Berry's phase in optical resonance

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We examine the role of Berry's geometrical phase [Proc. R. Soc. London, Ser. A **392**, 45 (1984)] in the coherent excitation of atoms. The dynamics of a two-level atom prepared in a superposition of its eigenstates will be modified in a nontrivial fashion by the Berry phase. We extend Berry's formalism to treat the Liouville equation for the density matrix. This extension allows us to discuss statistical mixtures and to incorporate Berry's phase within the well-known Bloch-vector description of optical resonance. Furthermore, our density-matrix formalism provides the means of investigating Berry's phase in open, dissipative systems.

I. INTRODUCTION

In 1984, Berry[1] published a remarkable paper demonstrating a new feature in the well-known adiabatic theorem.[2,3] This new element is a phase factor acquired by the adiabatically evolved wave function. This phase, named after Berry, develops in addition to the familiar dynamical phase. The significance of Berry's phase is that it is observable in the interference of two identically prepared systems only one of which is adiabatically varied (Fig. 1). When the Hamitonian returns to its original form (at $t=T$), the relative phase between the two systems becomes observable in an interference type experiment. Berrys's discovery has already made an impact in a wide variety of areas within quantum physics. We will not attempt to review these developments here but recommend to the reader the review by Aitchison.[4]

The interference experiment depicted in Fig. 1 is not the only way that Berry's phase can be significant. It may also appear when two identically prepared systems are both subjected to the influence of adiabatically varying Hamiltonians.[5] In this case the phase will appear in an interference experiment when both Hamiltonians acquire the same form (though not necessarily the initial form) after different variations. It is not even necessary for the Hamiltonian to return to its original form, as the phase difference between different states (produced by adiabatic evolution) can be understood in relation to Pancharatnam's connection.[5]

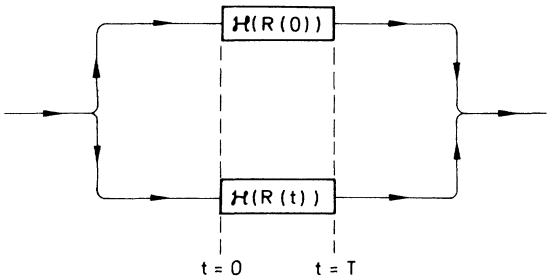

FIG. 1. Schematic diagram of an idealized interference experiment to measure Berry's phase. The time-dependent Hamiltonian is cycled so that $\mathcal{H}(\mathbf{R}(t))=\mathcal{H}(\mathbf{R}(0))$. [After Aitchison (Ref. 4)].

Yet another possibility is to consider a single quantum system prepared in a superposition of the eigenstates of its Hamiltonian.[6] As the Hamiltonian is varied each eigenstate may acquire a Berry phase. The differences between these phases are then observable at all times in the measurable properties of the system. This situation commonly occurs in optical resonance and Berry phase effects have been shown to affect the atomic inversion of an atom undergoing Rabi oscillations in an adiabatically varying laser field.[6]

In this paper we discuss the role of Berry's phase in optical resonance. We concentrate on the simplest model of optical resonance in which transitions between two atomic levels are driven by resonant interaction with an intense laser field.[7] In this model (Sec. II), the adiabatic variation of the laser phase causes a modification of the Rabi oscillations. This modification is attributable to the development of Berry phases by the dressed atomic energy levels.[6]

The density matrix is widely used in the solution of quantum optical problems. We show (Sec. III) how Berry's formalism may be extended to the Liouville equation for the density matrix. We employ this formalism to describe the two-level Bloch vector dynamics of our optical resonance model with the atom prepared in a mixed state. Use of the density matrix also allows us to describe

(Sec. IV) the quantum dynamics of open-dissipative systems. We add spontaneous emission to our two-level model and obtain the necessary conditions for the survival of Berry's phase in the presence of damping. Finally, we demonstrate in three appendixes the independence of the dynamics of Berry phases in the adiabatic limit (Appendix A) and in the adiabatic and weak damping limit (Appendix B) in the case of spontaneous emission. Appendix C gives the restrictions on the experimental parameters imposed by adiabaticity.

II. BERRY'S PHASE AND SUPERPOSITION STATES IN OPTICAL RESONANCE

Berry's phase is now well known and we do not propose to give a detailed discussion of it here. Instead, we recommend to the readers the original source[1] and the very readable description by Aitchison.[4] However, in order to establish our notation, we briefly highlight the main features of Berry's phase before turning our attention to superposition states and optical resonance.

In its original simplest form, Berry's phase forms part of the adiabatic theorem and is therefore an element in an approximate solution of the Schrödinger equation. the adiabatic theorem requires that a system initially prepared in an eigenstate of its Hamiltonian ($|m(0)\rangle$) will remain in the connected (instantaneous) eigenstate ($|m(t)\rangle$) as the Hamiltonian is slowly varied. In this adiabatic limit the wave function will develop in time to

$$|\psi(t)\rangle = a_m(t)\exp\left[\frac{-i}{\hbar}\int_0^t E_m(t')dt'\right]|m(t)\rangle \ , \tag{2.1}$$

where $E_m(t)$ is the instantaneous eigenenergy of the state and the probability amplitude a_m obeys the equation

$$\dot{a}_m = -a_m\langle m|\dot{m}\rangle \ . \tag{2.2}$$

The normalization of the state $|m\rangle$ ensures that $\langle m|\dot{m}\rangle$ is purely imaginary, as it must be for probability to be conserved. This looks like the arbitrary phase of quantum mechanics and it was always assumed that $|m\rangle$ could be multiplied by a suitable phase factor to make $\langle m|\dot{m}\rangle$ vanish.[2,3] Berry's insight was that interference experiments, of the type depicted in Fig. 1, make this phase an observable property and that it must therefore be more than just an arbitrary phase.

