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General theory of a convective nucleus of water in a nonsteady state and under nonlinear conditions at temperature ranges that include the density maximum

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A model of free convection is developed for the case of pure water contained in a cylindric reservoir submitted to arbitrary boundary conditions. In the temperature region where water has a density maximum a quadratic $\rho(T)$ function is assumed and the asymptotic solution is found as a sample calculation. Discontinuities in the $T(z,t)$ behavior observed experimentally are fully explained in terms of a function that includes the small coefficient of the quadratic term of $\rho(T)$ in the denominator with dramatic effects in the vicinity of the density maximum.

I. INTRODUCTION

If we put a sample of pure water in a gravitational field submitted to varying temperatures through a thermostat at T_s, free convection takes place and a distribution of temperature varying with space and time is obtained. If this experiment is carried on in a range of temperatures including 4 °C, where water has a density maximum, dramatic discontinuities in the $T(z,t)$ curves are observed.[1,2] During the past few years the free-convection problem in nonsteady conditions in a vertical cylinder has received considerable attention.

Mouton and De Roeck first studied experimentally the heating behavior of various liquids and, on the basis of their measurements, developed a model of convection which divides the cylinder into two regions, a boundary layer moving upwards at the wall and a nucleus moving downward along the axis.[3]

Their equations, developed for a Boussinesq liquid, failed to explain the inflection point in the $T(t)$ curves, but were an important breakthrough in the problem. In a previous paper we used their model coupled with more complete equations applied to a non-Boussinesq fluid, obtaining fair agreement between experiment and theory for water outside the region of maximum density.[4] Since then, a paper by Rahm on free convection in alcohol has appeared in which a quite different point of view is adopted to interpret the heating of the fluid: the cylinder is subdivided into three regions, a boundary layer, a "buoyancy layer," and a homogeneous region which s progressively stratified by the incoming buoyancy layer.[5]

Rahm's analysis refers explicitly to a Boussinesq fluid with a linear temperature-dependent density and provides solutions in qualitative agreement with experiment and with numerical solutions of the time-dependent Navier-Stokes equation.[6]

In order to interpret the convective behavior of water in the region of maximum density we not only must consider a non-Boussinesq fluid, but also must solve the proper hydrodynamic equations, taking into account the lateral interactions between the boundary layer and the nucleus through an appropriate velocity field that depends on time and height.

In this paper we present a theory of the nonsteady and nonlinear convective nucleus that is valid under very general boundary conditions when a particular model of convection is assumed and when the fluid may have a density maximum in the range considered.

II. MODEL OF CONVECTION

The system consists of a cylinder full of water with a given initial temperature distribution at $t=t_0$, which can exchange heat with a thermostat at temperature T_s through its lateral wall, vertical and of high conductance. In the gravity field, free convection takes place in the cylinder, and temperature varies as a function of space and time. Since water has a density maximum at 4 °C, we concentrate on this temperature region and solve the hydrodynamic problem by assuming a modified model of convective motion which has previously been used to fit successfully convection data in the temperature region where density is a monothonic function.[2-4] We introduce the following assumptions.

(a) The volume of water is subdivided into three regions, the boundary layer, the central nucleus, and the (small) intermediate region which connects smoothly the previous two regions.

(b) The convective nucleus is perturbed by the motion of the fluid and is described by the scalar field $T=T(z,t)$ and the vectorial field

$$\mathbf{U}=(0,0,-V_z(z,t)) ,$$

where the perturbation is expressed by a dependency of the velocity V upon the level z, measured from the top, which is small when compared with the time dependency.

(c) The density maximum is fitted by a parabola although, in principle, it is possible to apply a more precise formula,[7] with undesired complication.

(d) The solutions must describe not only the density maximum region but also the region outside, i.e., the equations must converge to functions where the parabolic expression of the density plays a role vanishingly small at temperatures far from 4 °C. This will be done by forcing the parabolic term of the density equation to become smaller and smaller where the fluid convection cannot be modified by the density maximum.

