 \,, \tag{34}$$

both the definitions are equivalent. λ_j can be calculated in the form

$$\lambda_j = 1 + 2[B_j(t) - |C_j(t)|] \tag{35}$$

and squeezing can be observed under the condition

$$B_j(t) < |C_j(t)| \,. \tag{36}$$

We mention that the principal squeezing is not affected by the transformations (7).

On the basis of results given in Ref. 50 we can establish the entropy of considered fields in the form

$$H_j(t) \equiv H = h(x), \quad h(x) = -\ln \frac{x^x}{(1+x)^{1+x}} \tag{37}$$

where

$$x = [K_j(t) + B_j(t) + \tfrac{1}{4}]^{1/2} - \tfrac{1}{2} \,. \tag{38}$$

B. Two-mode cases

In order to describe the statistical properties of two-mode fields formed by signal modes 1 and 2 or pump modes 3 and 4, we consider the special case of the normal quantum characteristic function (9) (Ref. 48)

$$C_{\mathcal{N}}(\beta_j,\beta_{j+1},t) = \exp\left[-\sum_{k=j,j+1} B_k(t)|\beta_k|^2 + \left(\frac{1}{2}\sum_{k=j,j+1} C_k^*(t)\beta_k^2 - B_{j,j+1}(t)\beta_j^*\beta_{j+1} + C_{j,j+1}^*\beta_j\beta_{j+1} + \text{c.c.} \right) + \sum_{k=j,j+1} [\xi_k^*(t)\beta_k - \text{c.c.}] \right], \quad j = 1,3 \,. \tag{39}$$

Denoting

$$\begin{aligned} L_j = {} & K_jK_{j+1} - 2B_jB_{j+1}(|C_{j,j+1}|^2 + |B_{j,j+1}|^2) + 2B_j(C_{j+1}C_{j,j+1}B_{j,j+1} + \text{c.c.}) \\ & + 2B_{j+1}(C_jC_{j,j+1}B_{j,j+1}^* + \text{c.c.}) - (C_jC_{j+1}C_{j,j+1}^2 + C_jC_{j+1}^*B_{j,j+1}^{*2} + \text{c.c.}) + (|C_{j,j+1}|^2 - |B_{j,j+1}|^2)^2 \,, \end{aligned} \tag{40}$$

and assuming

$$\begin{aligned} & K_{j+1} > 0, B_{j+1} + \mathrm{Re}C_{j+1} > 0 \,, \\ & L_j > 0, B_j + \mathrm{Re}C_j + \frac{1}{K_{j+2}}\{ -B_{j+1}|B_{j,j+1} + C_{j,j+1}|^2 + \mathrm{Re}[C_{j+1}^*(B_{j,j+1}^* + C_{j,j+1}^*)^2]\} > 0 \,, \end{aligned} \tag{41}$$

we determine the associated quasidistribution $\phi_{\mathcal{N}}$ in the form

$$\begin{aligned} \phi_{\mathcal{N}}(\alpha_j,\alpha_{j+1},t) = {} & \frac{1}{\pi^2(L_j)^{1/2}} \\ & \times \exp[-|\alpha_j - \xi_j(t)|^2E_1 - |\alpha_{j+1} - \xi_{j+1}(t)|^2E_2 \\ & + (\tfrac{1}{2}\{[\alpha_j^* - \xi_j^*(t)]^2E_3 + [\alpha_{j+1}^* - \xi_{j+1}^*(t)]^2E_4\} + [\alpha_j^* - \xi_j^*(t)][\alpha_{j+1}^* - \xi_{j+1}^*(t)]E_5 + [\alpha_j - \xi_j(t)] \\ & \times [\alpha_{j+1}^* - \xi_{j+1}^*(t)]E_6 + \text{c.c.})], \quad j = 1,3 \end{aligned} \tag{42}$$

