Dye-laser equation with saturation and its best Fokker-Planck equation

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We carry out numerical simulations of the evolution and steady-state properties of a dye laser operating near resonance. The model we use is fully two dimensional (i.e., it includes the intensity and the phase of the radiation) and does not involve the usual low-intensity expansions commonly used in the literature. The equation for the complex electric field is driven by white additive noise (quantum fluctuations due to spontaneous emission) and by colored multiplicative noise (fluctuations of the gain parameter). We note that the simulations agree with the experimental results of Zhu, Yu, and Roy [Phys. Rev. A **34**, 4333 (1986)] for parameter values used in their experiments. We compare the simulation results with predictions of the "best Fokker-Planck equation" (BFPE) for the steady-state distribution of the laser intensity and find remarkable quantitative agreement for large ranges of parameters values, even for values that are in principle beyond the range of validity of this equation. We conclude that the BFPE can be safely invoked to make predictions for this system.

I. INTRODUCTION

The single-mode dye laser operating near resonance serves as a generic system for the study of the effects of noise on physical systems.[1-9] This system has been particularly useful for a number of reasons. First, the equation describing it is sufficiently simple to be amenable to analysis and/or numerical simulations. Second, experimental measurements on dye lasers provide practical results against which to measure the success of the models and of the calculations based on these models.[2-6] Third, the noises driving the dye laser include many of the features that are currently of great interest in theoretical studies: the complex field amplitude is driven by white additive noise (spontaneous quantum fluctuations) and by multiplicative colored noise (correlated fluctuations of the gain parameter).

A number of theoretical models have been used to describe the single-mode dye laser near threshold.[1-12] Most of these treatments have made one or another of three approximations or assumptions: they have expanded the equations in a power series in the complex electric field, retaining contributions up to cubic order; they have only included one or the other of the noise sources; they have approximated the two-dimensional problem (intensity and phase) by a one-dimensional problem (intensity). These approximations have been made in part for historical reasons and in part for analytic tractability.

Our purpose in this paper is twofold: to analyze the behavior of the two-component dye laser near resonance using the full unexpanded equation for the complex electric field[1,7,8] via numerical simulations, and to test the validity and success of the "best Fokker-Planck equation"[13-15] (BFPE) in predicting the behavior of the dye laser. We find that variation of the intensity or of the correlation time of the colored noise may induce similar effects in the distribution of the laser intensity in the steady state. We also find that the BFPE reproduces the numerical simulations with extreme accuracy over wide ranges of parameter values, even in regimes where one might expect it to fail.

In Sec. II we present the Langevin equation for the complex field amplitude that forms the basis of our further analysis. In Sec. III we obtain the BFPE and present explicit solutions of it in the steady state for certain ranges of parameter values. Section IV presents our numerical simulations and the comparison of these simulations with the results of the BFPE treatment. A summary of conclusions is presented in Sec. V.

II. LANGEVIN EQUATION

The dimensionless equation of motion for the complex field amplitude $E(t)$ of a single-mode dye laser operating near resonance is currently thought to be of the form[1,4,7,8]

$$\frac{dE}{dt}=\left[\frac{A_g}{1+|E|^2}-a_l\right]E+\frac{E}{1+|E|^2}p(t)+q(t)\ , \qquad (2.1)$$

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where A_g and a_l are the gain and loss parameters, respectively. The functions $p(t)$ and $q(t)$ represent the fluctuations of the gain parameter and fluctuations due to spontaneous emission. These fluctuations are assumed to be complex zero-centered Gaussian processes. The additive fluctuations $q(t)=q_1(t)+iq_2(t)$ are δ correlated in time (i.e., white), with

$$\langle q_i(t)q_j(t')\rangle = 2D_w\delta_{ij}\delta(t-t') \tag{2.2}$$

with $i,j=1$, 2, and D_w the intensity of the additive fluctuations. The multiplicative fluctuations $p(t)=p_1(t)+ip_2(t)$ are colored and are assumed to have an exponential correlation function,

$$\langle p_i(t)p_j(t')\rangle = \delta_{ij}(D/\tau)e^{-|t-t'|/\tau} \rightarrow 2D\delta_{ij}\delta(t-t') \text{ as } \tau\rightarrow 0 . \tag{2.3}$$

The relation between the dimensionless equations (2.1)–(2.3) and a corresponding dimensional form is discussed in Appendix A.

The change of variables to polar coordinates according to

$$E(t)=|E(t)|e^{i\phi(t)} \tag{2.4}$$

and introduction of the intensity

$$I(t)=|E(t)|^2 \tag{2.5}$$

allows us to rewrite (2.1) as

$$\frac{dI}{dt}=2\left[-a_l+\frac{A_g}{1+I}\right]I+\frac{2I}{1+I}p_1(t) +2\sqrt{I}\,[q_1(t)\cos\phi+q_2(t)\sin\phi] , \tag{2.6a}$$

$$\frac{d\phi}{dt}=\frac{1}{1+I}p_2(t)+\sqrt{I}\,[q_2(t)\cos\phi-q_1(t)\sin\phi] . \tag{2.6b}$$

Notice that (2.6) is the full two-dimensional Langevin equation for the dye laser. This equation includes saturation which is kept only to leading order in I in Lamb's equation. One can rewrite (2.6) in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} I \\ \phi \end{bmatrix} = \begin{bmatrix} 2I[-a_l+A_g/(1+I)] \\ 0 \end{bmatrix} + \frac{1}{1+I}\begin{bmatrix} 2I & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \frac{1}{\sqrt{I}}\begin{bmatrix} 2I\cos\phi & 2I\sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} . \tag{2.7}$$

By introducing the vectors $\mathbf{x}^T=(I,\phi)$, $\mathbf{p}^T=(p_1,p_2)$, $\mathbf{q}^T=(q_1,q_2)$, and $G^T(\mathbf{x})=[(2\{-a_l+A_g[1/(1+I)]\})I,0]$, where T denotes the transpose, and the matrices

$$\mathbf{g}^c(\mathbf{x})=\frac{1}{1+I}\begin{bmatrix} 2I & 0 \\ 0 & 1 \end{bmatrix} , \tag{2.8}$$

$$\mathbf{g}^w(\mathbf{x})=\frac{1}{\sqrt{I}}\begin{bmatrix} 2I\cos\phi & 2I\sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} , \tag{2.9}$$

one can rewrite (2.7) as

$$\dot{\mathbf{x}}=\mathbf{G}(\mathbf{x})+\mathbf{g}^c(\mathbf{x})\mathbf{p}(t)+\mathbf{g}^w(\mathbf{x})\mathbf{q}(t) . \tag{2.10}$$

The superscripts c and w denote, respectively, the functional coefficients of the *colored* and *white* fluctuations.