III. MATHEMATICAL TREATMENT OF THE MODEL

The fundamental equations of hydrodynamics applied to the described model reduce to the following three equations (see the Appendix): the Fourier equation,

$$\frac{\partial T}{\partial t}-V_z\frac{\partial T}{\partial z}=\chi\frac{\partial^2 T}{\partial z^2}+\nu^*\left[\frac{\partial V_z}{\partial z}\right]^2+\frac{p}{\rho c}\left[\frac{\partial V_z}{\partial z}\right], \tag{1}$$

where ν^* is a viscosity parameter,

$$\nu^*=\frac{10\mu}{3c_v}+\frac{\xi}{\rho c_v},$$

and χ is the thermal diffusion coefficient, the Navier-Stokes equation,

$$-\frac{\partial V_z}{\partial t}+V_z\frac{\partial V_z}{\partial z}=g-\frac{\partial p}{\partial z}\frac{1}{\rho}-\nu\frac{\partial^2 V_z}{\partial z^2}, \tag{2}$$

where ν is a viscosity parameter,

$$\nu=\frac{4\tilde{\nu}}{3}+\frac{\xi}{\rho},$$

and g is the constant of gravity, and the continuity equation,

$$\frac{\partial\rho}{\partial t}-\frac{\partial(\rho V_z)}{\partial z}=0. \tag{3}$$

Equations (1)–(3) may be satisfied by a unique solution only providing the appropriate boundary conditions.

If we indicate with the symbols $\tilde{p}$, $\tilde{T}$, V, respectively, the variables hydrostatic pressure, temperature, and speed of the fluid in the nucleus at temperatures outside the density maximum, and with the symbols p, T, V_z, the same variables in the region of maximum density, we must find solutions which satisfy the following relationships:

$$\lim_{k\to 0}\begin{bmatrix}\rho\\ p\\ V_z\\ T\end{bmatrix}=\begin{bmatrix}\tilde{\rho}\\ \tilde{p}\\ V\\ \tilde{T}\end{bmatrix}, \tag{4}$$

where k is the positive temperature coefficient in the density equation

$$\rho(T)=\rho(T_s)(1+\beta_k\Delta T-k\Delta T^2), \tag{5}$$

where β_k is the thermal expansion coefficient, function of k, and $\Delta T=|T-T_s|$ is the initial temperature difference, in modulus.

The hydrodynamic equations for ρ, p, T, and V are[2,4] the Fourier equation,

$$\frac{\partial\tilde{T}}{\partial t}-V\frac{\partial T}{\partial z}=\chi\frac{\partial^2\tilde{T}}{\partial z^2}, \tag{6}$$

the Navier-Stokes equation,

$$-\dot{V}=g-\frac{1}{\tilde{\rho}}\frac{\partial\tilde{p}}{\partial z}, \tag{7}$$

and the continuity equation,

$$\frac{\partial\tilde{\rho}}{\partial t}-V\frac{\partial\tilde{\rho}}{\partial z}=0, \tag{8}$$

and the boundary conditions

$$\tilde{T}(z,t_0)=T_s+G(z), \tag{9}$$

$$V(t_0)=V_0, \tag{10}$$

$$\tilde{\rho}(T)=\tilde{\rho}(T_s)[1-\beta(\tilde{T}-T_s)], \tag{11}$$

where, considering that in the regions where convection is not modified by the density maximum we can assume $k\to 0$, we write

$$\beta=-\lim_{k\to 0}\beta_k. \tag{12}$$

The solutions of Eqs. (6)–(8) with the conditions (9)–(11) are straightforward by the method of characteristic functions:

$$\int_0^{x(t)}\frac{dy}{[G^*(yh/\pi+\delta)]^{1/3}}=a(t-t_0), \tag{13}$$

$$\tilde{T}(z,t)=T_s+\sum_{n=-\infty}^{\infty}\alpha_n\exp\{in[x(t)+\pi z/h]\}, \tag{14}$$