where

$$\begin{aligned} E_1 &= \frac{1}{L_j}\{B_jK_{j+1} - [B_{j+1}(|B_{j,j+1}|^2 + |C_{j,j+1}|^2) - (C_{j+1}C_{j,j+1}B_{j,j+1} + \text{c.c.})]\} \,, \\ E_2 &= \frac{1}{L_j}[B_{j+1}K_j - B_j(|B_{j,j+1}|^2 + |C_{j,j+1}|^2) + (C_jC_{j,j+1}B_{j,j+1}^* + \text{c.c.})] \,, \\ E_3 &= \frac{1}{L_j}(C_jK_{j+1} - 2B_{j+1}C_{j,j+1}^*B_{j,j+1} + C_{j+1}^*C_{j,j+1}^{*2} + C_{j+1}B_{j,j+1}^2) \,, \\ E_4 &= \frac{1}{L_j}(C_{j+1}K_j - 2B_jC_{j,j+1}^*B_{j,j+1}^* + C_jB_{j,j+1}^{*2} + C_j^*C_{j,j+1}^{*2}) \,, \\ E_5 &= \frac{1}{L_j}[C_{j,j+1}^*(B_jB_{j+1} + |B_{j,j+1}|^2 - |C_{j,j+1}|^2) - B_jC_{j+1}B_{j,j+1} - B_{j+1}C_jB_{j,j+1}^* + C_jC_{j+1}C_{j,j+1}] \,, \\ E_6 &= \frac{1}{L_j}[B_{j,j+1}^*(B_jB_{j+1} - |B_{j,j+1}|^2 + |C_{j,j+1}|^2) - B_jC_{j+1}C_{j,j+1} - B_{j+1}C_j^*C_{j,j+1}^* + C_j^*C_{j+1}B_{j,j+1}] \,. \end{aligned} \tag{43}$$

For the sake of simplicity, the explicit dependence of the coefficients $B_j, C_j, B_{j,j+1}, C_{j,j+1}$ on t and that of the coefficients E on j are omitted. The explicit expression (42) for $\phi_{\mathcal{N}}$ is conditioned by the nonappearance of a nonclassical behavior of the radiation of the appropriate two-mode fields.

To establish the photon statistics, we have used two-mode formulas:

$$p(n,t)=\frac{1}{n!}\frac{d^n C_{\mathcal{N}}(\lambda,t)}{d(-\lambda)^n}\bigg|_{\lambda=1}, \tag{44}$$

$$\langle W^k\rangle_{\mathcal{N}}=\frac{d^k C_{\mathcal{N}}(\lambda,t)}{d(-\lambda)^k}\bigg|_{\lambda=0}, \tag{45}$$

where [see Ref. 23, Sec. (8.5)]

$$\begin{aligned} C_{\mathcal{N}}(\lambda,t)&=\frac{1}{(\pi\lambda)^2}\int\int\exp\left[-\frac{1}{\lambda}(|\beta_j|^2+|\beta_{j+1}|^2)\right]C_{\mathcal{N}}(\beta_j,\beta_{j+1},t)d^2\beta_j d^2\beta_{j+1} \\ &=\frac{1}{\lambda^2 R_1^{1/2}}\exp(R_2/R_1)\ ; \end{aligned} \tag{46}$$

the coefficients a_l and b_l of the polynomials $R_1=\sum_{l=0}^4 b_l\lambda^{-l}$, $R_2=\sum_{l=0}^3 a_l\lambda^{-l}$ are expressed in terms of $B_j, B_{j+1}, C_j, C_{j+1}, B_{j,j+1}, C_{j,j+1}, \xi_j, \xi_{j+1}$, $j=1,3$ in a complicated form and we refer the reader to Ref. 48. The derivatives (44) and (45) were performed explicitly. Concerning a numerical calculation of these derivatives, they were sufficient only up to about $n=4$. Of course, formula (46) can be written in a form of the fourfold convolution of the Laguerre polynomials[48] which provide a more complicated way to obtain $p(n,t)$ and $\langle W^k\rangle_{\mathcal{N}}$ because the roots of the polynomial R_1 must be calculated.

In spite of the complicated form of the formula for $\langle W^k\rangle_{\mathcal{N}}$, we can easily establish the normal moments of the first and second orders according to definition (45)

$$\begin{aligned} \langle W(t)\rangle_{\mathcal{N}}&=\sum_j [B_j(t)+|\xi_j(t)|^2]\ , \\ \langle W^2(t)\rangle_{\mathcal{N}}&=\sum_j \{\tfrac{1}{2}(E_j-1)^2+\tfrac{1}{2}(F_j-1)^2+2A_{1j}(E_j-1) \\ &\qquad +2A_{2j}(F_j-1)+[B_j(t)+|\xi_j(t)|^2]^2\}\ . \end{aligned} \tag{47}$$

For the variance of the integrated intensity it holds that

$$\begin{aligned} \langle[\Delta W(t)]^2\rangle_{\mathcal{N}}=\sum_j \{&B_j^2(t)+|C_j(t)|^2+2B_j(t)|\xi_j(t)|^2 \\ &+2\,\mathrm{Re}[C_j^*(t)\xi_j^2(t)]\}\ . \end{aligned} \tag{48}$$

In (47) and (48) the summation over j includes two cases, either $j=1,2$ or $j=3,4$ denoting, respectively, either signal fields or pump fields.