III. BEST FOKKER-PLANCK EQUATION

In this section we obtain the BFPE associated with the two-dimensional Langevin equation (2.10).[13–15] The Liouville operator L_0 associated with the systematic portion of (2.10) and the corresponding operators L_c and L_w associated with the stochastic portions of (2.10) are

$$L_0=-\nabla_{\mathbf{x}}\cdot\mathbf{G}(\mathbf{x}) , \tag{3.1}$$

$$L_c(t)=-\nabla_{\mathbf{x}}\cdot\mathbf{g}^c(\mathbf{x})\mathbf{p}(t) , \tag{3.2}$$

$$L_w(t)=-\nabla_{\mathbf{x}}\cdot\mathbf{g}^w(\mathbf{x})\mathbf{q}(t) , \tag{3.3}$$

where $\nabla_{\mathbf{x}}\equiv(\partial/\partial I,\partial/\partial\phi)$. We have shown elsewhere that the BFPE is given by[14,15]

$$\frac{\partial}{\partial t}W_t=-\nabla_{\mathbf{x}}\cdot\mathbf{G}(\mathbf{x})W_t+D_w[\nabla_{\mathbf{x}}\cdot\mathbf{g}^w(\mathbf{x})]^T[\nabla_x\cdot\mathbf{g}^w(\mathbf{x})]^TW_t + \int_0^t dt'\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle \times W_t[1+O(D\tau)] , \tag{3.4}$$

where $W_t\equiv W(\mathbf{x},t|\mathbf{x}_0)$ is the conditional probability density. Written out explicitly, the first and second terms on the right-hand side of (3.4) are

$$-\nabla_{\mathbf{x}}\cdot\mathbf{G}(\mathbf{x})W_t=-2\frac{\partial}{\partial I}\left[-a_l+\frac{A_g}{1+I}I\right]IW_t , \tag{3.5}$$

$$D_w[\nabla_{\mathbf{x}}\cdot\mathbf{g}^w(\mathbf{x})]^T[\nabla_{\mathbf{x}}\cdot\mathbf{g}^w(\mathbf{x})]^TW_t = D_w\left[4\frac{\partial}{\partial I}\sqrt{I}\frac{\partial}{\partial I}\sqrt{I}-2\frac{\partial}{\partial I}+\frac{1}{I}\frac{\partial^2}{\partial\phi^2}\right]W_t . \tag{3.6}$$

The last term in (3.4) is evaluated in Appendix B using the cumulant resummation technique to yield

$$\int_0^t dt'\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle W_t = \left[4\frac{\partial}{\partial I}\frac{I}{1+I}\frac{\partial}{\partial I}\frac{I}{1+I}D_{11}(I)+\frac{D_{22}(I)}{(1+I)^2}\frac{\partial^2}{\partial\phi^2}\right]W_t , \tag{3.7}$$

where $D_{11}(I)$ and $D_{22}(I)$ are related to the generating function for the infinite series in τ that arises in the resummation technique. Using Eqs. (B9a) and (B9b), one shows that the diffusion function turns out to be independent of ϕ, a result difficult to anticipate. In Appendix B we show that $D_{22}(I)=D$. The function $D_{11}(I)\equiv D(I)$, on the other hand, is more complicated and can only be given in closed form in special cases. When $A_g=0$ we find the positive-definite diffusion function

$$D(I)=\frac{D}{1+2a_l\tau}\,{}_2F_1\left[1,1;2+\frac{1}{2a_l\tau};\frac{1}{1+I^2}\right] , \tag{3.8}$$

where F is a hypergeometric function. For $A_g\gg a_l$ we obtain

$$D(I)=D_c \equiv D/(1+2a_l\tau) , \quad (3.9)$$

which is independent of the intensity.

The three contributions (3.5), (3.6), and (3.7) determine the two-dimensional BFPE. It is noteworthy that due to the phase independence of the diffusion functions $D_{ii}(I)$ one can eliminate the angle variable ϕ by integrating the equation. This property is well known in the white-noise limit and has been assumed (but never before shown to be true) in the presence of colored fluctuations.[8] We thus define a reduced probability density

$$P_t \equiv P(I,t|I_0) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi_0 W(I,\phi,t|I_0,\phi_0) \quad (3.10)$$

which then satisfies the evolution equation

$$\frac{\partial}{\partial t}P_t = -2\frac{\partial}{\partial I}\left[\left[-a_l+\frac{A_g}{1+I}\right]I+D_w\right] +4D_w\frac{\partial}{\partial I}\sqrt{I}\frac{\partial}{\partial I}\sqrt{I}P_t +4\frac{\partial}{\partial I}\frac{I}{1+I}\frac{\partial}{\partial I}\frac{I}{1+I}D(I)P_t . \quad (3.11)$$

We wish to stress that (3.11) is a consequence of the full bidimensional dye-laser equation (with saturation), in contrast with the phenomenological extension of the white-noise equation used by Aguado and San Miguel.[7]

Equation (3.11) reduces to the appropriate equation in the white-noise limit when $D(I)\to D$. When $A_g \gg a_l$ the state independence of the diffusion function implies that the BFPE (3.11) has the white-noise-limit form but with the renormalized diffusion coefficient (3.9). In this case the steady-state solution of (3.11) is given by the expression

$$P_{ss}(I)=N(1+I)(I+\chi_1)^{\alpha}(I+\chi_2)^{\beta}\exp(-a_l I/2D_w) , \quad (3.12)$$

where N is the normalization constant and we have defined the following parameter combinations:

$$\chi_{1,2}=1+\frac{D_c}{2D_w}\pm\frac{D_c}{2D_w}(1+4D_w/D_c)^{1/2} , \quad (3.13a)$$

$$\alpha=S/2D_w-\beta-1 , \quad (3.13b)$$

$$\beta=\frac{A_g+2D_w-S\chi_1}{2D_c(1+4D_w/D_c)^{1/2}}-1 , \quad (3.13c)$$

with

$$S=A_g+2D_w+a_l D_c/D_w \quad (3.13d)$$

and D_c given in (3.9). Note that $P_{ss}(I)$ is well defined for all positive values of the intensity and that $P_{ss}(0)=N\chi_1^{\alpha}\chi_2^{\beta}\neq 0$.