$$\alpha_n=[1/(zh)]\int_{-h}^{h}G^*(z)\exp(-in\pi z/h)dz, \tag{15}$$

$$V(t)=\dot{x}(t)h/\pi, \tag{16}$$

$$\tilde{p}(z,t)=[h\dot{x}(t)/\pi+g]\times\int_0^z\rho(T_s)\{1-\beta[\tilde{T}(z,t)-T_s]\}dz+p_0, \tag{17}$$

$$\tilde{\rho}(T)=\rho(T_s)\{1-\beta[\tilde{T}(z,t)-T_s]\}, \tag{18}$$

where

$$a=\frac{k^*2\pi R\mu N_{\mathrm{Gr}}^{1/3}N_{\mathrm{Pr}}^{-2/3}}{h(R-\delta)^2[G^*(h/2)]^{1/3}},$$

$G^*(z)$ is the boundary condition for temperature, h is the level of the liquid contained in the cylinder, k^* is a pure constant, and δ is the average thickness of the boundary layer. According to Eq. (4) we write the new boundary conditions for Eqs. (1)–(3):

$$T(z,t_0)=T_s+G(z), \tag{19}$$

$$\epsilon_k(z,t_0)=0, \tag{20}$$

$$\epsilon_k(\delta,t)=U_k^{(1)}(t),\quad \epsilon_k(h-\delta,t)=U_k^{(2)}(t), \tag{21}$$

$$\rho(z,t_0)=\rho(T_s)\{1+\beta_k[T(z,t_0)-T_s]-k[T(z,t_0)-T_s]^2\}, \tag{22}$$

where the function $\epsilon_k(z,t)$ is defined by

$$V_z(z,t)=V(t)+\epsilon_k(z,t). \tag{23}$$

Equation (20) indicates that the nucleus at $t=t_0$ has the same velocity $V(t_0)$ as before.

Moreover, the condition (4) implies that

$$\lim_{k\to 0}\epsilon_k(z,t)=0 \ . \tag{24}$$

By substituting Eq. (23) in Eq. (2) we obtain

$$-V(t)-\frac{\partial\epsilon_k}{\partial t}+(V+\epsilon_k)\frac{\partial\epsilon_k}{\partial z}=g-\frac{\partial p}{\partial z}\frac{1}{\rho}-\nu\frac{\partial^2\epsilon_k}{\partial z^2} \ . \tag{25}$$

According to Eqs. (7) and (25) we have

$$g-\frac{1}{\bar\rho}\frac{\partial\bar p}{\partial z}=g-\frac{1}{\rho}\frac{\partial p}{\partial z}-\nu\frac{\partial^2\epsilon_k}{\partial z^2}+\frac{\partial\epsilon_k}{\partial t}-(V+\epsilon_k)\frac{\partial\epsilon_k}{\partial z} \ , \tag{26}$$

which becomes

$$\frac{\partial\epsilon_k}{\partial t}-\nu\frac{\partial^2\epsilon_k}{\partial z^2}=F(z,t)+(V+\epsilon_k)\frac{\partial\epsilon_k}{\partial z} \ , \tag{27}$$

where

$$F(z,t)=\frac{\partial p}{\partial z}\frac{1}{\rho}-\frac{\partial\bar p}{\partial z}\frac{1}{\bar\rho}\cong 0 \tag{28}$$

and

$$(V+\epsilon_k)\frac{\partial\epsilon_k}{\partial z}\ll\frac{\partial\epsilon_k}{\partial t}$$

because $V\gg\epsilon_k$.

