Screening in multifractal growth

Robin Ball and Martin Blunt*
Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, England
(Received 6 July 1988)

For any multifractal growth process we calculate how the probability of advance of a fixed site on the boundary of the structure changes as the fractal increases in size. We are then able to find expressions for the dimension of the active zone of the fractal and the distribution of ages of points from which growth occurs in terms of the scaling function $f(\alpha)$. For the case of diffusion-limited aggregation (DLA) and the screened-growth model, we offer a geometrical interpretation of the results. For DLA in arbitrary space dimensions we find a relation between the third moment of the probability distribution and the Hausdorff dimension D, which generalizes a result by Halsey [Phys. Rev. Lett. **59**, 2067 (1987)].

I. INTRODUCTION

Recently a large number of experimental and numerical studies have investigated the growth of structures whose surface is fractal.[1] The fractal can be considered to be composed of N discrete sites of unit size. If the overall extent of the structure is R, then N scales as R^D, where, by definition, D is the fractal or Hausdorff dimension. We can assign a probability p_i that growth will next occur from a given site i. The moments of the distribution of probabilities (the growth-site probability distribution, GSPD) are defined by

$$Z(q)=\sum_{i=1}^{N} p_i^q \tag{1}$$

and may scale with the system size as $R^{-\tau(q)}$, where $\tau(q)$ is a spectrum of moments describing the growth.[2] The GSPD can be interpreted in terms of a multifractal formalism:[2] the number N_α of sites with a growth probability $p_i \sim R^{-\alpha}$ for some small range of α from α to $\alpha+\delta\alpha$ will scale as $R^{f(\alpha)}\delta\alpha$. Then it is easy to show that[2]

$$\tau(q)=q\alpha_q - f_q(\alpha) , \tag{2}$$

where the subscripts refer to the value of $\partial f/\partial\alpha$ at which f and α are evaluated. The scaling function $f(\alpha)$ provides a statistical description of the evolution of fractal growth processes. Objects with a nontrivial $f(\alpha)$ are termed multifractals. The concept was first introduced by Mandelbrot to describe turbulence.[3] Multifractal behavior has been observed, and $f(\alpha)$ measured, in many cases: numerically for diffusion-limited aggregation (DLA),[4–6] for viscous fingering at finite viscosity ratios,[7] for the dielectric breakdown model,[5,6] for the screened-growth model,[8] for Meakin's multifractal Eden model,[9,10] in the electrical current and flicker noise distributions through percolating structures,[11,12] in hydrodynamic force distributions on fractal aggregates in solution,[13,14] and in the stress around fractals in an elastic medium,[15] as well as experimentally for turbulent flows,[16,17] viscous fingering at large viscosity ratios,[18,19] diffusion controlled crystallization[20] and heterogeneous reaction kinetics,[21] and analytically for a Laplacian field near an adsorbing random polymer coil,[22,23] for viscous fingering and DLA,[24] for random resistor networks,[25,26] and for models of turbulence.[17,27] Reference 28 is a review of the physical systems which can be described by a multifractal formalism and mentions examples in turbulence, chaos, temporal intermittency in disordered systems and percolation, as well as fractal growth processes.

A generic plot of the scaling function is shown in Fig. 1. All but an infinitesimal fraction of the growing sites have a growth singularity α_0. However, the majority of the growth occurs on the ensemble of points whose contribution to the first moment of the GSPD is dominant, i.e., the sites with singularities α_1. As $\tau(1)$ is defined to be zero for a normalized probability distribution, $f(\alpha)$ has a slope of unity where α and $f(\alpha)$ are equal ($\alpha_1=f_1$). For all other points, $f(\alpha)$ is less than α.

