Free-energy fluctuations in a one-dimensional random Ising model

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The statistical characteristics of the free-energy fluctuations in a one-dimensional random-bond Ising model are investigated by using the fluctuation-spectrum theory developed for the global characterization of fluctuations. As illustrations the characteristic function, the order-q free energy, and the fluctuation spectrum are calculated by generating the random coupling through chaotic dynamics.

There has been much effort devoted to the study of disordered magnetic systems.[1–7] There is also an increasing interest in the fluctuation properties of random magnetic systems with quenched randomness in connection with the fractal dimensions.[8] Very recently, a new statistical approach to the characterization of free-energy fluctuations of disordered systems has been developed by studying the sample-to-sample fluctuations of the free energy.[9,10] This development is based on the establishment of the multifractal theory of strange sets[9] and the fluctuation-spectrum theory of temporal fluctuations.[11] The purpose of the present paper is to discuss the characteristics of the free-energy fluctuations in a one-dimensional random-bond Ising system by considering how the free-energy fluctuation is reduced as the system size is increased.

The Hamiltonian for a one-dimensional random Ising chain of N spins is given by

$$H_N=-\sum_{i=1}^{N} J_i\sigma_i\sigma_{i+1}, \tag{1}$$

where $\sigma_i=\pm 1$ is the value of the spin on the site i, and J_i is the strength of the coupling between nearest-neighboring spins, σ_i and σ_{i+1}. In our system the coupling J_i is a random variable statistically independent of the site i and is called the random bond. We note here that the summation in Eq. (1) is extended over the whole spins for both cases of a linear chain and a ring. Once a distribution of random bonds $\{J_i\}$ is given, the partition function of the N spin system is calculated as

$$Z_N(\beta)=\sum_{\{\sigma_i=\pm 1\}} \exp(-\beta H_N) , \tag{2}$$

where $\beta(=1/k_BT)$ is the inverse temperature of the system. We can easily obtain the following expression for two cases of the one-dimensional random Ising chain and ring from Eq. (2) as

$$Z_N(\beta)=2^N\psi_N\prod_{i=1}^{N}\cosh(\beta J_i), \tag{3}$$

where

$$\psi_N=\begin{cases}1 \quad (\text{for chain}) \\ 1+\nu_1\nu_2\nu_3\cdots\nu_N \quad (\text{for ring}) ,\end{cases} \tag{4}$$

with $\nu_i=\tanh(\beta J_i)$. It is noted that as $N\to\infty$, $\psi_N\to 1$ for both cases of a one-dimensional Ising system. If a set of random bonds $\{J_i\}$ is determined, we can calculate the partition function $Z_N(\beta)$ for both systems, as described above.

We mention here that $Z_N(\beta)$ is a random variable and that the variable $g_N(\beta)$ defined through[11]

$$\frac{Z_N(\beta)}{Z_{N-1}(\beta)}=e^{-\beta g_N(\beta)}, \tag{5}$$

where

$$g_N(\beta)=-\frac{1}{\beta}\ln[2\cosh(\beta J_N)\psi_N/\psi_{N-1}] , \tag{6}$$

is statistically independent of N for a large N. The $g_i(\beta)$ may be regarded as "the free energy of the ith bond." The free energy per spin $f_{\beta,N}$ for a given sample is given by

$$f_{\beta,N}=-\frac{1}{N}\beta^{-1}\ln Z_N(\beta)=\frac{1}{N}\sum_{i=1}^{N}g_i(\beta) . \tag{7}$$