In addition, experimentally it has been observed[3] that the velocity perturbation is much larger in time than in space, when compared together. Thus

$$\frac{\partial\epsilon_k}{\partial t}=\nu\frac{\partial^2\epsilon_k}{\partial z^2} \ , \tag{29}$$

which is to be solved with the boundary conditions (20) and (21)

The solution of Eq. (29) is[8]

$$\epsilon(z,t)=\int_0^t\int_\delta^{h-\delta}\left\{[2/(h-\delta)]\sum_{n=1}^{\infty}\exp\{[-\pi n/(h-\delta)^2]\nu(t-\tau)\}\sin[\pi nz/(h-\delta)]\sin[\pi ny/(h-\delta)]f(y,\tau)\right\}dy\,d\tau \ , \tag{30}$$

where

$$f(y,\tau)=-\dot U_k^{(1)}-\frac{y}{h-\delta}(\dot U_k^{(2)}-\dot U_k^{(1)}) \ . \tag{31}$$

From Eqs. (16) and (23), combined with (30), we determine $V_z(z,t)$. As a consequence, through V_z, we can deduce the expression for $\rho(z,t)$ by solving Eq. (3) with the initial condition (22). This task is noticeably simplified because the term $\partial(\epsilon_k\rho)/\partial z$ is negligible compared to the other terms. With this assumption Eq. (3) becomes

$$\frac{\partial\rho}{\partial t}-V(t)\frac{\partial\rho}{\partial z}=0 \ . \tag{32}$$

The solution is

$$\rho(z,t)=\rho(T_s)+\sum_{n=-\infty}^{+\infty}\alpha_n\exp(in\pi z/h)\exp[-inx(t)] \ , \tag{33}$$

where $(h\dot x)/\pi=V(t)$ and

$$\alpha_n=1/(2h)\int_{-h}^{h}\rho(z,t_0)\exp(-in\pi z/h)dz \ , \tag{34}$$

$$\rho^*(z,t_0)\begin{cases}=\rho_1 & \text{for } 0\le z<\delta\\ =\rho(z,t_0)-\rho(T_s) & \text{for } \delta\le z\le h-\delta\\ =\rho_2 & \text{for } h-\delta<z\le h\\ =0 & \text{for } -h\le z<0 \ ,\end{cases} \tag{35}$$

where ρ_1 and ρ_2 are the small density perturbations in the boundary layer region.

In what follows we choose $\rho_1=\rho_2\cong 0$. Once $\rho(z,t)$ is known, through Eqs. (33)–(35) we are able to calculate the function $T(z,t)$ by means of the state equation (5), obtaining

$$T(z,t)=T_s+\frac{\beta_k\pm\{\beta_k^2-4k[\rho(z,t)-\rho(T_s)]/\rho(T_s)\}^{1/2}}{2k} \ . \tag{36}$$

We show that Eq. (4) is satisfied by Eq. (36) as requested, by calculating $\lim_{k\to 0}T(z,t)$ and verifying that it is coincident with $\bar T$.

We split Eq. (34) into two terms using Eqs. (25) and (22):

$$\alpha_n=\alpha_{n1}+\alpha_{n2} \ ,$$

with

$$\alpha_{n1}=[1/(2h)]\int_{-h}^{h}\beta_k\rho(T_s)\Delta T(z,t_0)\exp(-in\pi z/h)dz \ , \tag{37}$$

$$\alpha_{n2}[1/(2h)]\int_{-h}^{h}k\rho(T_s)[\Delta T(z,t_0)]^2\exp(-in\pi z/h)dz \ . \tag{38}$$

Through Eq. (33), we write

$$\rho(z,t)=\tilde\rho_k(z,t)+\hat\rho_k(z,t) \ , \tag{39}$$

where

$$\tilde\rho_k(z,t)=\rho(T_s)+\sum_{n=-\infty}^{+\infty}\alpha_{n1}\exp(in\pi z/h)\exp[-inx(t)]$$

and

$$\hat{\rho}_k(z,t)=\sum_{n=-\infty}^{+\infty}\alpha_{n2}\exp(in\pi z/h)\exp[-inx(t)] \ .$$