Berry's phase is the argument of the probility amplitude a_m and obeys the equation of motion

$$\dot{\gamma}_m = i\langle m|\dot{m}\rangle \ . \tag{2.3}$$

Berry[1] has given a simple and appealing description of the geometrical nature of this phase by considering the variation of the Hamiltonian as changes in the parameters upon which the Hamiltonian depends. We follow Berry's notation and represent these parameters as a vector $\mathbf{R}$. The variation of the Hamiltonian may be usefully pictured as the motion of a "Hamiltonian vector" in its parameter space. The dynamics of the Berry phase may be expressed in terms of the variation of these parameters,[1]

$$\dot{\gamma}_m = i\langle m(\mathbf{R})|[\nabla_{\mathbf{R}}|m(\mathbf{R})\rangle]\cdot\dot{\mathbf{R}} \ . \tag{2.4}$$

This equation may be integrated with the result

$$\gamma_m(t) = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \mathbf{A}_m\cdot d\mathbf{R} \ , \tag{2.5}$$

where $\mathbf{A}_m(\mathbf{R}) = i\langle m(\mathbf{R})|[\nabla_{\mathbf{R}}|m(\mathbf{R})\rangle]$ is a "pseudovector potential" and plays a similar role to that of the vector potential in electromagnetic theory. Equation (2.5) illustrates the geometrical nature of Berry's phase very nicely; it depends only on the geometry of the Hamiltonian trajectory in its parameter space and not on the time taken to execute it. For a complete loop in parameter space the Hamiltonian returns to its original form and we have

$$\gamma_m(c) = \oint \mathbf{A}_m\cdot d\mathbf{R} = \int\int (\nabla_{\mathbf{R}}\times\mathbf{A}_m)\cdot d\mathbf{S} \ . \tag{2.6}$$

If we carry the electromagnetic analogy a little further, we see that the Berry phase for a circut in parameter space is equal to the surface integral of the pseudo magnetic field $\mathbf{B}_m = \nabla_{\mathbf{R}}\times\mathbf{A}_m$. Berry's phase is clearly similar to the famous Aharonov-Bohm phase.[8] Indeed, the Aharonov-Bohm phase is a special example of Berry's phase.[1] If we change the phase of our eigenstate $|m\rangle$ then we will change the Berry phase γ_m. This is akin to making a gauge choice in electromagnetism. If we change the phase of $|m\rangle$ so that

$$|\tilde{m}\rangle = \exp[i\mu(\mathbf{R})]|m\rangle \tag{2.7}$$

then the pseudovector potential will change and with it γ_m,

$$\tilde{\mathbf{A}}_m = \mathbf{A}_m - \nabla_{\mathbf{R}}\mu \ . \tag{2.8}$$

This transformation only affects longitudonal part of $\mathbf{A}_m$. The curl of the longitudonal part of $\mathbf{A}_m$ is zero and therefore the Berry phase for a complete loop in parameter space is uniquely defined. The form of this phase is particularly simple near a point in parameter space where two eigenstates become degenerate.[1] In this case, the two states (that become degenerate) acquire phase equal to plus or minus one half the solid angle subtended by the Hamiltonian trajectory at the degeneracy. This is a further confirmation of the geometrical nature of Berry's phase.

We have already noted that a single quantum system prepared in a superposition of the eigenstates of its Hamiltonian may exhibit Berry phase effects at all times.[6] Such a superposition state can be prepared by a sudden variation of the Hamiltonian. If the new Hamiltonian is varied adiabatically then we can apply Berry's results to write the time-evolved wave function,

$$|\psi(t)\rangle = \sum_n a_n(0)\exp\left[\frac{-i}{\hbar}\int_0^t E_n(t')dt'\right] \times \exp[i\gamma_n(t)]|n(\mathbf{R}(t))\rangle \ , \tag{2.9}$$

where $a_n(0)$ is the initial amplitude for being in each of the eigenstate $|n\rangle$. There is only one arbitrary phase in quantum mechanics and therefore, the relative phases between the eigenstate are observable quantities. These relative phases depend on the differences between the Berry phases associated with each eigenstate. In this way the

evolution of observables will depend on the Berry phase. We can illustrate this principle with some results from optical resonance.

The fundamental model of optical resonance consists of a single two-level atom interacting with a classical laser field.[7] In the rotating-wave approximation the Hamiltonian for this model is

$$\mathcal{H}=\tfrac{1}{2}\hbar\Delta\sigma_3+\hbar\lambda[\sigma_+\exp(-i\phi)+\sigma_-\exp(i\phi)]\ , \tag{2.10}$$

where the atom is described by Pauli raising and lowering operators σ_+, and σ_- and the inversion operator σ_3. The deturning Δ, the coupling strength λ, and the phase ϕ all depend on the properties of the laser. Adiabatic variation of these parameters can induce a Berry phase.[6] If the atom is prepared in superposition of the eigenstates of $\mathcal{H}$ then this variation can lead to observable Berry-phase effects at all times. Consider our two-level atom in its ground state in the absence of any applied laser field. At time $t=0$ we suddenly switch on the laser field and the Hamiltonian subsequently evolves under the action of $\mathcal{H}$. The field is applied too quickly for the atom to respond adiabatically and the initial atomic state will be a superposition of the eigenstates of $\mathcal{H}$. In Ref. 6 we showed that subsequent adiabatic variation of $\mathcal{H}$ led to an observable modification of the probability for being in the bare atomic ground state. In particular, if the Berry phases are due to a variation of the laser phase only then the probability for being in the ground state at time t is

$$p_g(t)=1+\frac{\lambda^2}{2\Lambda}\left[\cos\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi(t)\right]-1\right]\ , \tag{2.11}$$

where $2\hbar\Lambda$ is the difference between the eigenenergies of $\mathcal{H}$ ($\Lambda^2=\lambda^2+\Delta^2/4$) and $\delta\phi(t)=\phi(t)-\phi(0)$ is the accumulated phase change. It should be noted that $\delta\phi$ can increase without limit beyond 2π. This ground-state probability exhibits the familiar Rabi oscillations at the frequency 2Λ. However, the term proportional to $\delta\phi$ is geometrical in origin and arises from the difference between the Berry phase acquired by the two eigenstates of $\mathcal{H}$. This is a clear illustration of Berry's phase in the simplest model of optical resonance. We emphasize the point that the geometrical phase causes a measurable effect on the inversion at all times and not simply when the Hamiltonian has returned to its original form. We note that although the phase variation-induced correction to p_g is geometrical in origin it also has a dynamical interpretation in terms of a frequency-induced effect.[6,9]

III. DENSITY MATRIX AND TWO-LEVEL BLOCH-VECTOR DYNAMICS

The quantum description of statistical mixtures is most commonly achieved using the density matrix In this section we develop a density matrix description of Berry's phase and apply it to the dynamics of the optical Bloch vector. A density matrix formulation also allows us to describe dissipative effects associated with open systems. We will investigate the role of damping and spontaneous emission in Sec. IV, but we concentrate here on nondissipative dynamics.

