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Attractor size in intermittent turbulence

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We estimate the number of degrees of freedom in intermittent fully developed turbulence using the multifractal formalism. We find that for any reasonable distribution $f(\alpha)$, the number of degrees of freedom is decreased by intermittency for space dimensionality $D<4$, and increased for $D>4$. This is in agreement with a result of Kraichnan [Phys. Fluids **28**, 10 (1985)] obtained for the special case of fractally homogeneous turbulence. We calculate the reduction in attractor size in three dimensions and find it to be very small. The possible dynamical significance of our result is briefly discussed.

In the multifractal formalism,[1,2] the dissipation ϵ_r averaged over a D-dimensional volume of size r, varies as

$$\epsilon_r \sim \langle\epsilon\rangle (r/L)^{\alpha-D}, \tag{1}$$

where $\langle\epsilon\rangle$ is the global average of the rate of dissipation, L is the external length scale of the flow, and α is a locally defined random variable. The usual Kolmogorov microscale is defined by

$$\bar{\eta} = (\nu^3/\langle\epsilon\rangle)^{1/4}, \tag{2}$$

where ν is the kinematic viscosity of the fluid. In the 1941 Kolmogorov[3] theory, $\bar{\eta}$ is the eddy size at which the cascade terminates, but in the presence of intermittency this smallest eddy size will fluctuate spatially due to fluctuations in ϵ. A reasonable estimate of the local value of η is given by

$$\eta = (\nu^3/\epsilon_\eta)^{1/4}, \tag{3}$$

where ϵ_η is obtained by setting $r=\eta$ in Eq. (1). Substituting this result into Eq. (3) and solving for η in terms of $\bar{\eta}$, gives

$$\eta/L = (\bar{\eta}/L)^{\gamma} \tag{4a}$$

with

$$\gamma = 4/(4+\alpha-D). \tag{4b}$$

Therefore, for any given value of α, we know the size at which an eddy will become stable.

To estimate the number of degrees of freedom in the flow, we assume that once an eddy has reached a size equal to η, it will not decay any further and can be counted approximately as a single degree of freedom. We thus need to know only how many such eddies of size η are present in the flow. In the multifractal formalism, the number of eddies of a given size η can be written as

$$N_\eta(\alpha) \sim (\eta/L)^{-f(\alpha)}. \tag{5}$$

The total number of degrees of freedom $N_{\rm tot}$ will then be given by the sum over all possible values of α:

$$N_{\rm tot} \sim \int N_\eta(\alpha)\, d\alpha \sim \int (\bar{\eta}/L)^{\delta}\, d\alpha, \tag{6a}$$

where

$$\delta = -4f(\alpha)/(4+\alpha-D). \tag{6b}$$

As usual, we evaluate the integral of Eq. (6a) in the limit of small $\bar{\eta}/L$ by steepest descent. It is convenient to write the results in terms of the moment exponents[4] D_q, which corresponds to the Legendre transform of $f(\alpha)$ according to

$$q = \partial f(\alpha)/\partial\alpha, \tag{7a}$$

$$(q-1)D_q = q\alpha - f(\alpha). \tag{7b}$$

Carrying out the steepest-descent evaluation, we obtain

$$N_{\rm tot} \sim (L/\bar{\eta})^{4Q}, \tag{8}$$

where Q is the solution of

$$Q(4-D) = (1-Q)D_Q, \tag{9}$$

or

$$Q = D_Q/(D_Q+4-D). \tag{10}$$

In the absence of intermittency $D_q=D$ for all q, and $Q=D/4$. Equation (8) then reduces to the usual Landau estimate of the number of degrees of freedom.[3] For fractally homogeneous turbulence, where one assumes that all turbulent activity is concentrated on a single fractal of dimension $D(\beta)$ (the β model[5]), $D_q=D(\beta)$ for all q, so that

$$Q = D(\beta)/[D(\beta)+4-D]. \tag{11}$$

Equation (11) has the property that $Q<D/4$ for $D<4$, and $Q>D/4$ for $D>4$, in agreement with Kraichnan's result[6] for this model of fractally homogeneous turbulence.

Our new result is that this crossover generalizes to an arbitrary physically reasonable $f(\alpha)$ curve. For intermittent turbulence, $D_q<D$ for all positive q. From Eq. (10) it thus follows that $Q<D/4$ for $D<4$, and $Q>D/4$ for $D>4$. Thus intermittency reduces the number of degrees of freedom when $D<4$, and increases it when $D>4$, whatever the precise multifractal distribution of the dissipation.

To estimate this reduction for $D=3$, we rewrite Eq. (10) as

$$\Delta Q = D/4 - Q = (4-D)(D-D_Q)/[4(D_Q+4-D)] , \quad (12)$$

and Eq. (8) as

$$N_{\rm tot} \sim (L/\bar{\eta})^{3-4\Delta Q} . \quad (13)$$

For values of q near $\frac{3}{4}$, the D_q curve can be approximated quite well[2] by the log-normal result

$$D_q = D - (\mu/2)q , \quad (14)$$

where μ is the intermittency exponent. Experimentally,[2] for $D=3$, $\mu \approx 0.25$, and $D_Q - D$ in the denominator of Eq. (12) is very small compared to 4. Equation (12) thus gives

$$\Delta Q \approx 3\mu/128 \approx 0.0059 . \quad (15)$$

The reduction in number of degrees of freedom due to intermittency is thus quite small.

It is interesting to speculate whether $D=4$ is a transition dimensionality for turbulence in a deeper sense. There is no known theoretical reason why intermittency cannot exist for $D>4$ even if it increases the attractor size compared to the 1941 Kolmogorov theory. It is perhaps worth reviving an older argument[7] which suggested that $D=4$ might be a transition dimensionality. It was suggested there that the basic small scale dynamical variable is the cube of a velocity derivative. This has some plausibility since the skewness of the velocity derivative plays an important dynamical role as the rate of production of mean-square vorticity. The natural scaling behavior of this quantity in the 1941 Kolmogorov theory is r^{-2}. Its square will then scale as r^{-4}, and its volume integral will diverge for $D<4$. In some loose sense, fluctuations of this dynamically relevant quantity will be dangerous only for $D<4$. Thus it is possible that intermittency vanishes for $D>4$, and that this could be a guide to a deeper dynamical theory.

Although this conjecture remains highly speculative, the result that attractor size is decreased by intermittency for $D<4$, and increased for $D>4$, appears to be quite robust. It does not depend on the model used to describe intermittency, but only on the assumption that the minimum eddy size varies as the $-\frac{1}{4}$ power of the average dissipation rate.

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