We might wish to know how the growth probability at a fixed site changes as the cluster evolves, i.e., the total number of sites N increases. We shall present here two derivations of the rate of change of the growth-site singularity, $d\alpha/dN$. We are then able to deduce a series of

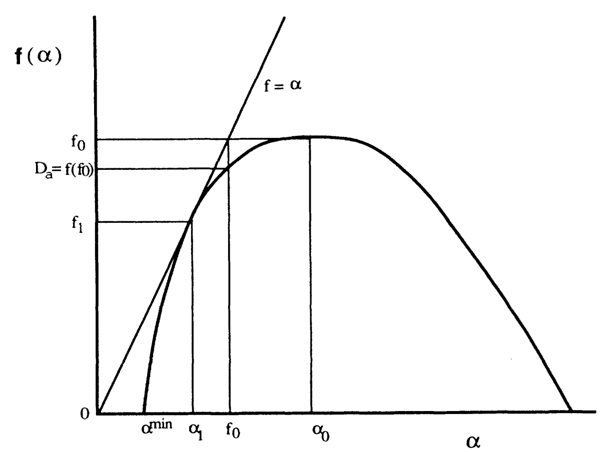

FIG. 1. A schematic plot of the scaling function.

powerful, general, relations between exponents describing the evolution of fractal structures.

In all that follows we shall assume that the fractal structure under investigation is all growing surface. Of the growth models mentioned previously, the only exception is viscous fingering at a finite viscosity ratio, which produces a compact displacement with a fractal boundary.[29,7] The derivations below are also general to any space dimension d.

II. A CONSERVATION EQUATION

Imagine that we have a flow $\mathbf{v}$ of conserved material with a density ρ and source density s, then the familiar conservation equation gives us

$$\nabla\cdot(\rho\mathbf{v})=-\frac{\partial\rho}{\partial t}+s\ . \tag{3}$$

We shall now develop an analogy. Growth will typically occur on unscreened sites of a cluster where the growth singularity α is small. The particle which has just been added will tend to screen the substrate from which it grew, as well as the neighboring sites. This will lower the probability of subsequent growth and thus the value of α for a fixed site will have increased. Similarly, the growth probability at a given fixed site i will decrease as growth occurs throughout the structure, but located preferentially at the extremities; the "active zone" of rapidly advancing points will, on average, eventually move away from point i. Initially exposed points become screened as the fractal develops. The deposition of new particles occurs primarily at α_1, but the dominant fraction of all sites has a singularity α_0. We need to explain, therefore, how the majority of particles grown at α equal to α_1 eventually form a set of singularity α_0.

N can be treated as a "time" variable, α a "space" variable, determining the position of a given particle on the $f(\alpha)$ curve as the cluster grows. Thus the "flow" is $d\alpha/dN$. If we add one particle, then the probability of landing on a site of singularity α is $R^{-\alpha}$ and there are $R^{f(\alpha)}$ such sites. Thus the source density is $R^{f-\alpha}$. The total density or quantity of material is $R^{f(\alpha)}$ and so the "flux" is $R^{f(\alpha)}d\alpha/dN$. The total number of sites is conserved. Hence we find, from (3),

$$\frac{\partial}{\partial\alpha}\left[R^{f(\alpha)}\frac{d\alpha}{dN}\right]=-\frac{\partial}{\partial N}R^{f(\alpha)}+R^{f(\alpha)-\alpha}\ . \tag{4}$$

Since $\partial f/\partial N$ is zero at fixed α and, by definition, N scales as R^D, we obtain

$$R^{f(\alpha)}\frac{d\alpha}{dN}\sim\int_{\alpha'}\{R^{f(\alpha')-\alpha'}-[f(\alpha')/D]R^{f(\alpha')-D}\}d\alpha'\ . \tag{5}$$

For large R the integral is evaluated carefully by steepest descents.

(i) When α is less than α_0, we take upper and lower limits on the integral to be α and α_∞, respectively. The integrand is dominated by the first term where $f(\alpha')-\alpha'$ is largest. This is simply when α' is equal to α, for α less than α_1. We find that $d\alpha/dN$ scales as $R^{-\alpha}$.

(ii) When α lies between α_1 and α_0, the integrand is largest and of order 1 where α' equals α_1, because, as we mentioned previously, $f(\alpha)-\alpha$ is zero when α is α_1. Thus $d\alpha/dN$ scales as R^{-f}.

(iii) For α greater than α_0, the integral is easiest to perform between $\alpha_{-\infty}$ and α. Then the exponent in the second term is larger. The exponent $f(\alpha')-D$ is largest when α' equals α ($\partial f/\partial\alpha$ is negative) and we find $d\alpha/dN$ scaling as R^{-D}.