The Hamitonian evolution of the density matrix is governed by the Liouville equation

$$i\dot{\rho}=\mathcal{L}\rho\ , \tag{3.1}$$

where $\mathcal{L}\equiv[\mathcal{H},\cdot]/\hbar$ is the Liouvillian superoperator. The eigenvalues of the Liouvillian are the differences between the eigenvalues of the Hamiltonian. Therefore, the eigenvalues of the Liouvillian are the transition frequencies of the system. For this reason the Liouvillian is sometimes called the "line operator" (particularly in the older literature).

The Liouvillian superoperator has eigenmatrices χ_n which obey the eigenvalue equation

$$\mathcal{L}\chi_n=\lambda_n\chi_n\ . \tag{3.2}$$

These eigenmatrices are not in general Hermitian. The Hermitian conjugate matrices are also eigenmatrices but with eigenvalues of opposite sign.

$$\mathcal{L}\chi_n^+=-\lambda_n\chi_n^+\ . \tag{3.3}$$

We note that it is not generally possible to prepare a state described by a single eigenmatrix. The Hermiticity of the density matrix requires that both χ_n and χ_n^+ must be present. Nevertheless, we proceed by considering a single eigenmatrix and construct the full density matrix later.

If the Liouvillian is slowly varied then a nondegenerate eigenmatrix will evolve into the connected eigenmatrix

$$\chi_n\rightarrow\xi_n=\exp\left[-i\int_0^t\lambda_n(t')dt'\right]\exp[i\gamma_n(t)]\chi_n(t)\ , \tag{3.4}$$

where γ_n is the Berry phase associated with this eigenmatrix. We note that γ_n will be equal to the difference between the geometrical phases associated with a pair of eigenstates. Substitution of this expression into the Liouville equation gives

$$\dot{\gamma}_n\chi_n(\mathbf{R})=i\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\cdot\dot{\mathbf{R}}\ , \tag{3.5}$$

where we have introduced the parameter vector $\mathbf{R}$ in analogy with the wave function analysis. We convert this into an equation for γ_n by using an inner product defined to be[10]

$$\langle\chi_n,\chi_m\rangle\equiv\tfrac{1}{2}\mathrm{Tr}(\chi_n^+\chi_m)=\delta_{nm}\ . \tag{3.6}$$

This relation allows us to project Eq. (3.5) along χ_n to provide an equation for γ_n

$$\dot{\gamma}_n=i\langle\chi_n(\mathbf{R}),\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\rangle\cdot\dot{\mathbf{R}}\ . \tag{3.7}$$

The formal solution of this equation is

$$\gamma_n=\frac{i}{2}\,\mathrm{Tr}\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\chi_n^+(\mathbf{R})[\nabla_{\mathbf{R}}\chi_n(\mathbf{R})]\cdot d\mathbf{R}\ . \tag{3.8}$$

This is the geometrical phase that is acquired by the eigenmatrix due to the variation of the Liouvillian. We note that, as with the wave-function analysis,[1] this phase is dependent only on the path in parameter space and not on the time required to execute it. As before, the orthonormality of the χ_n quarantees the reality of γ_n. In general, the density matrix will consist of weighted sum of independently evolving eigenmatrices (Appendix A).

The old problems associated with the apparent arbitrariness of the Berry phases persist in the density matrix formulation. However, as before, a careful analysis will always lead to a unique geometrical contribution to any expectation value. If the Hamiltonian is varied around a closed loop in parameter space then the Berry phase acquired by the eigenmatrix χ_n is

$$\gamma_n(c)=i\oint_c \langle \chi_n(\mathbf{R}),\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\rangle\cdot d\mathbf{R} = i\int\int \nabla_{\mathbf{R}}\times\langle \chi_n(\mathbf{R}),\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\rangle\cdot d\mathbf{S} . \quad (3.9)$$

As with the wave function analysis, the Berry phase for a complete loop in parameter space is unique and independent of the choice of phase of the eigenmatrix.

It is important to note that there is a class of eigenmatrices that do not develop Berry phases. These eigenmatrices are those that are diagonal in the eigenstate representation. They are all degenerate eigenmatrices of $\mathcal{L}$ with eigenvalue zero. The Hermiticity of the density matrix requires these terms to be real and so they cannot develop Berry phases. Moreover, these diagonal eigenmatrices represent the eigenstate populations and cannot vary in time if the adiabatic approximation is to hold.

We now outline a theoretical framework in which Berry phases are incorporated in the density matrix dynamics. Following the wave-function problem, we consider and N-level atom and its density matrix given by

$$\rho(t)=\frac{1}{N}1+\sum_{n=1}^{N^2-1}\tfrac{1}{2}C_n e^{-i\lambda_n t}e^{i\gamma_n(t)}\chi_n(\mathbf{R}(t))+\text{H.c.} \quad (3.10)$$

The preceding expansion has been chosen to be consistent with the Hermiticity and the trace properties of the density matrix. The eigenmatrices $\chi_n(\chi_n^+)$ may be expressed as weighted sums of the SU(N) generators:

$$\chi_n=\sum_{m=1}^{N^2-1}a_{nm}S_n . \quad (3.11)$$

The Hermitian conjugate expression is

$$\chi_n^+=\sum_{m=1}^{N^2-1}a_{nm}^*S_m , \quad (3.12)$$

where 1 is the $N\times N$ unit matrix and the commutation relations of the generators are

$$[S_n,S_m]=2if_{nml}S_l \quad (3.13)$$

and f_{nme} are the structure constants of the SU(N) group.

This type of expansion is a kind of normal-modes expansion for the density matrix and we can call the $\chi_n(\chi_n^+)$ eigenmatrices "normal matrices" of the Liouvillian. Accordingly,

$$\rho^{(-)}\equiv\sum_{n=1}^{N^2-1}\tfrac{1}{2}C_n e^{i\lambda_n t}e^{i\gamma_n(t)}\chi_n(\mathbf{R}(t)) \quad (3.14)$$

can be considered as a "negative frequency part" and its H.c. $\rho^{(+)}\equiv(\rho^{(-)})^+$ as a "positive frequency part". In the following we specialize to an $N=2$ level atom.