It is a simple matter to find the number and location of the extrema of the stationary distribution. $I=0$ is always an extremum, and the others are given by the solutions of the equation

$$I^3-\left[\frac{A_g}{a_l}-3\right]I^2-\left[\frac{2A_g}{a_l}-3\right]I+\left[\frac{2D_c-A_g}{a_l}+1\right]=0 . \quad (3.14)$$

The additive noise (through D_w) has no influence on the extrema of $P_{ss}(I)$. If we now assume that $A_g>3a_l$ [which is consistent with the condition $A_g \gg a_l$ necessary for the applicability of the BFPE with $D(I)$ given by (3.9)], it is easy to see that there is a critical value for the diffusion coefficient, given by

$$D_c^{\rm cr}=\frac{A_g-a_l}{2} . \quad (3.15)$$

If $D_c \le D_c^{\rm cr}$ then the stationary distribution has only one extremum (a maximum) aside from $I=0$ (which is a minimum). On the other hand, if $D_c>D_c^{\rm cr}$ then, in addition to the maximum at $I=0$, the distribution either has two extrema (a maximum and a minimum) or none at all. It is interesting to note how the shape of the stationary distribution is modified when the parameters characterizing the multiplicative noise are changed. From (3.9) and (3.15) two interesting possibilities are apparent. (i) If we fix τ, then it is always possible to change the shape of the distribution by increasing the noise intensity D. The change of shape occurs at the renormalized value

$$D_c=(1+2a_l\tau)(A_g-a_l)/2 . \quad (3.16)$$

(ii) If we fix D_c, then a change in distribution shape with changing τ is *only* possible if $D_c>D_c^{\rm cr}$. In this latter case the change of shape occurs at the value

$$\tau=[2D_c-(A_g-a_l)]/[2a_l(A_g-a_l)] . \quad (3.17)$$

Thus, although in some parameter regimes similar changes in the steady-state distribution can be accomplished by changing either the renormalized noise intensity or its correlation time, in general the role of these two parameters, D_c and τ, may be quite different.

In closing this section we note that the BFPE for this system does not suffer from the problem of negative diffusion functions and the consequent sustained probability densities that have elicited criticism of the method.[16,17] Such problems would have arisen had we used the expanded version (A5) of the model (2.1); since the intensity in this system is not bounded, such an expansion is not appropriate over the entire phase space and should be avoided in any case.

IV. NUMERICAL SIMULATIONS

We have carried out direct numerical simulations of Eq. (2.1) and in this section we compare the results of these simulations with those of the BFPE. In some of our simulations we used the sets of parameters reported by Roy *et al.*[4,5] in their dye-laser experiments at 6% and at 20% above threshold.[18] In the 6% case the dimensionless parameter values are $a_l=116.7$, $A_g=123.7$, $D=0.05$, $\tau=0.05$, $D_w=1.84\times10^{-10}$, with $T^*=10^{-5}$ (see Appendix A: this implies that our dimensional time unit is 10^{-5} sec). For operation at 20% above threshold, the parameter values are $a_l = 108$, $A_g = 129.6$ $D = 0.15$, $\tau = 0.042$, $D_w = 4.38 \times 10^{-9}$.

In Figs. 1(a) and 1(c) we depict the transient behavior obtained from the simulations, which agrees very well with the experimental results reported by Zhu, Wu, and Roy.[4] In particular, the simulations reproduce the observed time at which the intensity begins to increase sharply, and they also reproduce the time that it takes for the intensity to reach the steady-state level of operation. With our scaling the steady state values 0.06 and 0.2 of the intensity are related, respectively, to the 6% and 20% operation levels of the laser. In Figs. 1(b) and 1(d) we show the corresponding steady-state probability distribution obtained from the simulations. Note that with these parameter values we cannot apply the BFPE with the effective diffusion coefficient (3.9) since the condition $A_g \gg a_l$ is not satisfied.

In Fig. 2 we simultaneously explore the behavior of the dye-laser steady-state intensity distribution as we vary various system parameters in order to find the conditions that lead to a "noise-induced phase transition,"[19] and we also test the validity of the BFPE (3.11) with (3.9). Here we have taken $A_g = 5$, $a_l = 1$, $D_w = 1$ and we have varied the parameters D and τ of the multiplicative noise. Notice the transition occurring both vertically (changing D) and horizontally (changing τ): the absolute maximum and the number of relative maxima of the distribution change when one changes D or τ. This confirms the recent results of Aguado *et al.*[8] for a one-dimensional version of the unexpanded dye-laser model: they argue that it is not necessary to invoke noise color in order to get the noise-induced phase transition in which the maximum of the distribution jumps discontinuously from $I = 0$ to finite I, and that this transition can be achieved by changing the intensity of the white multiplicative noise.[8,20] We note that the agreement between the BFPE and the simulations is remarkably good for all the parameter values considered in this figure.[21] We thus conclude that when the condition $A_g \gg a_l$ is reasonably met the BFPE with the effective diffusion coefficient (3.9) is an excellent approximation for (at least the steady-state behavior of) the dye-laser equations (2.6a) and (2.6b).

Figure 3 shows the effect of the additive noise on the stationary distribution. These results should be com-