From Eq. (12) and observing that $\Delta T(z,t_0)=\Delta\tilde{T}(z,t_0)$, we have

$$\lim_{k\to 0}\rho(z,t)=\tilde{\rho}(z,t) \ .$$

As a consequence, by using Eq. (11), we obtain

$$\lim_{k\to 0}\frac{\tilde{\rho}_k(z,t)-\rho(T_s)}{\rho(T_s)}=-\beta\Delta T \ . \tag{40}$$

We are now able to calculate $\lim_{k\to 0}T$. By substituting Eq. (39) in Eq. (36) we have

$$T(z,t)=T_s+\frac{\beta_k\pm(\beta_k^2-4k\{[\tilde{\rho}_k(z,t)-\rho(T_s)]/\rho(T_s)\}-4k\hat{\rho}(z,t)/\rho(T_s))^{1/2}}{2k} \ , \tag{41}$$

which, through Eq. (40), gives

$$\lim_{k\to 0}T(z,t)=T_s+\Delta\tilde{T}=\tilde{T} \ . \tag{42}$$

The only unknown quantities are now $U_k^{(1)}$ and $U_k^{(2)}$, which can be determined through an additional flow equation at the interface between the nucleus and the boundary layer:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \ ,$$
$$\int_s \rho\mathbf{u}\cdot d\mathbf{S}=-\int_v\frac{\partial\rho}{\partial t}dv \ . \tag{43}$$

We calculate Eq. (43) by considering the volume v as illustrated in Fig. 1, formed by the upper (or lower) cylindrical part of the boundary layer. This choice is mandatory to obtain information from the continuity equation other than that from already known equations. We write

$$\int_{S_N}\rho\mathbf{u}_N\cdot d\mathbf{S}+\int_{S_B}\rho\mathbf{u}_B\cdot d\mathbf{S}=-\int_v\frac{\partial\rho}{\partial t}dv \ ,$$

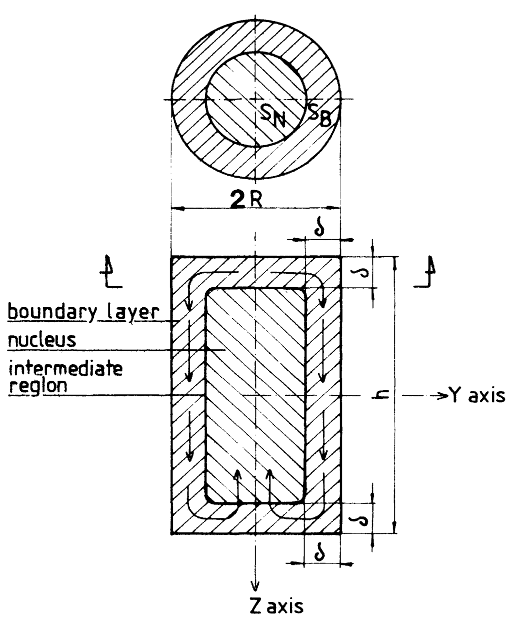

FIG. 1. Model of free convection in a vertical cylinder ($T_s<T$).

where the surfaces of integration S_N and S_B are chosen as described in Fig. 1 and $\mathbf{u}_N$ and $\mathbf{u}_B$ are the velocity vectors at S_N and S_B, respectively; v is the volume of the upper or lower cylindrical part of the boundary layer. Since the density ρ in v is almost equal to $\rho(T_s)$, we have

$$\frac{\partial\rho(T_s)}{\partial t}\cong 0, \quad \int_v\frac{\partial\rho}{\partial t}dv\cong 0 \ .$$

In scalar form and recalling that $u_N=V$, we obtain

$$\int_{S_N}-\rho u_N dS+\int_{S_B}\rho u_B dS=0 \ ,$$
$$-V\rho(\delta t)(R-\delta)^2\pi+\rho(T_s)\int_{S_B}u_B dS=0 \ , \tag{44}$$