The preceding formalism enables us to include Berry's phase within the Bloch-vector description of optical resonance. The Liouvillean superoperator, formed from the Hamiltonian (3.10) is easily diagonalized. One eigenmatrix is the identity and this has eigenvalue zero. The remaining three eigenvalues are $\lambda_1=0$ and $\lambda_{2,3}=\pm 2\Lambda$. The corresponding eigenmatrices may be expressed in terms of the SU(2) generators (Pauli matrices):

$$\chi_n=\sum_{i=1}^{3}a_{ni}\sigma_i , \quad (3.15)$$

where the coefficients a_{ni} are

$$\begin{aligned}
a_{11}&=\frac{\lambda}{\Lambda}\cos\phi ,\\
a_{12}&=\frac{\lambda}{\Lambda}\sin\phi ,\\
a_{13}&=\frac{\Delta}{2\Lambda} ,\\
a_{21}&=\frac{1}{2^{1/2}\Lambda}\left[\frac{-\Delta}{2}\cos\phi+i\Lambda\sin\phi\right] ,\\
a_{22}&=\frac{1}{2^{1/2}\Lambda}\left[\frac{-\Delta}{2}\sin\phi-i\Lambda\sin\phi\right] ,\\
a_{23}&=\frac{\lambda}{2^{1/2}\Lambda} ,\\
a_{31}&=\frac{1}{2^{1/2}\Lambda}\left[\frac{-\Delta}{2}\cos\phi-i\Lambda\sin\phi\right] ,\\
a_{32}&=\frac{1}{2^{1/2}\Lambda}\left[\frac{-\Delta}{2}\sin\phi+i\Lambda\cos\phi\right] ,\\
a_{33}&=\frac{\lambda}{2^{1/2}\Lambda} .
\end{aligned} \quad (3.16)$$

We can see from these coefficients that χ_1 is Hermitian and that $\chi_2^+=\chi_3$.

We are now in a position to calculate the Berry phases associated with the eigenmatrices. We shall see that χ_1 does not develop a Berry phase. We use Eq. (19) to provide the geometrical phases $(\gamma_1,\gamma_2,\gamma_3)$

$$\gamma_n=\frac{i}{2}\,\mathrm{Tr}\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\chi_n^+(\mathbf{R})[\nabla_{\mathbf{R}}\chi_n(\mathbf{R})]\cdot d\mathbf{R} . \quad (3.17)$$

As in our earlier example, we keep the amplitude and detuning constant and vary only the laser phase. From (21) and (22) we obtain the expression

$$\gamma_n=i\int_{\phi(0)}^{\phi(t)}\left[a_{n1}^*\frac{\partial a_{n1}}{\partial\phi}+a_{n2}^*\frac{\partial a_{n2}}{\partial\phi}+a_{n3}^*\frac{\partial a_{n3}}{\partial\phi}\right]d\phi . \quad (3.18)$$

Simple manipulation gives the results

$$\begin{aligned}
\gamma_1&=0\\
\gamma_{2,3}&=\pm\frac{\Delta}{2\Lambda}[\phi(t)-\phi(0)] .
\end{aligned} \quad (3.19)$$

We note that γ_2 and γ_3 are equal to the term appearing in the ground-state probability (2.11). In the earlier treatment, the contribution was due to the difference between

the Berry phases associated with the two eigenstates.[6] Here, however, the geometrical phases themselves have this form. This is because the eigenmatrices have phases that are the differences between the phases of the eigenstates.

An important feature of the density matrix is that it allows us to treat statistical mixtures. We can illustrate this ability by choosing our atom to be initially prepared in a mixed state with probability P of being excited

$$\rho(0)=\tfrac{1}{2}\mathbf{1}+(P-\tfrac{1}{2})\sigma_3 \ . \tag{3.20}$$

This initial condition corresponds to the coefficients

$$C_1=(P-\tfrac{1}{2})\frac{\Delta}{2\Lambda}$$

$$C_2=C_3=(P-\tfrac{1}{2})\frac{\lambda}{2^{1/2}\Lambda} \ . \tag{3.21}$$

Substituting these coefficients into Eq. (3.14) gives the time-evolved density matrix. The projections of the density matrix along the three pseudospin directions are the components of the Bloch vector[4]

$$u\equiv \mathrm{Tr}[\rho(t)\sigma_1] \ ,$$

$$=2(P-\tfrac{1}{2})\left\{\frac{\Delta\lambda}{2\Lambda^2}\cos\phi\left[1-\cos\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi\right]\right]\right.$$

$$\left.+\frac{\lambda}{\Lambda}\sin\phi\sin\left[2\Lambda t-\frac{\Delta}{2}\Lambda\delta\phi\right]\right\} \ ,$$

$$v\equiv \mathrm{Tr}[\rho(t)\sigma_2]$$

$$=2(P-\tfrac{1}{2})\left\{\frac{\Delta\lambda}{2\Lambda^2}\sin\phi\left[1-\cos\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi\right]\right]\right.$$

$$\left.-\frac{\lambda}{\Lambda}\cos\phi\sin\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi\right]\right\} \ , \tag{3.22}$$

$$w\equiv \mathrm{Tr}[\rho(t)\sigma_3]$$

$$=2(P-\tfrac{1}{2})\left[\frac{\Delta^2}{4\Lambda^2}+\frac{\lambda^2}{\Lambda^2}\cos\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi\right]\right] \ ,$$

where $\phi=\phi(t)$ and $\delta\phi=\phi(t)-\phi(0)$. Here u and v are the in-phase and in-quadrature components of the dipole and w is the atomic inversion. If the atom is initially prepared in its ground state ($P=0$) then the ground-state probability [$p_g=\frac{1}{2}(1-w)$] agrees with our earlier result (2.11).

The Bloch-vector components precess around the Hamiltonian vector with the characteristic Rabi frequency Λ. In addition, the Hamiltonian is slowly rotating in parameter space. The motion of the Hamiltonian is reflected in the two dipole components u and v. These terms depend directly on the projection of the Hamiltonian vector onto σ_1 and σ_2 and are therefore dependent on the sine and cosine of the instantaneous laser phase. The adiabatic theorem has a natural geometrical interpretation in the Bloch-vector picture (Fig. 2). The validity of the theorem requires that the rotational velocity of the Bloch vector ($\sim\Lambda$) should be much greater than the precessional velocity of the Hamiltonian ($\sim\dot{\phi}$) about which it precesses.