The stationarity of $g_N(\beta)$ for a large N indicates that as N becomes large, the fluctuation of $f_{\beta,N}$ reduces, and $f_{\beta,N}$ approaches an average value $f_{\beta,\infty}$. This should be compared with the ergodicity assumption in the case of temporal fluctuations. Namely, there is no sample-to-sample fluctuation of the free energy in the thermodynamic limit. If J_i is independent of i $(J_i=J)$, then $g_i(\beta)$ does not depend on i. However, for a random-bond system J_i depends on i and the randomness in $\{J_i\}$ produces the randomness in $\{g_i(\beta)\}$, and therefore in $f_{\beta,N}$. In Fig. 1, the bond-to-bond fluctuations of $g_i(\beta)$ are illustrated, where the distribution of random bonds J_i is governed by the one-dimensional chaotic map

$$x_{N+1}=f(x_N)=Ax_N(1-X_N), \tag{8}$$

where A is the amplitude of the map. The ith bond J_i is

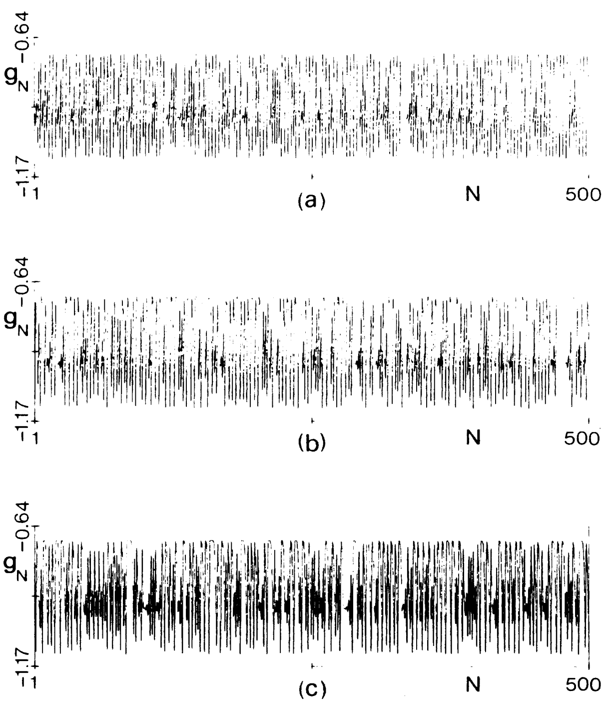

FIG. 1. Typical temporal evolutions of the dynamical variable $g_N(\beta)$ ($\equiv g_N$) for (a) $A=3.88$, (b) $A=3.98$, and (c) $A=3.9999$ with $\beta=1.0$ and $\omega=1.0$.

determined by $J_i=\omega x_i$, where ω is a constant. Thus we can obtain a set of random bonds $\{J_i\}$ from Eq. (8) under a suitable initial condition x_0. Depending on the value of A, the $\{x_N\}$ have different statistical characteristics. Then these cause the differences of the statistical property of $\{g_i(\beta)\}$, depending on A.

In order to study the global characteristics of the above free-energy fluctuations, we introduce the characteristic function λ_q as[10,11]

$$\lambda_q(\beta)=\lim_{N\to\infty}\frac{1}{N\beta q}\ln\langle[Z_N(\beta)]^{-q}\rangle\ , \quad \frac{d\lambda_q(\beta)}{dq}\ge 0 \qquad (9)$$

where q is a dimensionless parameter ($-\infty<q<\infty$). The symbol $\langle\cdots\rangle$ denotes the ensemble average over a set of distribution for random bonds $\{J_i\}$. The present approach for dealing with the fluctuation of free energies can be applied to both uncorrelated bond randomness as in Refs. 1–7 and correlated bond randomness as in our case [Eq. (8)]. Especially for an uncorrelated bond randomness, Eq. (9) can be written as

$$\lambda_q(\beta)=(\beta q)^{-1}\ln[2^{-q}\langle\cosh^{-q}(\beta J)\rangle]\ .$$

As q goes to zero, Eq. (9) becomes

$$\lambda_0(\beta)=-\lim_{N\to\infty}\frac{1}{N\beta}\langle\ln Z_N(\beta)\rangle\ . \qquad (10)$$