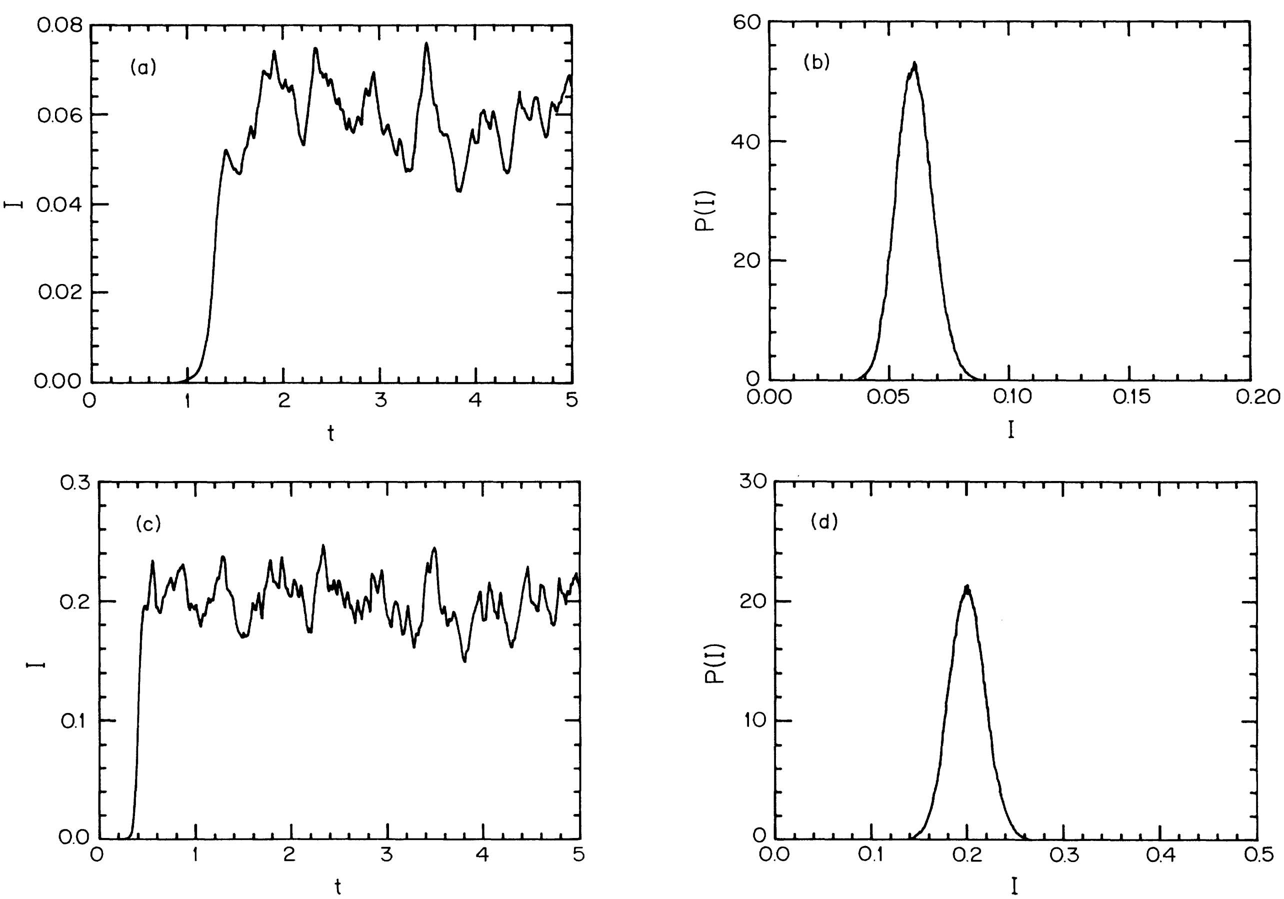

FIG. 1. Simulation results based on the two-variable laser model (2.6), using parameter values corresponding to the experiments of Zhu, Wu, and Roy. (a) and (c) show typical intensity trajectories as a function of time for operation at 6% above threshold (a) and 20% above threshold (c). (b) and (d) show the corresponding intensity distributions in the steady state.