Our Bloch-vector solution displays a geometrical Berry phase in addition, to the Rabi oscillation and the conventional precessional terms [$\cos\phi(t)$ and $\sin\phi(t)$ in u and v]. This phase is proportional to the overall change in the laser phase. After the Hamiltonian has completed a loop in the Bloch-vector space ($\delta\phi=2\pi$) the dynamical phase will have been shifted by the density matrix Berry phase

$$\gamma_{2,3}=\pm\frac{\pi\Delta}{\Lambda} \ . \tag{3.23}$$

IV. OPEN SYSTEMS AND SPONTANEOUS EMISSION

In this chapter we extend our earlier analysis to incorporate damping and utilize our method to include spontaneous emission within the two-level model discussed above. The statistics of open systems form an important problem in quantum optices and in other branches of quantum mechanics. Such systems are commonly described by using the reduced density matrix. This reduced density matrix is obtained by tracing the complete density matrix over all systems except the one in question.[11] In the weak coupling limit, the reduced density

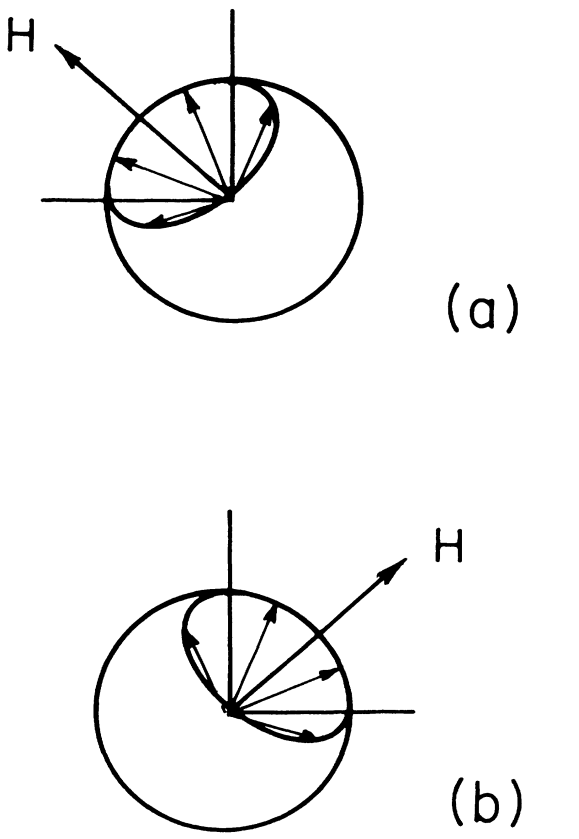

FIG. 2. Bloch sphere. The coherence vector (u,v,w) processes around the Hamiltonian vector $\mathcal{H}$ many times in the time taken for the Hamiltonian to process from position (a) to position (b).

matrix ρ obeys the equation of motion

$$i\dot{\rho}=\mathcal{L}\rho+iL\rho \ . \tag{4.1}$$

Here, $\mathcal{L}$ is the Liouvillian superoperator for the system in question and L is the effective superoperator describing the influences of all the remaining systems. The application of Berry's formalism to this system proceeds as before. The non-Hermitian superoperator $\mathcal{L}+iL$ must be diagonalized and a Berry phase calculated for each eigenmatrix.

The total superoperator $(\mathcal{L}+iL)$ has eigenmatrices χ_n obeying the eigenvalue equation

$$(\mathcal{L}+iL)\chi_n=\tilde{\lambda}_n\chi_n \ , \tag{4.2}$$

where $\tilde{\lambda}_n$ is in general a complex eigenvalue. If the Liouvillian is slowly varied then a nondegenerate eigenmatrix χ_n [that is, $\mathrm{Re}(\tilde{\lambda}_n)\neq\mathrm{Re}(\tilde{\lambda}_m)$ for $m\neq n$] will evolve into the connected eigenmatrix

$$\chi_n\rightarrow\xi_n=\exp\left[-\int_0^t\tilde{\lambda}_n(t')dt'\right]\exp[i\gamma_n(t)]\chi_n(t) \ . \tag{4.3}$$

The adiabatic correction for this eigenmatrix (γ_n) is given by

$$\dot{\gamma}_n=i\langle\chi_n(\mathbf{R}),\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\rangle\cdot\dot{\mathbf{R}} \ , \tag{4.4}$$

where the inner product is defined as before. We note that γ_n will in general be complex in this dissipative system.

For our two-level model the Liouvillian is derived from the Hamiltonian (2.10) as before. The additional (non-Liouvillian) term arises from tracing the total density matrix over all the modes of the surrounding vacuum. The non-Liouvillian term is

$$L\rho=\Gamma\sigma_-\rho\sigma_+-\frac{\Gamma}{2}(\sigma_+\sigma_-\rho+\rho\sigma_+\sigma_-) \ , \tag{4.5}$$

where Γ is the spontaneous emission rate.

In practice, the determination of the eigenmatrices of the superoperator $\mathcal{L}+iL$ may be complicated. However, in the weak damping regime we can obtain satisfactory results by working with the eigenmatrices of the Liouvillean superoperator. The phase acquired by an eigenmatrix has two contributions; one from the variation of $\mathcal{L}$ and one due to the presence of damping. In Appendix B we show that, in the weak damping limit, the total Berry phase is

$$\gamma_n(t)=\gamma_{n,\,\mathrm{Berry}}(t)-i\gamma_{n,\mathrm{DIS}}(t) \ , \tag{4.6}$$

where $\gamma_{n,\,\mathrm{Berry}}$ is the familiar Berry phase

$$\gamma_{n,\,\mathrm{Berry}}(t)=i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\langle\chi_n(\mathbf{R}),\nabla_{\mathbf{R}}\chi_n(\mathbf{R})\rangle\cdot d\mathbf{R} \tag{4.7}$$

and $\gamma_{n,\,\mathrm{DIS}}$ is a damping-induced correction

$$\gamma_{n,\,\mathrm{DIS}}(t)=\int_0^t dt'\langle\chi_n(\mathbf{R}),L\chi_n(\mathbf{R})\rangle \ . \tag{4.8}$$

The preceding expression for the total phase holds only for the eigenmatrices of the Liouvillian that have nonzero eigenvalue (that is for $n=2,3$ in Sec. III). The total phase for the eigenmatrix with zero eigenvalue χ_1 is (Appendix B)

$$\gamma_1(t)=\gamma_{1,\,\mathrm{Berry}}(t)-i\left[\gamma_{1,\,\mathrm{DIS}}(t)+\ln\left[1+\int_0^t\frac{dt''}{C_1}\langle\chi_1,L1\rangle e^{-i\gamma_{1,\,\mathrm{Berry}}(t'')}e^{-\gamma_{1,\,\mathrm{DIS}}(t'')}\right]\right] \ . \tag{4.9}$$