In order to investigate squeezing properties, we have found

$$\begin{aligned} \langle(\Delta^{\hat{Q}_{j,j+1}}_{\hat{P}_{j,j+1}})^2\rangle=2[&1+B_j+B_{j+1}+2\,\mathrm{Re}B_{j,j+1} \\ &\pm\mathrm{Re}(C_j+C_{j+1}+2C_{j,j+1})]\ , \\ &\qquad j=1,3 \end{aligned} \tag{49}$$

where

$$\begin{aligned} \hat{Q}_{j,j+1}&=\hat{A}_j+\hat{A}_{j+1}+\hat{A}_j^\dagger+\hat{A}_{j+1}^\dagger\ , \\ \hat{P}_{j,j+1}&=-i(\hat{A}_j+\hat{A}_{j+1}-\hat{A}_j^\dagger-\hat{A}_{j+1}^\dagger)\ , \\ [\hat{Q}_{j,j+1},\hat{P}_{j',j'+1}]&=4i\delta_{jj'}\ . \end{aligned} \tag{50}$$

According to the standard definition, squeezing can be observed if

$$\min\{\langle(\Delta\hat{Q}_{j,j+1})^2\rangle,\ \langle(\Delta\hat{P}_{j,j+1})^2\rangle\}<2\ . \tag{51}$$

As to the definition of principal squeezing,[51] this occurs under the condition

$$\begin{aligned} \lambda_{j,j+1}=\min\{&\langle(u_1\Delta\hat{Q}_{j,j+1}+u_2\Delta\hat{P}_{j,j+1})^2\rangle\ ; \\ &u_1^2+u_2^2=1\}<2\ . \end{aligned} \tag{52}$$

$\lambda_{j,j+1}$ can be expressed explicitly as follows

$$\begin{aligned} \lambda_{j,j+1}=2[&1+B_j(t)+B_{j+1}(t)+2\,\mathrm{Re}B_{j,j+1}(t) \\ &-|C_j(t)+C_{j+1}(t)+2C_{j,j+1}(t)|]\ . \end{aligned} \tag{53}$$

The principal squeezing coincides with the standard one supposing

$$\begin{aligned} \langle\Delta\hat{Q}_{j,j+1}\Delta\hat{P}_{j,j+1}\rangle=2\,\mathrm{Im}[&C_j(t)+C_{j+1}(t) \\ &+2C_{j,j+1}(t)]=0\ . \end{aligned} \tag{54}$$

Condition (52), necessary and sufficient for the occurring squeezing effect, can be rewritten in the form

$$\begin{aligned} B_j(t)+B_{j+1}(t)+4\,\mathrm{Re}B_{j,j+1}(t)<|&C_j(t)+C_{j+1}(t) \\ &+2C_{j,j+1}(t)|\ . \end{aligned} \tag{55}$$

The entropy of fields under consideration can be treated analogously as in the one-mode case in a closed form.[45] Introducing the notation

$$M \equiv M_j = B_j^2 + B_{j+1}^2 - |C_j|^2 - |C_{j+1}|^2 + 2(|B_{j,j+1}|^2 - |C_{j,j+1}|^2) ,$$

$$\begin{aligned} N \equiv N_j = {} & B_j^2 B_{j+1}^2 - 2B_j B_{j+1}(|B_{j,j+1}|^2 + |C_{j,j+1}|^2) + (|B_{j,j+1}|^2 - |C_{j,j+1}|^2)^2 \\ & - B_j^2 |C_{j+1}|^2 - B_{j+1}^2 |C_j|^2 + |C_j|^2 |C_{j+1}|^2 + 2B_j(C_{j+1} B_{j,j+1} C_{j,j+1} + \text{c.c.}) \\ & + 2B_{j+1}(C_j B_{j,j+1}^* C_{j,j+1} + \text{c.c.}) - (C_j C_{j+1} C_{j,j+1}^2 + C_j C_{j+1}^* B_{j,j+1}^{*2} + \text{c.c.}), \quad j = 1,3 \end{aligned} \tag{56}$$