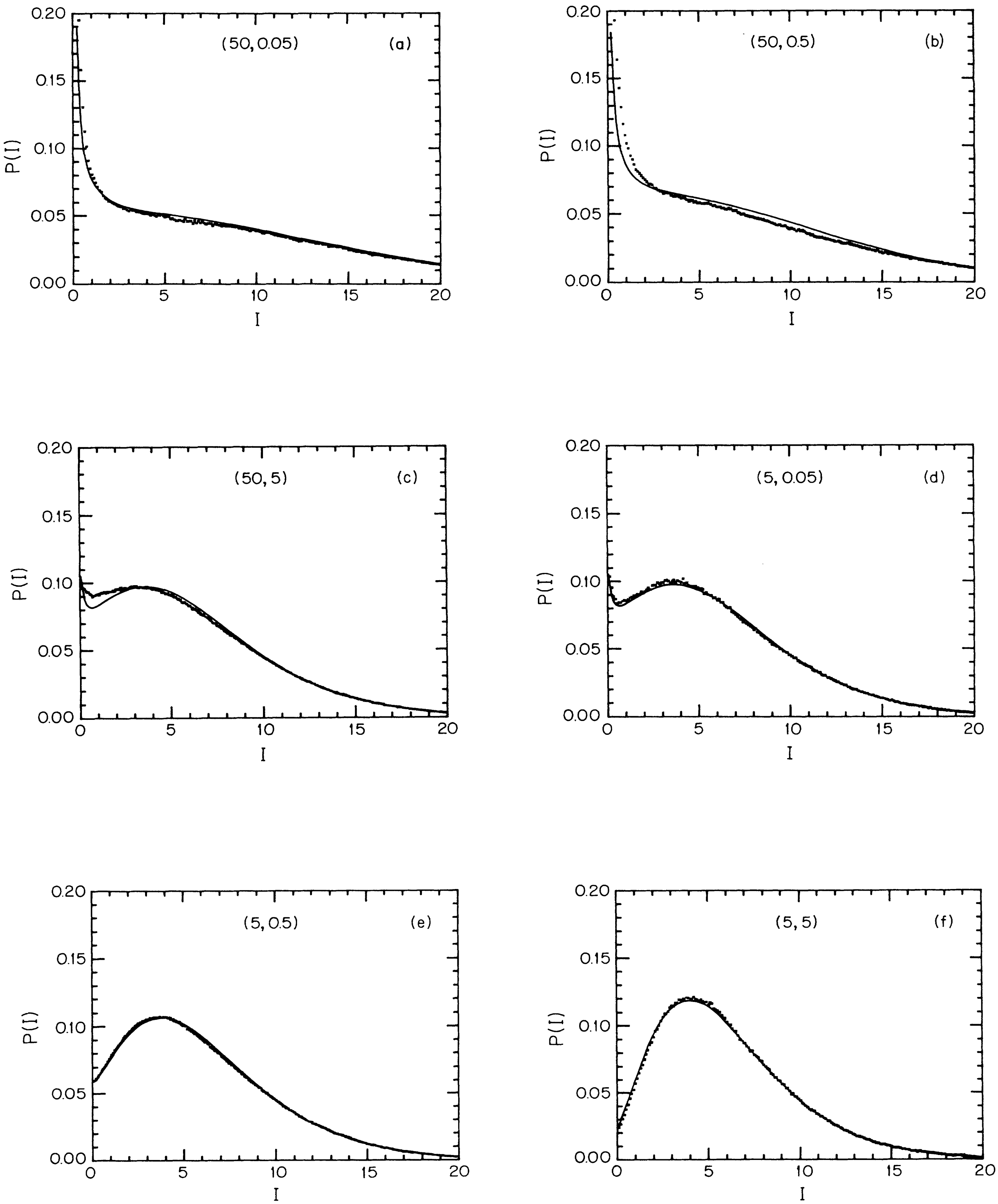

FIG. 2. Comparison of simulation results (dots forming jagged line) and the predictions of the BFPE for the intensity distribution in the steady state calculated from the model (2.6). The values $A_g = 5$, $a_l = 1$ and $D_w = 1$ are used throughout, and the values of D and τ are varied. The pair (D, τ) is indicated in each panel.

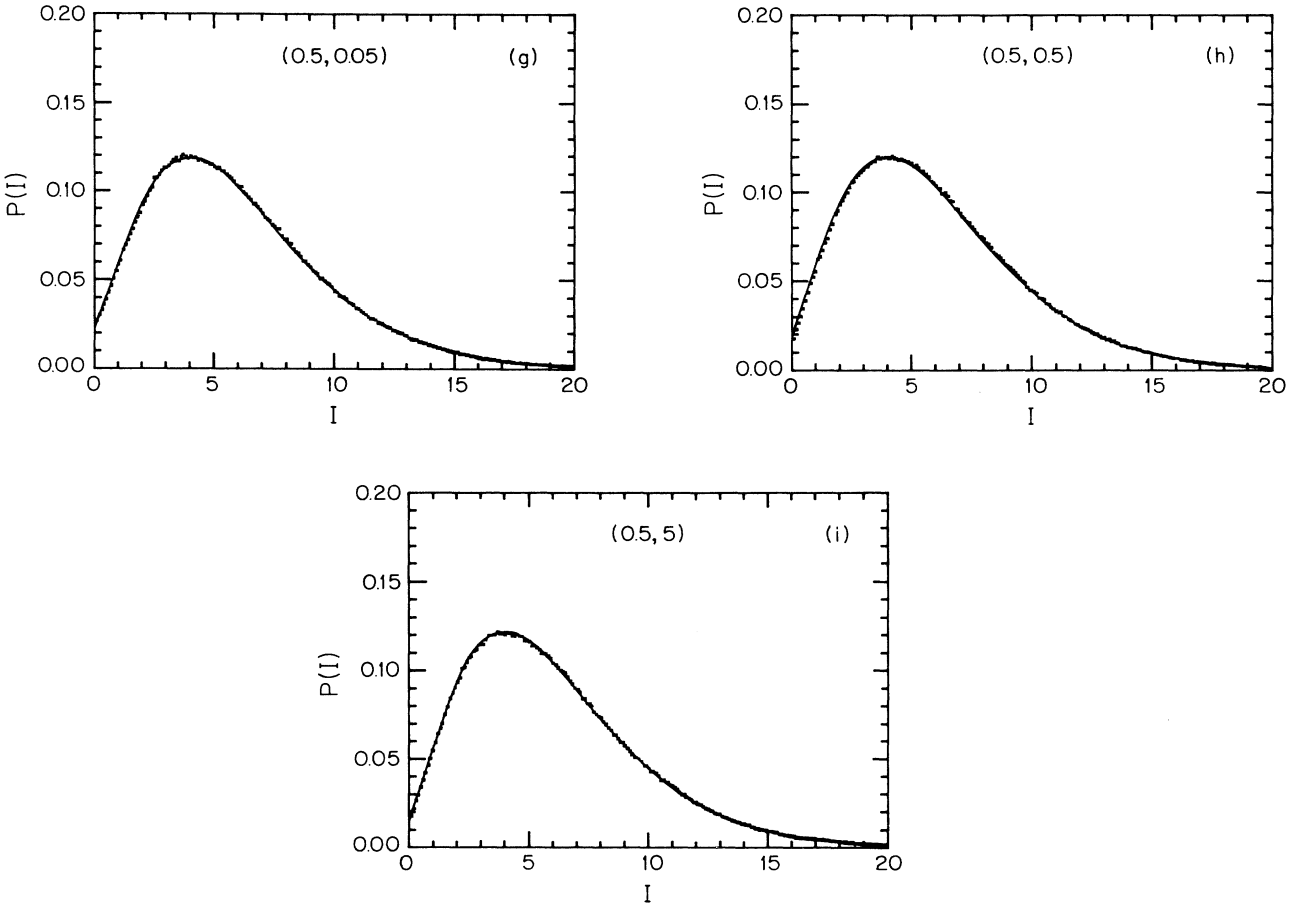

FIG. 2. (*Continued*).

pared with those of Figs. 2(d), 2(e), and 2(f) for the same values of the other parameters but with a much stronger additive noise. In particular, comparison of Fig. 3(b) with Fig. 2(e) shows the dramatic effect of the additive noise when the renormalized multiplicative noise intensity is close to the value (3.16). Note that even in these cases the BFPE provides an excellent approximation and captures the essential details of the process.[22]

Figure 4 shows the effect of the additive noise on the transient behavior of the laser for the parameter values $A_g=100$, $a_l=1$, $D=0.5$, $\tau=0.5$, and $D_w=10^{-6}$ [Fig. 4(a)] or $D_w=1$ [Fig. 4(b)]. As the intensity of the white additive noise increases, the transient behavior of the intensity becomes of shorter duration but the steady-state output of the laser is more erratic.

V. CONCLUSIONS

We have simulated the two-dimensional model (2.1) or, equivalently, (2.6), that describes a single-mode dye laser operating near resonance. We have avoided small-intensity expansions that may give rise to difficulties in the application of approximate analytic methods for the analysis of this problem. We have also obtained analytic solutions for the problem within the framework of the BFPE.

Our simulations agree well with experimental results where the latter are available, thus providing a check for the model and for our simulations. We are not able to find explicit analytic results within the BFPE formalism for these experimental parameter values, and so we choose others where we are able to find analytic expressions in order to compare the theoretical results with those of the simulations. The agreement between the two is remarkable, even in parameter regimes where the BFPE ceases in principle to be valid. This agreement includes regimes where the number of extrema of the intensity distribution in the steady-state changes from one to three, and also to regimes of vanishing additive noise where the distribution increases very sharply with intensity as $I \to 0$.