As we noted in Sec. III, χ_1 is constrained to be Hermitian by the Hermiticity of the density matrix. Therefore $\gamma_{1,\,\mathrm{Berry}}=0$ and $\gamma_1(t)$ must be imaginary,

$$\gamma_1(t)=-i\left[\gamma_{1,\,\mathrm{DIS}}(t)+\ln\left[1+\int_0^t\frac{dt''}{C_1}\langle\chi_1,L1\rangle e^{-\gamma_{1,\,\mathrm{DIS}}(t'')}\right]\right] \ . \tag{4.10}$$

We know all the necessary quantities for the computations of $\gamma_{1,2,3}$ and only need to choose an initial condition to determine C_1. We consider the simple initial condition of the atom prepared in a mixed state with probability P of being excited; see Eq. (3.20) with corresponding expansion coefficients given in Eq. (3.21). The dissipative contributions to γ_n are both *path* and *time dependent.* Therefore we require a specific development at the variation of the Hamiltonian. For simplicity we return to the variation of the laser phase and choose this variation to be linear in time,

$$\phi(t)=\phi(0)+\frac{2\pi t}{T} \ . \tag{4.11}$$

Here T is the time needed to complete one loop in parameter space. It is now straightforward to calculate the dissipative part of the phases with the result

$$\gamma_{1,\,\mathrm{DIS}}(t)=-K\Gamma t \ , \tag{4.12}$$

$$\gamma_{2,3;\,\mathrm{DIS}}(t)=-G\Gamma t \ , \tag{4.13}$$

where K and G are

$$K=\frac{2\lambda^2+\Delta^2}{4\lambda^2+\Delta^2} \ , \tag{4.14}$$

$$G=\frac{1}{2}\frac{6\lambda^2+\Delta^2}{4\lambda^2+\Delta^2} \ . \tag{4.15}$$

Before writing down the full Berry phase must complete the calculation of the dissipative part of γ_1 using Eq. (4.10). We find

$$\gamma_1(t)=iK\Gamma t-i\ln[1-\tfrac{1}{2}(P-\tfrac{1}{2})^{-1}\frac{1}{K}(e^{K\Gamma t}-1)] \tag{4.16}$$

and

$$\gamma_{2,3}(t)=\pm\frac{\Delta}{2\Lambda}\delta\phi(t)+iG\Gamma t \ . \tag{4.17}$$

The imaginary part of these phases represent the effective dissipation rate associated with each eigenmatrix.

Collecting the values eigenvalues, eigenmatrices, and Berry phases we can compute the time evolved density matrix from the Eq. (3.10). The component along the σ_3 matrix is the population inversion and has the form

$$\begin{aligned} w(t)&=\rho_{22}(t)-\rho_{11}(t) \\ &=2(P-\tfrac{1}{2})\left[\frac{\Delta^2}{4\Lambda^2}\left[\frac{2K(P-\frac{1}{2})+1}{2K(P-\frac{1}{2})}e^{-K\Gamma t}-\frac{1}{2K(P-\frac{1}{2})}\right]+\cos\left[2\Lambda t-\frac{\Delta}{2\Lambda}\delta\phi(t)\right]\frac{\lambda^2}{\Lambda^2}e^{G\Gamma t}\right] . \end{aligned} \tag{4.18}$$

Here the decaying exponentials come from the dissipation and the experimentally manifestable Berry phase gives rise to the term containing $\delta\phi$. We can check that $w(0)=2(P-\frac{1}{2})$ and $-1<w(\infty)=-\Delta^2/(\Delta^2+2\lambda^2)<0$ as they should be. Moreover, the asymptotic value $w(\infty)$ is in agreement with the steady-state solution w_{SS} of the Bloch equations in the weak damping limit,[7]

$$w_{SS}=\frac{\Delta^2+\frac{\Gamma^2}{4}}{\Delta^2+\frac{\Gamma^2}{4}+2\lambda^2}\rightarrow w(\infty) . \tag{4.19}$$

The final expression for the population difference, Eq. (4.18), also shows that the Berry phase causes a drifting of the population inversion. This drifting is observable only as long as the dissipation has not acted too strongly, that is, if $G\Gamma t\lesssim 1$. (As $K\sim G$ we also have $K\Gamma t\lesssim 1$.) Therefore, in addition to the adiabatic condition, we require a second condition for the observability of the Berry's phase in the presence of dissipation,

$$\frac{\pi}{\Lambda}\ll T\lesssim\frac{2}{\Gamma}\frac{4\lambda^2+\Delta^2}{6\lambda^2+\Delta^2} . \tag{4.20}$$

This condition is slighty stronger than the weak damping condition only. The Berry phase will be observable in the damped system if

$$\frac{\pi}{\Lambda}\ll T,\frac{1}{\Gamma} , \tag{4.21}$$

$$T\lesssim\frac{1}{\Gamma} . \tag{4.22}$$

V. CONCLUSIONS

The motivation of this work was to apply Berry's phase to problems in optical resonance. We have concentrated on the semiclassical model of a two-level atom driven by a laser with a varying phase. The density matrix allows us to model systems prepared in mixed states or undergoing dissipation. We have applied Berry's formalism to the Liouville equation for the density matrix. This development has allowed us to describe the two-level dynamics using the geometrical Bloch-vector picture. Dissipation via spontaneous emission will affect the Berry phase. We have shown that the phase will survive only if the time required to complete the loop in parameter space is less than the characteristic damping time.

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APPENDIX A

We show here that the Berry phases associated with nondegenerate eigenstates evolve independently of one another. This supplies the rigorous justification for the method used in Sec. III.