This is the average value of the free energy per spin $[\lambda_0(\beta)=f_{\beta,\infty}]$. Furthermore, $\lambda_q(\beta)$ is related to the asymptotic realization probability of the fluctuation of the free energy as follows. Let $\rho_N(f';\beta)$ be the probability density that $f_{\beta,N}$ takes the value between f' and $f'+df'$.[12] If $f_{\beta,N}$ has no fluctuation, $\rho_N(f';\beta)$ has a sharp peak at $f'=\lambda_0(\beta)$. We assume that the probability density is asymptotically expressed by

$$\rho_N(f';\beta)\sim\exp[-\beta\sigma_\beta(f')N] \qquad (11)$$

for a large N. By making use of the steepest-descent method, the fluctuation spectrum $\sigma_\beta(f)$ is easily obtained from the Legendre transformation of $\lambda_q(\beta)$. The relations among thermodynamic variables obtained from the characteristic function per spin are written as[9–11]

$$f_\beta(q)\equiv\frac{\partial}{\partial q}[q\lambda_q(\beta)]\ , \qquad (12)$$

$$\sigma_\beta(f_\beta(q))\equiv q^2\frac{\partial\lambda_q(\beta)}{\partial q}\ , \qquad (13)$$

$$\lambda_q(\beta)=f_\beta(q)-q^{-1}\sigma_\beta(f_\beta(q)). \qquad (14)$$

Hereafter $f_\beta(q)$ will be called the order-q free energy. The fluctuation spectrum $\sigma_\beta(f')$ is a concave function of f' and has a single minimal value $\sigma_\beta(f')=0$ at $f'=\lambda_0(\beta)=f_\beta(0)$. In the one-dimensional random Ising system, the behavior of the characteristic function $\lambda_q(\beta)$ is the same for both cases whether the bond is ferromagnetic or antiferromagnetic, i.e., Eq. (3) is invariant under the change $J_i\to -J_i$ for a large N.

Numerical results of the characteristic function $\lambda_q(\beta)$ are given in Fig. 2. The average free energy $f_\beta(0)$ $[=\lambda_0(\beta)]$ is numerically obtained as -0.895 ($A=3.88$), -0.878 ($A=3.98$), and -0.863 ($A=3.9999$). The order-q free energies per spin $f_\beta(q)$ are shown in Fig. 3 for several values of A. It is noted here that $\lambda_q(\beta)$ and $f_\beta(q)$ have similar q dependence, but the slope of $f_\beta(q)$ at $q=0$ is twice as steep as that of $\lambda_q(\beta)$. We also calculate the fluctuation spectrum of the free energy. In Fig. 4 the fluctuation spectra $\sigma_\beta(f)$ are shown for several values of A. $\sigma_\beta(f)$ expresses the rate of the decrease of the proba-

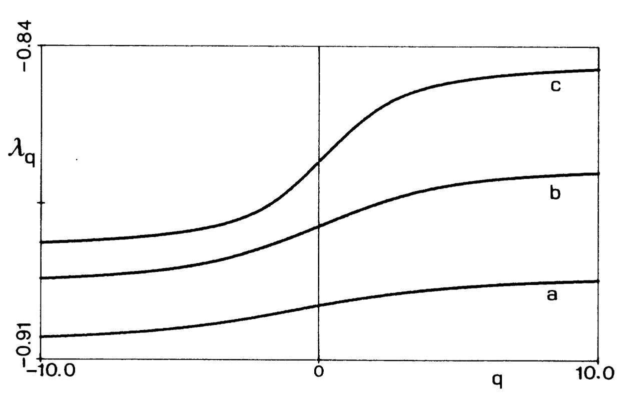

FIG. 2. Numerical results of the characteristic function $\lambda_q(\beta)$ ($\equiv\lambda_q$) for the configuration of random bonds generated by the logistic map (8). The values of A for a, b, and c are the same as in (a), (b), and (c) in Fig. 1, respectively. The number of spins contained in the one-dimensional chain is 250. The average $\langle\cdots\rangle$ is taken by the distribution of 1000 sets of random bonds $\{J_i\}$.