Because of the limitations on our analytic results, our analysis of the dye-laser system has been mostly limited to the high-gain regime, $A_g \gg a_l$. The behavior of the intensity distribution in this regime as the other parameters of the system are varied may be unimodal or bimodal, and in the unimodal cases the peak may occur at $I=0$ or at a finite intensity. For a fixed white-noise intensity and a fixed correlation time of the colored noise, increasing the intensity of the colored noise enhances the zero-intensity weight of the distribution. If we fix the intensity of both the white and the colored noise, then increasing

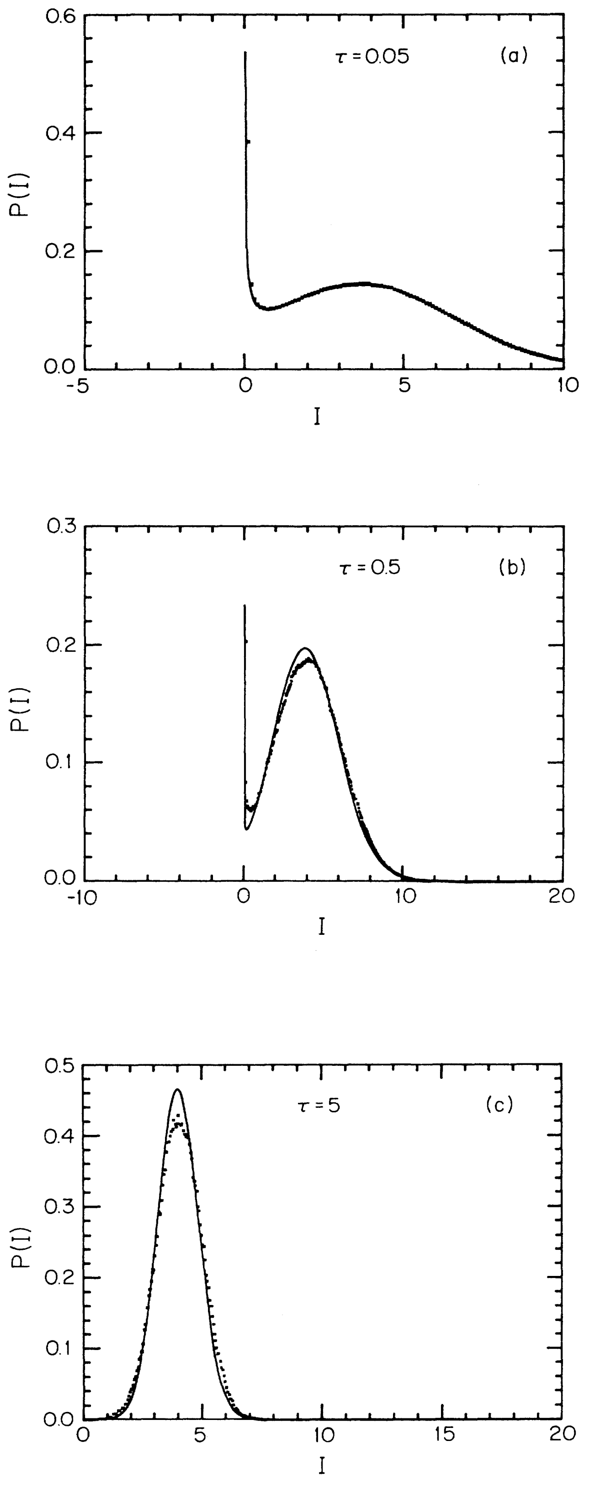

FIG. 3. Comparison of simulation results (dots forming jagged line) and the predictions of the BFPE for the intensity distribution in the steady state in the (near) absence of additive noise. In these calculations $A_g = 5$, $a_l = 1$, $D = 5$, $D_w = 0$ in the BFPE and $D_w = 10^{-6}$ in the simulations (see Ref. 22 for reasons for this small difference). The correlation time τ is changed as indicated on each panel.

the correlation time of the noise increases the finite-intensity contribution to the distribution. Finally, for fixed intensity and correlation time of the colored noise, increasing the white noise intensity favors the finite-intensity weight of the distribution.

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APPENDIX A

The equation of motion for the complex electric field $\Xi(T) = \Xi_1(T) + i\Xi_2(T)$ for a single-mode dye laser is given by[1,4,7,8]

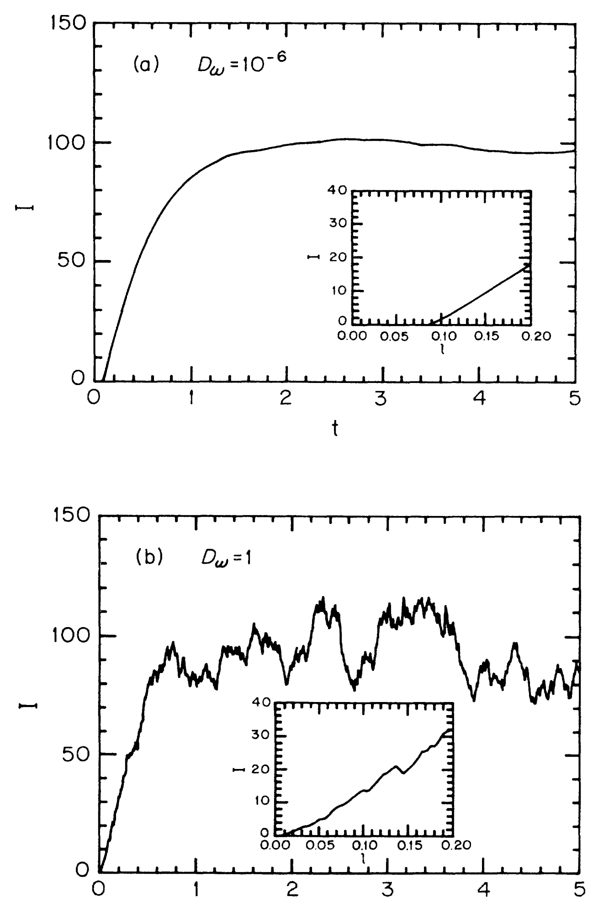

FIG. 4. Effect of additive noise on the transient behavior of the laser from simulations of the model (2.6). The figures show a typical intensity trajectory as a function of time for parameter values $A_g = 100$, $a_l = 1$, $D = 0.5$, $\tau = 0.5$. In (a), $D_w = 10^{-6}$; in (b), $D_w = 1$.