The integral in Eq. (43) may be calculated taking from the literature[3] the value of u_B as

$$u_B=u_0\lambda_0 \ ,$$

where λ_0 is a function of y/δ and

$$u_0=K_1 w^{-1}\nu N_{\mathrm{Gr}}^{1/2}(1+0.49N_{\mathrm{Pr}}^{2/3})^{-1/2} \ ,$$

where K_1 is a proportionality constant,

$$w=h-z \ ,$$

N_{Gr} is the Grashof number,

$$N_{\mathrm{Gr}}=\beta G^*(h/2)gh^3/\tilde{\nu}^2 \ ,$$

and N_{Pr} is the Prandtl number,

$$N_{\mathrm{Pr}}=\frac{\tilde{\nu}}{\chi} \ .$$

This formulation of u_N is deduced for a laminar flow and by approximating the problem to a plane surface at constant T_s. However, experiments indicate that the correct combination of N_{Gr} and N_{Pr} in free turbulent convection is $N_{\mathrm{Gr}}^{1/3}N_{\mathrm{Pr}}^{-2/3}$ (Ref. 3), i.e.,

$$\int_{S_B}u_B dS=2\pi RK^*\mu N_{\mathrm{Gr}}^{1/3}N_{\mathrm{Pr}}^{-2/3}\left[\frac{T(\delta,t)-T_s}{G^*(h/2)}\right]^{1/3}$$

The velocity V is given by

$$V(t)=\frac{2RK^*\mu\rho(T_s)N_{\mathrm{Gr}}^{1/3}N_{\mathrm{Pr}}^{-2/3}}{(R-\delta)^2\rho(\delta,t)[G^*(h/2)]^{1/3}}[T(\delta,t)-T_s]^{1/3} \ , \tag{45}$$

where R is the radius of the cylinder.

We observe that in Eq. (45) the ratio $\rho(T_s)/\rho(\delta,t)\cong 1$ to a very good approximation and, recalling that $V=\dot{x}h/\pi$, we have

$$\dot{x}=a[T(\delta,t)-T_s]^{1/3} , \tag{46}$$

where

$$a=\frac{2RK^*\mu\pi N_{\rm Gr}^{1/3}N_{\rm Pr}^{-2/3}}{h(R-\delta)^2[G^*(h/2)]^{1/3}},\quad x(t_0)=0 . \tag{47}$$

Recalling the definition of $U_k^{(1)}$ and $U_k^{(2)}$ [Eqs. (21) and (23)], we have

$$\begin{aligned} U_k^{(1)}&=\frac{h}{\pi}a_{(1)}[T(\delta,t)-T_s]^{1/3}V(t) ,\\ U_k^{(2)}&=\frac{h}{\pi}a_{(2)}[T(h-\delta,t)-T_s]^{1/3}-V(t) , \end{aligned} \tag{48}$$

where

$$a_{(1,2)}=\frac{2\pi RK^*\mu N_{\rm Gr}^{1/3}N_{\rm Pr}^{-2/3}}{h(R-\delta_{1,2})^2[G^*(h/2)]^{1/3}} .$$

Since $\lim_{k\to 0}U_k^{(1)}=\lim_{k\to 0}U_k^{(2)}=0$, we have as a consequence that $\lim_{k\to 0}\epsilon_k=0$ and $\lim_{k\to 0}V_z(z,t)=\dot{V}(t)$. It is also obvious that all the conditions (4) are verified.

The entire set of equations is now solved and can be applied to various practical and theoretical situations by changing the boundary conditions and by varying the choice of G^* and ρ^* at will.

A feature of Eq. (36) is that the small coefficient k is at the denominator so that, in the region of temperatures including the density maximum, sensible deviations from smoothness are expected. Outside of this region k becomes smaller and smaller but, as we have previously shown, $T(z,t)$ tends to $\tilde{T}$ as k goes to 0 [Eq. (40)].