We consider a Hamiltonian with N nondegenerate eigenstates. The Hamiltonian is slowly varied by changing the parameters $\mathbf{R}$ on which it depends. The wave function obeys the Schrödinger equation

$$\frac{i\partial}{\partial t}|\psi(\mathbf{R}(t))\rangle=\mathcal{H}(\mathbf{R}(t))|\psi(\mathbf{R}(t))\rangle . \tag{A1}$$

We assume the initial state to be a superposition of the initial eigenstates $[|n(\mathbf{R}(0))\rangle]$

$$|\psi(0)\rangle=\sum_{n=1}^{N}C_n|n(\mathbf{R}(0))\rangle . \tag{A2}$$

The Hamiltonian is allowed to slowly vary and the eigenstates acquire Berry phases. The generalization of Berry's ansatz for the wave function is

$$|\psi(t)\rangle=\sum_{n=1}^{N}C_n\exp\left[-i\int_0^t dt\,\lambda_n(t)\right]\times e^{i\gamma_n(t)}|n(\mathbf{R}(t))\rangle , \tag{A3}$$

where $\lambda_n(t)$ are the eigenenergies of the dressed states ($|n(\mathbf{R}(t))\rangle$). Substitution of this ansatz into the wave equation gives

$$C_n\dot{\gamma}_m e^{i\gamma_m}=i\langle n(\mathbf{R}(t))[\nabla_{\mathbf{R}}|m(\mathbf{R}(t))\rangle]\cdot\dot{\mathbf{R}}e^{i\gamma_n}C_m$$
$$+i\sum_{n\neq m}C_n\langle m(\mathbf{R}(t))|[\nabla_{\mathbf{R}}|n((\mathbf{R}))\rangle]\cdot\dot{\mathbf{R}}e^{i\gamma_n}\exp\left[-i\int_0^t dt[\lambda_n(t)-\lambda_n(t)]\right]. \tag{A4}$$

If we define $\chi_m\equiv e^{i\gamma_m}$ then Eq. (A4) becomes

$$\dot{\chi}_m=A_m(\mathbf{R}(t))\chi_m+\sum_{n\neq m}A_{mn}(\mathbf{R}(t))e^{-\Delta_{nm}(t)\chi_n}, \tag{A5}$$

where

$$A_m(\mathbf{R})(t))\equiv-\langle m(\mathbf{R}(t))|[\nabla_{\mathbf{R}}|m(\mathbf{R}(t))\rangle]\cdot\dot{\mathbf{R}}(t), \tag{A6}$$

$$A_{mn}(\mathbf{R}(t))\equiv-\frac{C_n}{C_m}\langle m(\mathbf{R}(t))|[\nabla_{\mathbf{R}}|n(\mathbf{R}(t))\rangle]\cdot\dot{\mathbf{R}}(t), \tag{A7}$$

$$\Delta_{mn}(t)\equiv\int_0^t dt[\lambda_n(t)-\lambda_m(t)]. \tag{A8}$$

The formal solution of Eq. (A5) is

$$\chi_m(t)=\exp\left[\int_0^t dt' A_m(\mathbf{R}(t'))\right]\left[1+\sum_{m\neq n}\int_0^t dt'' A_{mn}(\mathbf{R}(t''))\exp\left[-\int_0^{t''}dt' A_m(\mathbf{R}(t'))\right]e^{-i\Delta_{mn}(t'')}\chi_n(t'')\right]. \tag{A9}$$

Every term in the summation is negligibly small compared to unity. To show this we define the quantities

$$\dot{B}_{mn}(\mathbf{R}(t''))=A_{mn}(\mathbf{R}(t''))\exp\left[-\int_0^{t''}dt' A_n(\mathbf{R}(t'))\right]\chi_n(t''), \tag{A10}$$

$$C_{mn}(t'')=e^{-i\Delta_{mn}(t'')}. \tag{A11}$$

Here, B_{mn} has been defined in terms of its derivative because A_{mn} is obviously the derivative of a slowly varying function. We aim to show self-consistently that χ_m is also a slowly varying function. Therefore we assume that B_{mn} is a slowly varying function

$$\left|\frac{\dot{B}_{mn}}{B_{mn}}\right|\simeq\frac{1}{T}\ll|\lambda_n-\lambda_m|. \tag{A12}$$

Integration by parts of the last term in Eq. (A9) gives

$$\int_0^t dt''\dot{B}_{mn}(t'')C_{mn}(t'')=\frac{C_{mn}(t'')\dot{B}_{mn}(t'')}{-i[\lambda_n(t'')-\lambda_m(t'')]}\Bigg|_{t''=0}^{t''=t}-\int_0^t dt''\frac{\ddot{B}_{mn}(t'')C_{mn}(t'')}{-i[\lambda_n(t'')-\lambda_m(t'')]}$$
$$\simeq\frac{C_{mn}(t'')B_{mn}(t'')}{-i[\lambda_n(t'')-\lambda_m(t'')]T}\Bigg|_{t''=0}^{t''=t}-\int_0^t\frac{dt''C_{mn}(t'')}{-i[\lambda_n(t'')-\lambda_m(t'')]}\frac{\dot{B}_{mn}}{T}\ll1. \tag{A13}$$

These terms are negligibly small because the adiabatic theorem requires that $(\lambda_n-\lambda_m)T\gg1$. Dropping these small terms and taking the logarithm of Eq. (A9) gives the familiar result

$$\gamma_n(t)=i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\langle n(\mathbf{R})|[\nabla_{\mathbf{R}}|m(\mathbf{R})\rangle]\cdot d\mathbf{R}. \tag{A14}$$

The Berry phase for each nondegenerate eigenstate evolves independent of the others.

APPENDIX B

We demonstrate here that the dynamics of the geometrical phases γ_n can also be identified and separated in the dissipation problem of Sec. IV. This generalization is carried out under the condition of weak damping (to which we will give a precise meaning) and at the price of the introduction of a path-dependent imaginary part in Berry phase. For simplicity we restrict our analysis to the two-level atom problem.