In summary we find

$$u_T\sim\begin{cases}R^{-\alpha}, & \alpha\le\alpha_1 & (6a)\\ R^{-f(\alpha)}, & \alpha_1\le\alpha\le\alpha_0 & (6b)\\ R^{-D}, & \alpha\ge\alpha_0 & (6c)\end{cases}$$

where $d\alpha/dN$ is called u_T. This is the averaged total flow $d\alpha/dN$ derived from the conservation equation.

We might also like to know the typical, or most likely, value of $d\alpha/dN$ experienced by most of the sites with a given α. This will be called u_L. Consider the events which are likely to cause a considerable change in the growth probability at a site α (that is, events which make a contribution of order 1 to α). Firstly, these will arise from growth at or very near to the site in question [Fig. 2(a)]. The probability for this scales as $R^{-\alpha}$. Secondly,

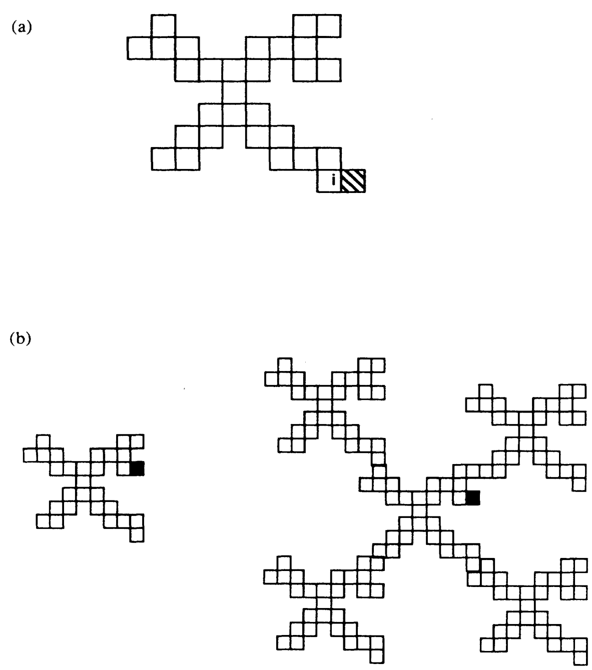

FIG. 2. Typical processes causing screening in multifractal growth. (a) Growth occurs at or near the site i. This event decreases the probability of further growth at i. (For mass fractals, it is unlikely that any site will ever be completely surrounded by neighbors.) (b) An exposed site with a large probability of growth no longer remains at the extremities of the cluster, once the cluster has grown appreciably in size.

any site will eventually become more screened, as it will become surrounded by sites throughout the cluster once the overall size R has increased considerably [Fig. 2(b)]. This will take of order R^D events, contributing a factor of R^{-D} for each new particle added. Hence we derive

$$u_L \sim \begin{cases} R^{-\alpha}, & \alpha \le D \quad (7a) \\ R^{-D}, & \alpha \ge D . \quad (7b) \end{cases}$$

This somewhat simple calculation does not reveal the intermediate regime found for u_T.

A physical interpretation of these results requires a more sophisticated treatment of the effects of growth at all distances throughout the fractal.

III. SCREENING ON INTERMEDIATE SCALES

So far we have only discussed growth singularities which are measured for the whole cluster. Imagine that we were to cover the fractal with boxes or blobs of size b. At the resolution of the boxes, we shall see a fractal of effective size R/b, with a multifractal GSPD; $(R/b)^{f(\alpha)}$ sites with probabilities $(R/b)^{-\alpha}$, with the same function $f(\alpha)$ as before. Within each box, we also see the same GSPD. For unit incident flux there are $b^{f(\alpha)}$ points with probabilities $b^{-\alpha}$. This is simply a statement of the scale invariance of a fractal structure.