$$\frac{d\Xi}{dT}=\left[-a_l^{\rm ex}+\frac{A_g^{\rm ex}+p_c^{\rm ex}(T)}{1+c^{\rm ex}|\Xi|^2}\right]\Xi+q_w^{\rm ex}(T)\ , \tag{A1}$$

with $A_g^{\rm ex}$, $a_l^{\rm ex}$, and $c^{\rm ex}$ the experimental gain, loss, and scaling parameters, respectively. Here both Ξ and the time T carry appropriate units. The additive complex noise $q_w^{\rm ex}(T)=q_1^{\rm ex}(T)+iq_2^{\rm ex}(T)$ is white, with correlation functions

$$\langle q_i^{\rm ex}(T)q_j^{\rm ex}(T')\rangle=2D_w^{\rm ex}\delta_{ij}\delta(T-T')\ , \tag{A2}$$

where $D_w^{\rm ex}$ is the strength of the noise. The multiplicative pump noise $p_c^{\rm ex}(T)=p_1^{\rm ex}(T)+ip_2^{\rm ex}(T)$ is colored, with correlation functions

$$\langle p_i(T)p_j(T')\rangle=\frac{D^{\rm ex}}{\tau^{\rm ex}}\delta_{ij}e^{-|T-T'|/\tau^{\rm ex}}\ , \tag{A3}$$

where $D^{\rm ex}$ and $\tau^{\rm ex}$ are the strength and correlation time of the colored noise.

The electric field $\Xi(T)$ and the time T are usually scaled according to personal convenience. A common scaling changes the electric field to dimensionless form but leaves the time as a dimensioned quantity. Thus we define the dimensionless complex field $E\equiv\sqrt{c}\,\Xi$, where c is a scaling factor of appropriate units. This yields the equation

$$\frac{dE}{dT}=\left[-a_l^{\rm ex}+\frac{A_g^{\rm ex}+p_c^{\rm ex}(T)}{1+(c^{\rm ex}/c)|E|^2}\right]E+q_w(T)\ , \tag{A4}$$

where $q_w=\sqrt{c}\,q_w^{\rm ex}$. In much of the literature it is assumed that the dimensionless quantity $c^{\rm ex}|E|^2/c\ll 1$ (which cannot be true for all E), and Eq. (A4) is then expanded as

$$\frac{dE}{dT}=\left[(A_g^{\rm ex}-a_l^{\rm ex})-\frac{c^{\rm ex}}{c}A_g^{\rm ex}|E|^2\right]E+p_c^{\rm ex}(T)E+q_w(T)\ . \tag{A5}$$

This is the equation used by Roy *et al.*,[4,5] and the relation between their parameters (denoted by a superscript R) and those defined above is

$$A^R=\frac{c^{\rm ex}}{c}A_g^{\rm ex}=\frac{c^{\rm ex}}{c}A_g^R\ , \tag{A6a}$$

$$a_0^R=A_g^{\rm ex}-a_l^{\rm ex}\ , \tag{A6b}$$

$$D^R=D^{\rm ex}\ , \tag{A6c}$$

$$\tau^R=\tau^{\rm ex}\ , \tag{A6d}$$

$$D_w^R=cD_w^{\rm ex}\ . \tag{A6e}$$

Note that in order to deduce the actual experimental quantities from the parameter values reported by Roy *et al.* requires knowledge of their choice for the scaling factors $c^{\rm ex}$ and c (or use of the information contained in Ref. 18).

Others in the literature prefer to use a completely dimensionless equation (as we do in our text), which requires the scaling of both the complex field and the time. We have chosen

$$E=(c^{\rm ex})^{1/2}\Xi,\quad t=T/T^*\ . \tag{A7}$$

The relevant times for the laser problem are of the order $T\sim 10\ \mu$sec. If we wish the relevant dimensionless time t to be of $O(1)$, the scaling factor T^* must be of the order of 10^{-5} sec. With this scaling we obtain Eq. (2.1) with parameters related to those of Roy *et al.* as follows:

$$A_g=T^*A_g^R=T^*\frac{c}{c^{\rm ex}}A^R\ , \tag{A8a}$$

$$a_l=T^*a_l^R\ , \tag{A8b}$$

$$D=T^*D^R\ , \tag{A8c}$$

$$\tau=\tau^R/T^*\ , \tag{A8d}$$

$$D_w=\frac{c^{\rm ex}}{c}T^*D_w^R\ . \tag{A8e}$$

Another widely used dimensionless form of the equation arises when the strength of the multiplicative fluctuations is chosen to be unity and is obtained with the scaling

$$E^S=(c^{\rm ex})^{1/2}\left[\frac{A_g^{\rm ex}}{D^{\rm ex}}\right]^{1/2}\Xi,\quad t^S=D^{\rm ex}T\ . \tag{A9}$$

We use the superscript S to indicate that this is the scaling that has been used by Aguado *et al.*[7,8] One obtains the dimensionless equation

$$\frac{dE^S}{dt^S}=\left[-a_l^S+\frac{A_g^S+p_c^S(t^S)}{1+(1/A_g^S)|E^S|^2}\right]E^S+q_w^S(t^S)\ , \tag{A10}$$

whose parameters are related to our dimensionless ones as follows:

$$A_g^S=A_g/D\ , \tag{A11a}$$

$$a_l^S=a_l/D\ , \tag{A11b}$$

$$D^S=1\ , \tag{A11c}$$

$$\tau^S=D\tau\ , \tag{A11d}$$

$$D_w^S=(A_g/D)D_w\ . \tag{A11e}$$

Yet another frequently used scaling to a dimensionless equation arises when the strength of the additive fluctuations is chosen to be unity, as done by Lett *et al.*,[6]

$$E^L=(c^{\rm ex}A_g^{\rm ex}u/c)^{1/2}\Xi,\quad t^L=T/u\ , \tag{A12}$$

where

$$u=\frac{T^*}{(D_wA_g)^{1/2}}=\frac{1}{(A^RD_w^R)^{1/2}}\ . \tag{A13}$$

This scaling yields

$$\frac{dE^L}{dt^L}=\left[-a_l^L+\frac{A_g^L+p_c^L(t^L)}{1+(1/A_g^L)|E^L|^2}\right]E^L+q_w^L(t^L)\ , \tag{A14}$$

whose parameters are related to our dimensionless ones as follows:

$$A_g^L=uA_g/T^*\ , \tag{A15a}$$

$$a_l^L = ua_l/T^* \ , \tag{A15b}$$

$$D^S = uD/T^* \ , \tag{A15c}$$

$$\tau^S = T^*\tau/u \ , \tag{A15d}$$

$$D_w^L = 1 \ . \tag{A15e}$$

APPENDIX B

To evaluate the integrand in Eq. (3.4) we apply the Baker-Campbell-Hausdorff formula