The substitution of the normal mode expansion (3.10) into the dissipative Liouville equation (4.1) leads (after projection on χ_n) to

$$i\dot{\gamma}_m e^{i\gamma_m}=-(\langle\chi_m,\nabla_{R\chi_m}\rangle\cdot\dot{\mathbf{R}}-\langle\chi_m,L\chi_m\rangle)$$
$$-\sum_{m\neq n}(\langle\chi_m,\nabla_{R\chi_n}\rangle\cdot\dot{\mathbf{R}}-\langle\chi_m,L\chi_n\rangle)e^{-i\gamma_n}\frac{C_n}{C_m}\exp\left[-i\int d\tau(\lambda_n-\lambda_m)\right]+\frac{1}{2C_m}\langle\chi_m,L1\rangle\exp\left[i\int d\tau\lambda_m\right] \tag{B1}$$

This equation is of the type of (A5). Therefore we can apply the formal solution (A9) and find:

$$\chi_m(t)=\exp\left[\int_0^t dt'[A_m(\mathbf{R}(t'))+A'_m(\mathbf{R}(t'))]\right]$$
$$\times\left[1+\sum_{m\neq n=2}^{3}\int_0^t dt''[A_{mn}(\mathbf{R}(t''))+A'_{mn}(\mathbf{R}(t''))]\exp\left[-\int_0^{t''}dt'[A_m(\mathbf{R}(t'))+A'_m(\mathbf{R}(t'))]\right]\right.$$
$$\left.\times e^{-i\Lambda_{mn}(t'')}\chi_n(t)+\int_0^t dt''D_m(\mathbf{R}(t''))\exp\left[i\int_0^{t''}d\tau\lambda_m(\tau)\right]\right], \tag{B2}$$

where

$$\begin{aligned}
&\chi_m(t)\equiv e^{i\gamma_m(t)},\\
&\Delta_{mn}(t)\equiv\int_0^t d\tau[\lambda_n(\tau)-\lambda_m(\tau)],\\
&A_m(t)\equiv-\langle\chi_m,\nabla_{\mathbf{R}}\chi_m\rangle\cdot\dot{\mathbf{R}},\\
&A'_m(t)\equiv\langle\chi_m,L\chi_m\rangle,\\
&A'_{mn}(t)\equiv\frac{C_n}{C_m}\langle\chi_m,L\chi_n\rangle,\\
&D_m(t)\equiv\frac{1}{2C_m}\langle\chi_m,L1\rangle
\end{aligned} \tag{B3}$$

We are going to show that the summations and (if $\lambda_m\neq0$), the last term are negligibly small compared to unity under the adiabatic and weak damping approximation.

We define the quantities:

$$\begin{aligned}
&\dot{B}_{mn}(\mathbf{R}(t''))\equiv A_{mn}(\mathbf{R}(t''))\exp\left[-\int_0^{t''}dt'[A_m(\mathbf{R}(t'))+A'_m(\mathbf{R}(t'))]\right]\chi_n(t''),\\
&\dot{B}'_{mn}(\mathbf{R}(t''))\equiv A'_{mn}(\mathbf{R}(t''))\exp\left[-\int_0^{t''}dt'[A_m(\mathbf{R}(t'))+A'_m(\mathbf{R}(t'))]\right]\chi_n(t''),\\
&C_{mn}(\mathbf{R}(t''))\equiv e^{-i\Delta_{mn}(t'')}.
\end{aligned} \tag{B4}$$

We explained in Appendix A why B_{mn} was defined in terms of its derivative, however, here we need an additional assumption to justify this choice and insure that $A'_m(\mathbf{R}(t))$ is slowly varying. Namely, we assume that

$$\|L\|\sim\Gamma\ll|\lambda_n-\lambda_m|. \tag{B5}$$

Obviously this assumption states that the damping is weak. If this is the case we can assume

$$\left|\frac{\dot{B}_{mn}}{B_{mn}}\right|,\left|\frac{\dot{B}'_{mn}}{B'_{mn}}\right|\ll|\lambda_m-\lambda_n|. \tag{B6}$$

As in appendix A an integration by parts shows that because of the adiabatic condition $(\lambda_n-\lambda_m)T\gg1$ and of (B5), each term of the summation in (B2) is negligibly small.

Finally, if $\lambda_m(t)\neq0$ we can apply the same argument to the last term in (B2) and neglect it as well. If on the other hand $\lambda_m(t)=0$ the last term is slowly varying. In this case we can keep it in the solution because it is consistent with the assumption of a slowly varying χ_m.

Taking the logarithm in the case $\lambda_m\neq0$ ($m=2,3$) gives the by now familiar result (4.7) and (4.8). When $\lambda_m\neq0$ ($m=1$) we must include the last term also and obtain the expression (4.9) for γ_1.

APPENDIX C

In this appendix we point out the restrictions of the adiabatic theorem on the variations of the laser phase and amplitude in the two-level atom problem of Sec. II. From Messiah's formulation of the adiabatic theorem[3] the following condition must be fulfilled:

$$P_{+\to-}\lesssim\max\left|\frac{\alpha_{-+}(t)}{\omega_{-+}(t)}\right|^2\ll1, \tag{C1}$$

where $P_{+\to-}$ is the probability of transition from one state to the other. According to Messiah's definition we have

$$\begin{aligned}
&\alpha_{-+}(t)=\langle-(\phi)|\frac{d}{dt}[|+(\phi)\rangle],\\
&\omega_{-+}(t)=E_--E_+.
\end{aligned} \tag{C2}$$

Using the eigenvalues and the eigenvectors[6] we find

$$\alpha_{-+}(t)=|\dot{\theta}+\cos\theta\dot{\phi}|\leq|\dot{\theta}|+|\dot{\phi}|, \tag{C3}$$

where

$$\cos\theta=\frac{E_+-\frac{1}{2}\hbar\Delta}{[2E_+(E_+-\frac{1}{2}\hbar\Delta)]^{1/2}},$$
$$\sin\theta=\frac{\hbar\lambda}{[2E_+(E_+-\frac{1}{2}\hbar\Delta)]^{1/2}}.$$

We can see that a simple sufficient condition (C1) is

$$\dot{\theta},\dot{\phi} \ll \Lambda \tag{C4}$$

since $\Lambda=(\lambda^2+\Delta^2/4)^{1/2}$ is obviously the half of the transition frequency between the dressed states. The statement $\dot{\phi} \ll \Lambda$ is directly obtained in terms of the laser phase, while the statement $\dot{\theta} \ll \Lambda$ is a condition on laser detuning and amplitude. With a little algebra one can show that we must have

$$\dot{\theta} = -\cos^2\theta \frac{\dot{\lambda}\Delta - \dot{\Delta}\lambda}{\left[\Lambda^2 - \frac{\Delta}{2}\Lambda\right]} \ll \Lambda \ . \tag{C5}$$

If $\lambda(t) \ll \Delta(t)$ then (C5) reduces to

$$|\dot{\theta}| = \left|\frac{\dot{\lambda}}{\Delta}\right| \ll \Delta \ . \tag{C6}$$

If on the other hand $\lambda(t) \gg \Delta(t)$ we get

$$|\dot{\theta}| = \left|\frac{\dot{\Delta}}{\lambda}\right| \ll \lambda \ . \tag{C7}$$

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