the entropy of two-mode fields formed by signal modes 1 and 2 and pump modes 3 and 4, respectively, develops according to the formula

$$H_{j,j+1}(t) \equiv H = H_j + H_{j+1} , \tag{57}$$

where

$$H_k = h(\Lambda_k'), \quad \Lambda_k' = \Lambda_k - \tfrac{1}{2}, \quad k = j, j+1$$

$$\Lambda_{j,j+1} = \left\{ \frac{M}{2} \mp \left[\left(\frac{M}{2} \right)^2 - N \right]^{1/2} \right\}^{1/2} ; \tag{58}$$

the function $h(x)$ is given in (37).

We restricted ourselves to two-mode signal and two-mode pump fields because the measurements of nonclassical effects in two-mode signal-pump fields are difficult to perform due to substantially different levels of intensities of signal and pump modes, respectively.

IV. NUMERICAL RESULTS

To demonstrate the complex formulas introduced above, we use the results of the short-length[28] and short-time[29,31,33] analyses in order to choose parameters giving rise to nonclassical behavior of the studied radiation. We assume four-wave mixing described by Hamiltonian (1) under the conditions

$$\xi_1(0) = 1, \quad \xi_2(0) = \exp\left[i\frac{\pi}{2} \right], \quad \xi_3(0) = 2 ,$$

$$\xi_4(0) = 2, \quad g = 1 \tag{59}$$

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0 ,$$

$$\langle n_{d_1} \rangle = \langle n_{d_2} \rangle = \langle n_{d_3} \rangle = \langle n_{d_4} \rangle = 0 . \tag{60}$$

In Fig. 1 we present the second reduced factorial moment (29) for $\tau = gt$ for single-signal modes 1,2 (curve a), single-pump modes 3,4 (curve b), two-mode signal field $\{12\}$ (curve c), and two-mode pump field $\{34\}$ (curve d). For this purpose formulas (26), (28), (47), and (48) have been used. Curve a indicates the super-Poisson behavior of single-signal modes. The tendency of single-pump modes to conserve coherence at the beginning of the process is obvious from the shape of curve b. The negative values of the second reduced factorial moment for two-mode fields $\{12\}$ and $\{34\}$ predict the nonclassical sub-Poisson effect. This effect is demonstrated for two-mode signal field in Fig. 2 visualizing the sub-Poisson distribution. It is a consequence of the coupling of modes. The time development of this photon number distribution according to formula (20) is obvious from Fig. 3(a). We can observe the initial tendency of two-mode signal field to the sub-Poisson behavior which changes later to the super-Poisson oscillating behavior reflecting the presence of competing states. The evolution of the photon number distribution of two-mode pump field under the same conditions can be seen from Fig. 3(b). As to Fig. 4, no squeezing has been found in single signal or pump modes (curves a and b) in agreement with the short-time analysis. Squeezing occurs in the two-mode pump field in the $\hat{P}$ quantity [curves d, Figs. 4(b) and 4(c)]. With respect to the principal squeezing definition, the conditions for observing this effect are also fulfilled for the two-mode signal field [curve c in Fig. 4(c)]. In Fig. 5 we can trace the time evolution of the entropy. The down-

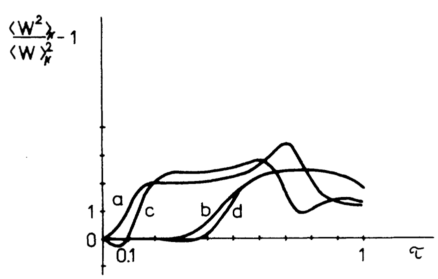

FIG. 1. Time development of the second reduced factorial moment for single-signal modes (curve a), single-pump modes (curve b), two-mode signal field (curve c), and two-mode pump field (curve d).