If the box has a growth singularity α_{box} and a site within the box has singularity α_{site}, then the value of α for that site, measured on the whole cluster, is given by

$$R^{-\alpha} = \left[\frac{R}{b} \right]^{-\alpha_{box}} b^{-\alpha_{\text{site}}} \quad (8)$$

as the total probability of growth at the site, $R^{-\alpha}$, is the product of the probability of growth somewhere in the box, and the probability specifically at the point α_{site} inside it. This is a very important equation as it assumes that the probability subdivides in a self-similar manner throughout the growing surface. This is implied by a growth process on a fractal boundary, and is also valid for many other multifractal systems, but is not implicit in the scaling function $f(\alpha)$. Equation (8) can be rewritten to find b in terms of α, α_{site}, and α_{box}. If b scales as R^y, then y is

$$y = (\alpha_{\text{box}} - \alpha)/(\alpha_{\text{box}} - \alpha_{\text{site}}) . \quad (9)$$

Moreover, we can define an exponent f_{eff}, for the subset of points with singularities α_{site} and α_{box} such that the total number of these sites is

$$R^{f_{\text{eff}}} = \left[\frac{R}{b} \right]^{f_{\text{box}}} b^{f_{\text{site}}} , \quad (10)$$

where f_{site} and f_{box} are $f(\alpha_{\text{site}})$ and $f(\alpha_{\text{box}})$, respectively. We find

$$f_{\text{eff}} = \frac{f_{\text{box}}(\alpha - \alpha_{\text{site}}) + f_{\text{site}}(\alpha_{\text{box}} - \alpha)}{\alpha_{\text{box}} - \alpha_{\text{site}}} . \quad (11)$$

The exponent f_{eff} is a weighted linear combination of the exponents f_{site} and f_{box}. Since the scaling function is convex, f_{eff} will always lie below $f(\alpha)$. Consequently the number of sites for which α_{site} and α_{box} are not equal is only an infinitesimal fraction of the total number of points with singularity α. We recover the "typical" set, where f_{eff} equals $f(\alpha)$ only when α equals either α_{site} or α_{box} and b is equal to R or 1, respectively; or when α_{site} and α_{box} are both equal to α, and f_{eff} is $f(\alpha)$ regardless of b; these are sites whose growth singularity is the same measured over all scales.

We shall now use this geometrical picture to derive the contribution to $d\alpha/dN$ from the deposition of particles distances b away. The probability of growth at any site α' in any box of size b is given by

$$P_1 \sim \left[\frac{R}{b} \right]^{f_{\text{box}} - \alpha_{\text{box}}} b^{f(\alpha') - \alpha'} . \quad (12)$$

A. Calculation for DLA

We now wish to estimate the change in the total flux, $d/dN(R^{f-\alpha})$, induced at a site α_{site} due to growth at a point with singularity α', all within the same box. For this we use a propagator approach. This is only valid for a Laplacian field with an adsorbing boundary condition; that is, DLA-like processes, but not other types of fractal growth model. The change in the local field is equal to the probability that growth occurs at α', then another walker arrives at the same site which could then have wandered to α_{site}; it is only via such events that the consequence of growth at α' can be felt at α_{site}. To reach α_{site}, the particle must escape the singularity α'.

When a particle which enters a box of size b is captured by the singularity, it will typically perform a random walk of length of order b within the box. Hence it visits of order b^2 sites of a total b^d outside the cluster before sticking, where d is the dimension of space. To escape the walker may first reach any of b^d sites—on entry it only sampled b^2. The probability to escape the singularity, without returning, along any given path is the same for entry along the same path. Thus as the probability for entry is $b^{-\alpha'}$, the probability for exit is $b^{-\alpha'+d-2}$. We find that the total probability to land at α', escape, and then land at any site with singularity α_{site} is (see Fig. 3)

$$P_2 \sim \left[\frac{R}{b} \right]^{-\alpha_{\text{box}}} b^{-2\alpha'+d-2+f_{\text{site}}-\alpha_{\text{site}}} . \quad (13)$$

Hence $d(R^{f-\alpha})/dN$, which scales as $R^{f-\alpha} d\alpha/dN + R^{f-D}$, is the contribution $P_1 P_2$ integrated over all combinations of α', α_{site}, and α_{box} with the box of size b which is given by Eq. (9).

All the integrals are performed by steepest descents. The α' integral is dominated by the maximum value of $f(\alpha') - 3\alpha'$. From (2), the largest value of this exponent is $-\tau(3)$. The growth most affecting the local field occurs at points whose growth singularity, within the box b, determines the third moment of the GSPD.