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots \ , \tag{B1}$$

where [,] denotes the commutator. If we identify A with L_0t' and B with $L_c(t-t')$, then one can prove by induction that the series (B1) can be written as[14,15]

$$e^{L_0t'}L_c(t-t')e^{-L_0t'} = \sum_{n=0}^{\infty} \frac{t'^n}{n!}\frac{\partial}{\partial x_i}F_{ij}^{(n)}(\mathbf{x})p_j(t-t') \ , \tag{B2}$$

where the matrix elements $F_{ij}^{(n)}$ satisfy the recurrence relations

$$F_{ij}^{(0)}(\mathbf{x}) = g_{ij}^c(\mathbf{x}) \ , \tag{B3}$$

$$F_{ij}^{(n)}(\mathbf{x}) = \sum_k \left[F_{kj}^{(n-1)}\frac{\partial G_i}{\partial x_k} - G_k\frac{\partial F_{ij}^{(n-1)}}{\partial x_k} \right] . \tag{B4}$$

In these expressions $G_i(\mathbf{x})$ and $g_{ij}^c(\mathbf{x})$ are the elements of the vector $\mathbf{G}(\mathbf{x})$ and of the matrix $\mathbf{g}^c(\mathbf{x})$, respectively. Using these results we obtain

$$\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle = \frac{D}{\tau}e^{-t'/\tau}\sum_{i,j,k}\frac{\partial}{\partial x_i}g_{ij}^c(\mathbf{x})\frac{\partial}{\partial x_k}F_{kj}(\mathbf{x},t') \ , \tag{B5}$$

where we have defined

$$F_{kj}(\mathbf{x},t) \equiv \sum_{n=0}^{\infty}\frac{t^n}{n!}F_{kj}^{(n)}(\mathbf{x}) \ . \tag{B6}$$

We further introduce the decomposition (which constitutes a definition of the functions H_{ij})

$$F_{kj}(\mathbf{x},t) = \sum_l g_{kl}^c(\mathbf{x})H_{lj}(\mathbf{x},t) \tag{B7}$$

so that

$$\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle = \frac{D}{\tau}e^{-t'/\tau}\sum_{i,j,k,l}\frac{\partial}{\partial x_i}g_{ij}^c(\mathbf{x})\frac{\partial}{\partial x_k}g_{kl}^c(\mathbf{x})H_{lj}(\mathbf{x},t') \ . \tag{B8}$$

The functions $H_{ij}(\mathbf{x},t)$ satisfy the partial differential equation

$$\frac{\partial H_{ij}}{\partial t} = \sum_{k,l,s}\left[(\mathbf{g}^c)_{il}^{-1}\left[\frac{\partial G_l}{\partial x_k}\right]g_{ks}H_{sj} - (\mathbf{g}^c)_{il}^{-1}\left[\frac{\partial g_{ls}}{\partial x_k}\right]H_{sj}G_k \right] - \sum_k \left[\frac{\partial H_{ij}}{\partial x_k}\right]G_k \tag{B9a}$$

with initial condition

$$H_{ij}(\mathbf{x},0) = \delta_{ij} \ . \tag{B9b}$$

The solutions of this partial differential equation are

$$H_{ij}(\mathbf{x},t) = 0 \text{ for } i \neq j \ , \tag{B10a}$$

$$H_{22}(\mathbf{x},t) = 1 \ , \tag{B10b}$$

and $H_{11}(\mathbf{x},t)$ can only be written in general in terms of a transcendental equation,

$$a[H_{11}(I,t)]^{A_g/a_l} - (a - a_lI^2)[H_{11}(I,t)]^{A_g/a_l-1} = a_lI^2e^{-2at} \ , \tag{B10c}$$

where we have noted that H_{11} turns out to be independent of ϕ in $\mathbf{x}=(I,\phi)$ and where $a \equiv A_g - a_l$. Explicit solutions are only possible for particular parameter values. If $A_g=0$, then we find

$$H(I,t) = (1+I^2)/(1+I^2e^{2a_lt}) \ ; \tag{B11}$$

if $A_g \gg a_l$ then

$$H(I,t) = e^{-2a_lt} \ . \tag{B12}$$

Associated with these H functions we define the diffusion function

$$D_{ij}(I,t) = \int_0^t dt'\, H_{ij}(I,t')\frac{D}{\tau}e^{-t'/\tau} \ , \tag{B13}$$

in terms of which

$$\int_0^t dt'\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle P_t = \sum_{i,j,k,s}\frac{\partial}{\partial x_i}g_{ij}^c(\mathbf{x})\frac{\partial}{\partial x_k}g_{ks}^c(\mathbf{x})D_{sj}(\mathbf{x},t)P_t \ . \tag{B14}$$

When the correlation time τ is short, it is usually argued that the upper limit in the integration in (B14) can be set to infinity, and the diffusion functions thus become independent of time.[13–15] The off-diagonal diffusion functions vanish because of (B10a). The result (B10b) immediately yields $D_{22}(I)=D$. The function $D_{11}(I) \equiv D(I)$ can be given explicitly only in the cases when H_{11} can be expressed explicitly, as in Eqs. (3.8) and (3.9). With these results we finally have

$$\int_0^\infty dt'\langle L_c(t)e^{L_0t'}L_c(t-t')e^{-L_0t'}\rangle P_t = \left[4\frac{\partial}{\partial I}\frac{I}{1+I}\frac{\partial}{\partial I}\frac{I}{1+I}D(I) + \frac{D}{(1+I)^2}\frac{\partial^2}{\partial\phi^2}\right]P_t \ , \tag{B15}$$

which is used in Eq. (3.7).

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