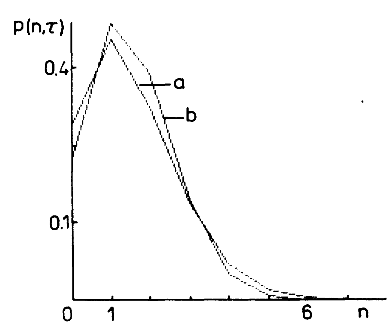

FIG. 2. Comparison of the photon number distribution of two-mode signal field (curve b) with the corresponding Poisson distribution (curve a) for the maximum quantum effect $[\langle(\Delta W)^2\rangle_{\mathcal{N}}/\langle W^2\rangle_{\mathcal{N}} = -0.18, \langle W\rangle_{\mathcal{N}} = 1.33, \tau = 0.06]$.

ward tendency of curve c may reflect the sub-Poisson behavior, although there need not generally be a direct connection between the sub-Poisson behavior and the decreasing entropy. Such a global characteristics as entropy cannot give so much information as detailed characteristics given by the photon number distributions. In agreement with results in Ref. 50, the squeezing process does not change the value of the von Neumann entropy.

We also investigated the role of the damping in the studied process. The nonclassical behavior is smoothed out with increasing damping as demonstrated on the squeezing effect in the $\hat{P}$ quantity in Fig. 6. Here we used the abbreviated notation $\{\gamma\}=a, \{\langle n_d \rangle\}=b$ which means that each quantity γ, i.e., $\gamma_j, j=1,2,3,4$ equals a and analogously each $\langle n_d \rangle$, i.e., $\langle n_{d_j} \rangle$, $j=1,2,3,4$ takes on the same value b, The noise in the signal modes increases more rapidly than the noise in the pump modes. The pump modes retain coherence until a certain time.

Similar behavior has been found if single-signal modes or single-pump modes or two-mode signal field start from vacuum fluctuations (spontaneous process).

V. CONCLUSION

The statistical properties of radiation in four-wave mixing evolving from the coherent states have been investigated in the framework of the model of the generalized superposition of coherent and chaotic fields. Special attention has been devoted to single-mode fields and two-mode signal and two-mode pump fields.

One-mode signal fields exhibit the super-Poisson behavior, whereas one-mode pump fields can be sub-Poissonian. For the two-mode signal field and the two-mode pump field, respectively, the sub-Poisson behavior occurs under certain conditions on phases of the initial complex field amplitudes.

In accordance with the short-time analysis there are no conditions for observing squeezing in single-signal modes. It is a consequence of the predominance of pumping

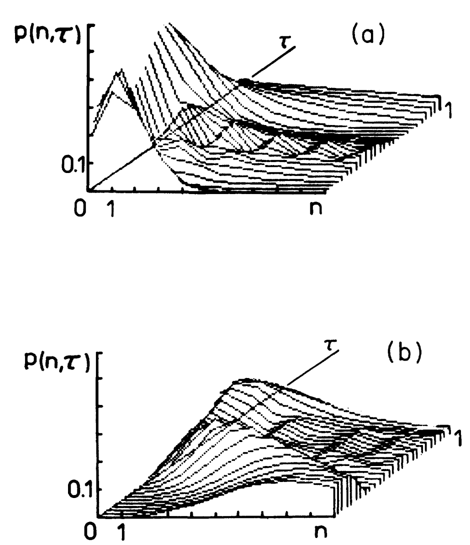

FIG. 3. (a) Time development of the photon number distributions of two-mode signal field and (b) of two-mode pump field.