After eliminating b, and using Eq. (8), we are left with

$$\frac{d}{dN}(R^{f-\alpha}) \sim \int \int R^{x-\alpha} d\alpha_{\text{site}} d\alpha_{\text{box}} , \quad (14)$$

where

$$x=\frac{\{(f_{\rm box}-\alpha_{\rm box})(\alpha-\alpha_{\rm site})+[f_{\rm site}-\tau(3)+d-2](\alpha_{\rm box}-\alpha)\}}{(\alpha_{\rm box}-\alpha_{\rm site})}\ . \tag{15}$$

First, however, we can calculate the contributions to $d\alpha/dN$ from the typical sets, with b either 1 or R. This should give us u_L.

We find that when b is 1, and α equals $\alpha_{\rm box}$, $u_L \sim R^{-\alpha}$, and when b equals R, and α is $\alpha_{\rm site}$, $u_L \sim R^{-\tau(3)+d-2}$. Comparison of this last result with Eq. (7b) demonstrates the result

$$D=\tau(3)-d+2\ . \tag{16}$$

For the case $d=2$, this result has already been derived, in a different manner, by Halsey,[30] with an extension to embrace the dielectric breakdown model. The best numerical value for $\tau(3)$ for DLA in two dimensions is 1.71 ± 0.01, while D is approximately 1.70.[4]

We now return to evaluating the integral over all possible box sizes. This is done by finding the maximum value of x: i.e., when $\partial x/\partial\alpha_{\rm site}$ and $\partial x/\alpha_{\rm box}$ are both zero. We find consistent solutions for x equal to zero, when α lies between α_1 and α_0: $\alpha_{\rm box}=\alpha_1$, $\alpha_{\rm site}=\alpha_0$ and $f_0=D=\tau(3)-d+2$. We find, as before, that u_T scales as $R^{-f(\alpha)}$ [Eq. 6(b)]. For values of α outside these limits, we again find expressions for u_T as in Eqs. (6a) and (6c).

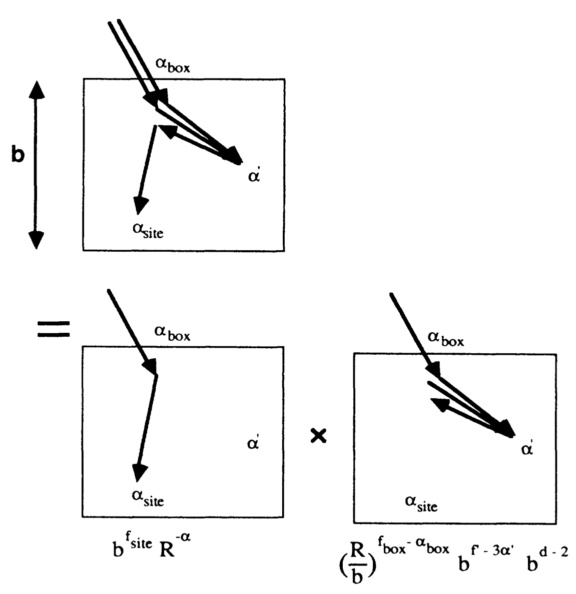

FIG. 3. A diagrammatic calculation of the effect of screening. The effect on the flux of particles at $\alpha_{\rm site}$ due to growth at α', within a box of size b is given by the probability that a particle deposits at α' multiplied by the probability that a walker again lands at α' and then wanders to the site at $\alpha_{\rm site}$: it is only via such events that the effect of the growth at α' can be detected.

B. The screened-growth model

Our analysis can also be performed for the screened-growth model.[31,32] The model generates clusters by the algorithm that the probability of growth from a site i is

$$p_i=\frac{\prod_j \exp(-r_{ij}^{-\epsilon})}{\sum_{i,j}\prod_j \exp(-r_{ij}^{-\epsilon})}\ , \tag{17}$$

where r_{ij} is the distance between two sites i and j on the cluster and the sum and products run over all sites on the cluster. Different structures are generated for different values of the exponent ϵ. Ignoring the normalization, the logarithm of p_i can be written as a sum of terms. It is then easy to see that the change in p_i at a point with singularity $\alpha_{\rm site}$ due to growth at α' a distance b away is simply $p_i b^{-\epsilon}$. Thus P_2 in Eq. (13) [using Eq. (8)] is given by

$$P_2\sim\left[\frac{R}{b}\right]^{-\alpha_{\rm box}} b^{f_{\rm site}-\alpha_{\rm site}-\epsilon}\ . \tag{18}$$