IV. SAMPLE CALCULATION: ASYMPTOTIC SOLUTION IN A FLUID WITH A DENSITY MAXIMUM

The ideal asymptotic solution of a convective nonsteady system ought to be found by introducing a boundary function with infinite temperature derivative, i.e., a square temperature step. However, to give significance to the density maximum of the liquid, we are compelled to introduce a step function with a large but finite slope as follows:

$$G^*(z)=\begin{cases} 0 \ \text{ for } -h\le z<2\delta/3 \\ \dfrac{3T_0}{4\delta}(z-2\delta/3) \ \text{ for } 2\delta/3\le z<2\delta \\ T_0 \ \text{ for } 2\delta\le z\le h-2\delta \\ \dfrac{3T_0}{4\delta}[-z+(h-2\delta/3)] \ \text{ for } h-2\delta<z\le h-2\delta/3 \\ 0 \ \text{ for } h-2\delta/3<z\le h \ , \end{cases} \tag{49}$$

where T_0 is the initial temperature, and the conditions defining the density maximum of the fluid are

$$\rho^*(z)=\begin{cases} \rho(0) \ \text{ for } -h\le z<2\delta/3 \\ \rho(0)[1+\beta_k f_1(z)-kf_1^2(z)] \ \text{ for } 2\delta/3\le z<2\delta \\ \rho(0)(1+\beta_k T_0-kT_0^2) \ \text{ for } 2\delta\le z\le h-2\delta \\ \rho(0)[1+\beta_k f_2(z)-kf_2^2(z)] \ \text{ for } h-2\delta<z\le h-2\delta/3 \\ \rho(0) \ \text{ for } h-2\delta/3<z\le h \ , \end{cases}$$

where

$$f_1(z)=\frac{3T_0}{4\delta}(z-2\delta/3) ,$$

$$f_2(z)=\frac{3T_0}{4\delta}[-z+(h-2\delta/3)] ,$$

and β_k and k are fixed parameters.

From Eq. (13), we have

$$x(t)=\begin{cases}\{[a(t-t_0)/\lambda]+\theta^{2/3}\}^{3/2}-\theta & \text{for } t_0<t<t_1\\ T_0^{1/3}a(t-t_1)+\theta/3 & \text{for } t_1<t<t_2\\ \frac{\alpha}{\lambda}(t-t_2)+\left[2\pi\left[1-\frac{7\theta}{\pi}\right]^{2/3}\right]^{3/2}-\pi\left[1-\frac{5\theta}{\pi}\right] & \text{for } t_2<t<t_3\\ \pi-5\theta & \text{for } t>t_3\ ,\end{cases}$$

where

$$a=\frac{2\pi RK^*\mu N_{\rm Gr}^{1/3}N_{\rm Pr}^{-2/3}}{h(R-\delta)^2T_0^{1/3}}\ ,$$

$$\theta=\frac{\pi\delta}{3h}\ ,$$

$$t_1=t_0+\frac{\lambda}{a}\left[\frac{\theta}{3}\right]^{2/3}n_0\ ,$$

$$t_2=t_1+\frac{\pi-12\theta}{aT_0^{1/3}}\ ,$$

$$t_3=t_2+\frac{\lambda}{a}(2\pi)^{2/3}\left[\left[1-\frac{5\theta}{\pi}\right]^{2/3}-\left[1-\frac{7\theta}{\pi}\right]^{2/3}\right]\ ,$$

$$\lambda=\frac{3}{2}\left[\frac{4\theta}{T_0}\right]^{3/2},\quad n_0=\frac{16^{1/3}-1}{9^{1/3}}=0.7307\ .$$

As a consequence, the solutions for $\rho(z,t)$ and $T(z,t)$ are

$$\rho(z,t)=\rho(0)+\sum_{n=-\infty}^{+\infty}\alpha_n\exp(in\pi z/h)\exp[inx(t)]\ , \tag{50}$$

$$T(z,t)=\frac{\beta_k\pm[\beta_k^2-4k[\rho(z,t)-\rho(0)]/\rho(0)]^{1/2}}{2k}\ ,$$

where the coefficients are the results of straightforward calculations, using Eq. (34).