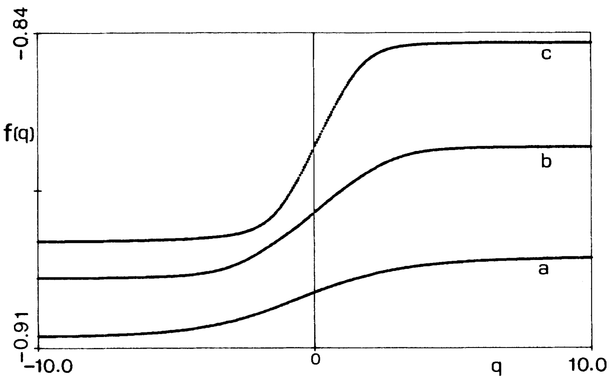

FIG. 3. Numerical results of the order-q free energy $f_\beta(q)$ [$\equiv f(q)$]. The values of A for a, b, and c are the same as in (a), (b), and (c) in Fig. 1, respectively.

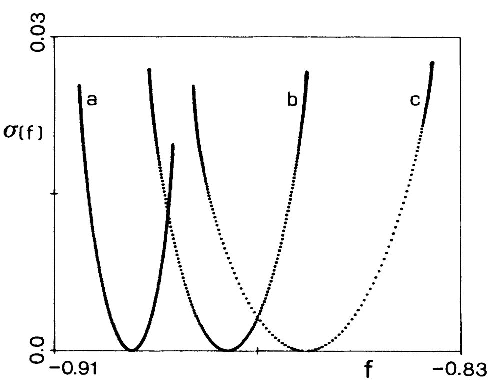

FIG. 4. Fluctuation spectra $\sigma_\beta(f)$ [$\equiv\sigma(f)$] of the free energy are shown for (a) $A=3.88$, (b) $A=3.98$, and (c) $A=3.9999$.

bility density that the free energy per spin $f_{\beta,N}$ of the N spin system takes the value f as the number of spins is increased. There appears the spectral structure of $\sigma_\beta(f)$ due to the bond randomness. The intensity of the randomness of $\{g_N(\beta)\}$ and thus $\{f_{\beta,N}\}$ is evaluated, roughly speaking, with the quantity

$$D\equiv\partial\lambda_q(\beta)/\partial q\,|_{q=0}=\partial f_\beta(q)/\partial q\,|_{q=0}/2$$

and the dispersion range

$$\Delta\equiv\lambda_\infty(\beta)-\lambda_{-\infty}(\beta)=f_\beta(\infty)-f_\beta(-\infty)\ .$$

The D determines the curvature of $\sigma_\beta(f')$ at its minimum at $f'=f_\beta(0)$, $\sigma_\beta''(f_\beta(0))=1/2D$. So, although in comparing three sequential fluctuations of $g_N(\beta)$ in Fig. 1 with each other it is hard to get an apparent difference among their statistical characteristics, one may clearly distinguish them with the quantities $\lambda_q(\beta)$, $f_\beta(q)$, and $\sigma_\beta(f)$. For three values of A, we numerically get $(D,\Delta)\simeq(0.0012,0.018)$ for $A=3.88$, $(0.0026,0.029)$ for $A=3.98$ and $(0.0063,0.053)$ for $A=3.9999$.

In the present paper we have discussed how we can single out the statistical characteristics of free-energy fluctuations in a one-dimensional random-bond Ising system. Its global property is shown to be described with the characteristic function $\lambda_q(\beta)$, the order-q free energy $f_\beta(q)$, and the fluctuation spectrum $\sigma_\beta(f)$. Finally we note that the present approach is also applicable to the random-field Ising system, where the Hamiltonian is expressed as

$$H_N=-\sum_{i=1}^{N}J\sigma_i\sigma_{i+1}+\sum_{i=1}^{N}h_i\sigma_i\ , \tag{15}$$

h_i being the quenched random field at the site i. In calculating the characteristic function $\lambda_q(\beta)$, the averaging procedure in the ensemble average $\langle\cdots\rangle$ should be carried out over the distribution of $\{h_i\}$.

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