We now perform the same analysis as before. The α' integral gives us $\tau(1)$, which is defined to be zero for a probability distribution. We then calculate u_L and compare with Eq. (7). This yields $D=\epsilon$: ϵ replaces $\tau(3)-d+2$ in Eqs. (13)–(16). This result has already been derived, in a different manner, by Meakin *et al.*[33]

This analysis gives us a powerful picture of the mechanism of growth in DLA-type processes and the screened-growth model. As has been mentioned earlier, the most exposed sites, with α less than α_1, become screened via growth events on the site or its near neighbors. When α reaches α_1, the points belong to the set which receives the dominant contribution to the flux, $R^{f-\alpha}$. Typically, most sites of a strength α when measured over the whole cluster, have the same singularity within boxes of all sizes, from 1 to R [$f_{\rm eff}$ equals $f(\alpha)$]. For such sites screening proceeds as above until α reaches D. However, it is now an atypical subset which dominates the screening and leads to the intermediate regime found for u_T. These points lie in a box of singularity α_1 and size b, and have a singularity α_0 within it. Screening occurs when particles land in such boxes, which capture all but a negligible fraction of the flux, and affects sites of singularity α_0 within them, which represent all but a negligible fraction of the total sites. The box gets bigger as α increases. Its size is unity when α is α_1 and R when α reaches α_0. Then appreciable screening occurs only as the cluster increases in overall extent.

We are now able to glean further information. The derivations below are general to any fractal growth process, and not just DLA or the screened-growth model.

IV. THE DIMENSION OF THE ACTIVE ZONE

If a cluster grows from a size R (the "old" cluster) to $R+\delta R$, where δ is a small, but finite fraction, then those sites on the old cluster from which growth has occurred are defined as the active zone.[34,35] If the number of such sites scales as R^{D_a}, then the exponent D_a describes the number of points on which growth will ever occur in a finite length of time. This quantity is easily measured both experimentally and numerically by comparing the difference between two pictures of the fractal taken a short interval apart. Consider, firstly, that there is no screening and so the value of α for every site is fixed throughout the growth. For a finite fractional increase in radius, of order δR^D new sites have been added. Consequently growth is likely to have occurred from all sites with growth singularity α (growth probability $R^{-\alpha}$) less than D. The number of such points is described by the maximum value of $f(\alpha)$ for α less than or equal to D. This is simply $f(D)$. There is also the possibility that an infinitesimal fraction of sites with larger f and α might affect the value of D_a. For α greater than D, a fraction of order $R^{-\alpha+D}$ sites will have grown of a total R^f. Note that as δ is small, the effect of the change in R during the small interval to $R+\delta R$ can be neglected. Both regimes are dominated by α equal to D and thus

$$D_a=f(D) . \tag{19}$$

The effect of screening can only be to decrease D_a, since growth at any given site always becomes less likely as the fractal evolves. We need to consider the slowest mechanism of screening experienced by the largest number of sites; i.e., u_L.

In fact it is easy to see that screening is irrelevant to the dimension of the active zone. The important contribution to D_a comes from sites with singularity α equal to D. Here $d\alpha/dN$ scales as R^{-D}. If we add a number of order R^p sites to a cluster, then α has altered by only an infinitesimal amount, R^{-D+p}, as p approaches D from zero. Therefore, in the limit of infinite clusters, we still expect to find the result of Eq. (19). Notice, however, that D_a is dominated by those sites that are just about to become screened. $f(D)$ is approximately 1.36 ± 0.02 for DLA in two dimensions,[6] while numerical estimates[34,35] suggest a value close to 1.0. It is possible that in this case, an exponent f for those sites which received the most flux, f_1, was measured. This is, indeed, known theoretically to be exactly 1 for DLA in two dimensions.[36]

V. THE HISTORY DISTRIBUTION

While the scaling function, measured at a fixed time during growth, and derived from the GSPD, is a powerful statistical characterization, for many discrete, particulate models, it is neither the most natural nor the most easily measured description of the evolution of the cluster. It is more straightforward to look at the order in which sites on a cluster are added. The age of a site can then be considered to be the number of new particles which have been added since the site first became part of the fractal.