It is also possible to calculate the functions $V_z(z,t)$ and $p(z,t)$ using Eqs. (30) and (1).

For brevity, without reporting numerical calculations, we notice that Eq. (50), when applied to water in a temperature range including 4 °C, gives rise to a time-dependent behavior of T with a functional form very different from the initial condition (49), formed by linear steps. The perturbed temperature distribution introduces in the $T(h/2,t)$ curve an inflection point due exclusively to the density maximum, as observed experimentally.[1,2]

V. CONCLUSIONS AND PERSPECTIVES

This work has several potential applications since it gives a solution to the problem of free convection in pure water in the nonlinear region and in nonsteady conditions in temperature ranges including the density maximum. To reach this goal within the limits of our model of convection, we rejected the simplifying hypothesis of a Boussinesq fluid to obtain equations including the small temperature coefficients of density. As a consequence, the equations obtained with our model of convection can be compared with Rahm's theory[5] only outside the density maximum region, but this comparison shall be carried out in a planned future paper along with a thorough discussion of the differences and similarities of the two models. This treatment may now be applied to the case of mixtures of water and heavy water subjected to convection under corresponding experimental conditions, in order to test the hypothesis of ideal mixing. Moreover, we expect from our approach a theoretical explanation of observed structural asymmetries observed in quasisteady water layers subjected to slow heating and cooling cycles in the vicinity of 4 °C.[9]

APPENDIX

The fundamental equations of hydrodynamics are as follows. The Fourier equation is

$$\rho c\frac{dT}{dt}=\nabla\cdot(\kappa\nabla T)-p\nabla\cdot\mathbf{U}+\mu\left\{2\left[\left[\frac{\partial U_x}{\partial x}\right]^2+\left[\frac{\partial U_y}{\partial y}\right]^2+\left[\frac{\partial U_z}{\partial z}\right]^2\right]+\left[\frac{\partial U_x}{\partial y}+\frac{\partial U_y}{\partial x}\right]^2+\left[\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right]^2+\left[\frac{\partial U_y}{\partial z}+\frac{\partial U_z}{\partial y}\right]^2-\tfrac{2}{3}(\nabla\cdot\mathbf{U})^2\right\}+\xi(\nabla\cdot\mathbf{U})^2\ ,$$

where $\nabla\cdot$ is the divergence operator, ∇ is the gradient operator, c is the constant volume heat capacity, κ is the thermal conductivity, p is the hydrostatic pressure, $\mathbf{U}$ is the velocity in the nucleus, μ is the coefficient of viscosity, ξ is the second viscosity, and ρ is the density of the liquid.

The Navier-Stokes equation is

$$\frac{d\mathbf{U}}{dt}=\mathbf{G}-(1/\rho)\nabla p+\tilde{\nu}\Delta^2\mathbf{U}+(\xi/\rho+\tilde{\nu}/3)\nabla\,\nabla\cdot\mathbf{U}\ ,$$

where $\mathbf{G}$ is the external force per unit mass and ν is a kinematic viscosity, $\tilde{\nu}=\mu/\rho$.

The continuity equation is

$$\nabla\cdot(\rho\mathbf{U})+\frac{\partial\rho}{\partial t}=0\ .$$

By choosing the z axis downwards and putting $T=T(z,t)$ and $\mathbf{U}=(0,0,-V_z)$, these equations become Eqs. (1)–(3). The Navier-Stokes equation gives, for the components x and y,

$$\frac{\partial P}{\partial x}=0,\quad \frac{\partial p}{\partial y}=0\ ,$$

i.e., $p=p(z,t)$.

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