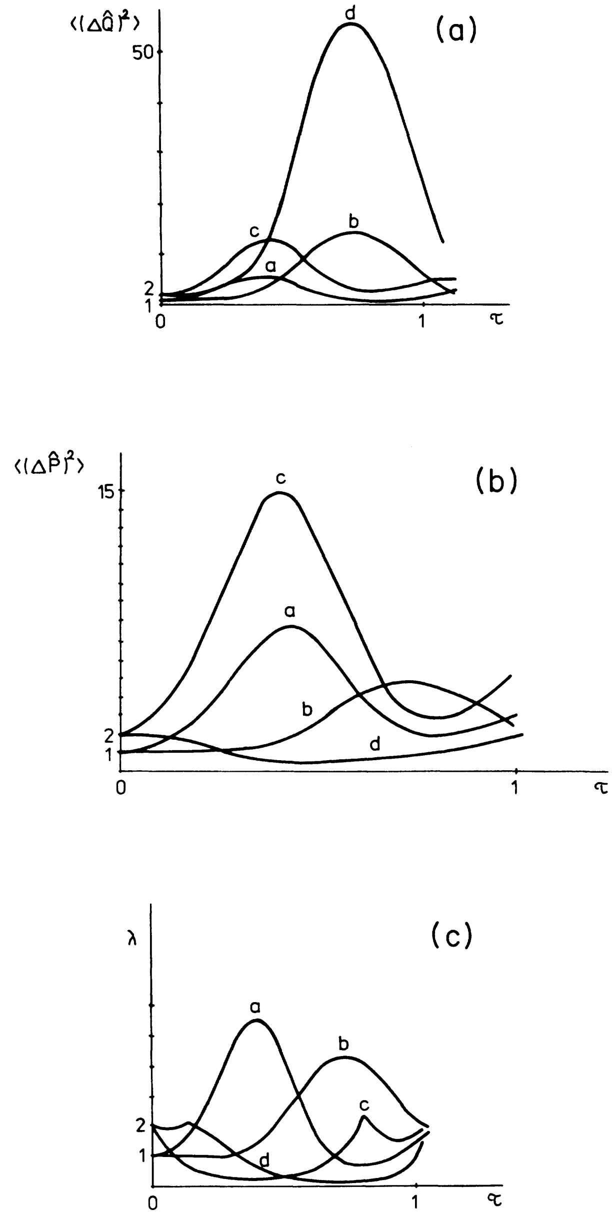

FIG. 4. (a) Time behavior of $\langle(\Delta\hat{Q})^2\rangle$, (b) $\langle(\Delta\hat{P})^2\rangle$, and (c) λ for single-signal modes (curves a), for single-pump modes (curves b), two-mode signal field (curves c), and for two-mode pump field (curves d).

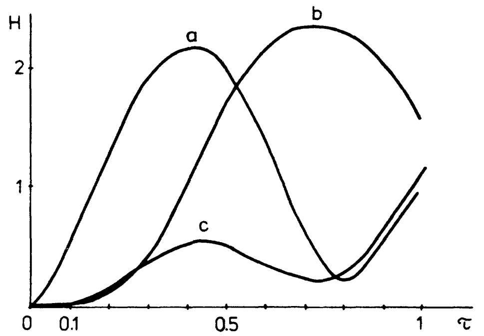

FIG. 5. Evolution of the entropy of single-signal modes (curves a), single-pump modes (curves b), and two-mode signal field (curve c).

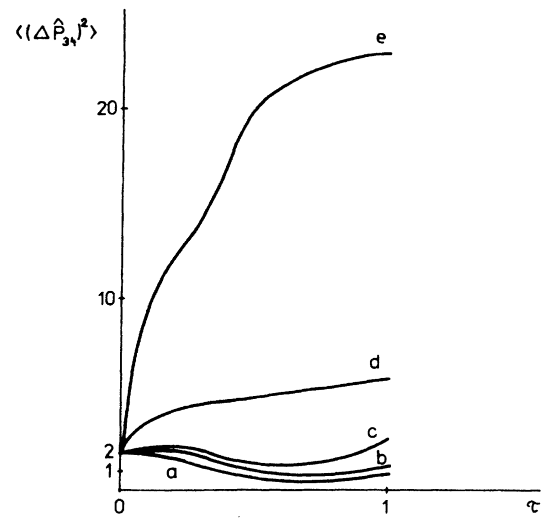

FIG. 6. Time development of the variance $\langle(\Delta\hat{P}_{34})^2\rangle$ for $\xi_1=1$, $\xi_2=\exp[i(\pi/2)]$, $\xi_3=\xi_4=2$, and $\{\gamma\}=0$, $\{\langle n_d\rangle\}=0$ (curve a), $\{\gamma\}=1$, $\{\langle n_d\rangle\}=0$ (curve b), $\{\gamma\}=1$, $\{\langle n_d\rangle\}=1$ (curve c), $\{\gamma\}=5$, $\{\langle n_d\rangle\}=1$ (curve d), $\{\gamma\}=5$, $\{\langle n_d\rangle\}=5$ (curve e).

modes over signal ones in the intensity of light. In single-pump modes squeezing is very probable, but this phenomenon is difficult to observe experimentally. This is caused by the fact that it is not admissible to approximate the pump modes by the pure coherent states but rather by the states of the signal plus noise. Squeezing occurs in two-mode signal and two-mode pump fields under certain phase conditions as a result of coupling of modes. Taking account of the lossy mechanism, we can see that the nonclassical effects are quickly smoothed out with increasing values of the damping constants and the mean numbers of the reservoir oscillators.

The closed-form formulas for the entropy of the appropriate optical fields allow us to trace the time evolution of this measure of noise and order in quantum optics.

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