We can now define a history distribution using a scaling hypothesis that there is a probability $R^{\beta-g(\beta)}\delta\beta$ that growth will next occur from a site with an age in the range R^β to $R^{\beta+\delta\beta}$, where R is the present radius of the cluster. The number of such sites is $R^\beta\delta\beta\ln R$. We shall now derive the relationship between $g(\beta)$ and $f(\alpha)$. Clearly, $g(\beta)$ is only defined for β less than or equal to D. For β less than D, the fractional difference between the present radius and the radius when the selected site was added is of order $R^{\beta/D-1}$, which is infinitesimally small for asymptotically large clusters. Thus we shall use the two radii interchangeably.[37]

Consider that the site, which now has a singularity α, was first added with singularity α'. In any one increment in the growth there was a probability $R^{-\alpha'}$ of growth on a site of singularity α', and then a probability $R^{-\alpha}$ of subsequent growth from the same site. We allow the first growth to have occurred at any time in the age range R^β to $R^{\beta+\delta\beta}$. This is approximately $R^\beta\delta_\beta\ln R$ growth events. Then $R^{\beta-g(\beta)}\delta\beta$ is the product of the probabilities of growth at both occasions integrated over all possible singularity strengths, α:

$$R^{\beta-g(\beta)}\delta\beta\sim\int R^{-\alpha'}R^{f(\alpha)-\alpha}R^\beta\delta_\beta\ln R\,d\alpha\ , \tag{20}$$

where

$$\alpha=\alpha'+\int_0^{R^\beta}\frac{d\alpha''}{dN}dN=\alpha'+\int_0^\beta R^{-\alpha''+\beta'}\ln R\ d\beta' \tag{21}$$

and α'', the growth singularity after the addition of N new points, is α' when N equals zero, and α when N is R^β. Since $g(\beta)$ is dominated principally by the sites which screen least quickly, u_L is substituted for $d\alpha/dN$. For small values of β, α and α' are approximately equal and $g(\beta)$ is given by the minimum of $2\alpha-f(\alpha)$. This is $\tau(2)$. Screening is unimportant, therefore, until α equals α_2, beyond which α and α' are not necessarily the same. Without screening, $g(\beta)$ would equal $\tau(2)$ for all values of β. The integral in (21) should yield a contribution of order 1. To within logarithmic factors this is achieved when the integrand maintains β' equal to α'' throughout any finite increment of β'. Thus α equals β and hence $g(\beta)$ is the minimum value of $\alpha'-f(\beta)+\beta$ for β greater than α_2, and α' less than or equal to β. This is simply $2\beta-f(\beta)$. As for the calculation of D_a, $g(\beta)$ is dominated by those sites for which screening is just about to become significant. We obtain (ignoring logarithmic factors)

$$g(\beta)=\begin{cases}\tau(2), & \beta\le\alpha_2 & \text{(22a)}\\ 2\beta-f(\beta), & D\ge\beta\ge\alpha_2. & \text{(22b)}\end{cases}$$

$g(\beta)$ and its first derivative are continuous at α_2.

The portion of the scaling function $f(\alpha)$ between α_2 and D should be recoverable from the history distribution. This can easily be tested numerically, and possibly experimentally for a variety of multifractal structures. The verification of the relation would justify the use of u_L in (21), rather than the result of a more detailed analysis of the screening of atypical subsets.

VI. CONCLUDING REMARKS

We have introduced a quantitative scaling description of screening, which is general to a large variety of previously studied systems. In particular, the derivation of the rate of change of growth singularity led us to determine several other exponents describing the growth. We found an expression for the dimension of the active zone and introduced a new family of exponents via the history distribution, which should yield a natural and accurate statistical analysis of fractal growth: Moreover, for DLA we found a relation between the third moment of the GSPD and D, which generalized the result derived by Halsey.[30] This leads to a variety of suggestions for further experimental, numerical, and theoretical studies. Firstly, the new scaling relations proposed here need to be tested against experiment and computer simulation. Secondly, the physical insight into the evolution of fixed sites on an advancing cluster should, we hope, prompt further theoretical developments. In particular, the effect of screening from growth on sites sitting in a hierarchy of boxes of different sizes and singularity strengths can now be calculated.

ACKNOWLEDGMENTS

We should like to thank M.E. Cates for helpful discussions and one of us (M.J.B.) gratefully acknowledges British Petroleum and the United Kingdom Science and Engineering Research Council (SERC) for financial support.

---

*Present address: BP Research Centre, Chertsey Road, Sunbury-on-Thames, Middlesex TW16 7LN, United Kingdom.

[1]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2]T. C. Halsey *et al.*, Phys. Rev. A **33**, 1141 (1986).

[3]B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).

[4]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[5]C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).

[6]Y. Hayakawa, S. Sato, and M. Matsushita, Phys. Rev. A **36**, 1963 (1987).

[7]M. J. Blunt and P. R. King, Phys. Rev. A **37**, 3935 (1988).

[8]P. Meakin, Phys. Rev. A **34**, 790 (1986).

[9]P. Meakin, J. Phys. A **20**, L771 (1987).

[10]P. Meakin, J. Phys. A **20**, L779 (1987).

[11]L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985).

[12]R. Rammal, C. Tannous, P. Breton, and A. M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985).

[13]P. Meakin and J. M. Deutch, J. Chem. Phys. **86**, 4648 (1987).

[14]T. A. Witten, Phys. Rev. Lett. **59**, 900 (1987).

[15]P. Meakin, Phys. Rev. A **36**, 453 (1985).

[16]C. Meneveau and K. R. Sreenivasan, in *Physics of Chaos and Systems far from Equilibrium,* edited by Minh-Dong Van and B. Nichols (North-Holland, Amsterdam, 1987).

[17]C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[18]J. Nittmann, H. E. Stanley, E. Touboul, and G. Daccord, Phys. Rev. Lett. **58**, 619C (1987).

[19]K. J. Måloy, F. Boger, J. Feder, and T. Jossang, in *Time Dependent Effects in Disordered Materials, NATO Advanced Study Institute, Series B: Physics,* edited by R. Pynn and T. Riste (Plenum, New York, 1987), pp. 111–138.

[20]S. Ohta and H. Honjo, Phys. Rev. Lett. **60**, 611 (1988).

[21]R. Kopelman, J. Stat. Phys. **42**, 185 (1986).

[22]M. E. Cates and T. A. Witten, Phys. Rev. Lett. **56**, 2497 (1986).

[23]M. E. Cates and T. A. Witten, Phys. Rev. A **35**, 1809 (1987).

[24]M. J. Blunt and R. C. Ball, J. Phys. A **20**, 5961 (1987).

[25]R. Blumenfeld, Y. Meir, A. Brook Harris, and A. Aharony, J. Phys. A **19**, L791 (1986).

[26]R. Blumenfeld, Y. Meir, A. Aharony, and A. Brook Harris, Phys. Rev. B **35**, 3524 (1987).

[27]R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).

[28]G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[29]P. R. King, J. Phys. A **20**, L529 (1987).

[30]T. C. Halsey, Phys. Rev. Lett. **59**, 2067 (1987).

[31]P. A. Rikvold, Phys. Rev. A **26**, 647 (1982).

[32]P. Meakin, Phys. Rev. B **28**, 6718 (1983).

[33]P. Meakin, F. Leyvraz, and H. E. Stanley, Phys. Rev. A **31**, 1195 (1985).

[34]P. Meakin and T. A. Witten, Phys. Rev. A **28**, 2985 (1983).

[35]P. Meakin, Phys. Rev. A **32**, 453 (1985).

[36]N. G. Makarov, Proc. London Math. Soc. **51**, 369 (1985).

[37]An alternative approach, which defines $R^{-g(\beta)}$ to be the probability that a site which grows on a cluster of size R will wait for R^{β} new particles to be added before growth again occurs from that site (the lifetime distribution), yields the same $g(\beta)$ as the analysis given here. The only difference is that now β need not be bounded by D. We find that $g(\beta)$ diverges for β greater than D.