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Mapping of classical canonical transformations to quantum unitary operators

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Classical linear transformations in coordinate-momentum phase space are mapped to unitary quantum-mechanical operators in Hilbert space to produce a generalization of squeeze operators. The classical to quantum transition is manifestly apparent in the derivation. The unitary operators are evaluated in the coherent-state representation using the "integration within ordered products" technique in the one- and two-mode cases. Application of these operators results in a new generalization of coherent states.

I. INTRODUCTION

In recent years, considerable interest has been expressed in squeezed states of the electromagnetic field.[1,2] Applications of squeezed light have been proposed in low-noise optical communications[3] and high-precision interferometry.[4] Single-mode squeezed states as generalizations of coherent states have also been discussed from a group-theoretical viewpoint by a number of authors.[5]

Squeezed states were introduced independently by Stoler[6] and Lu[7] and have been extensively discussed as eigenstates of the operator

$$a'=\mu a+\nu a^\dagger , \tag{1.1}$$

where a and $a^\dagger$ are, respectively, photon annihilation and creation operators and μ and ν are arbitrary complex numbers satisfying $|\mu|^2-|\nu|^2=1$, by Yuen,[8] who showed that such states could be produced by a two-photon lasing process or by parametric amplification. Squeezing has been demonstrated experimentally by several groups.[9]

The current literature deals almost exclusively with the restricted squeezed state that is an eigenstate of a' with $\mu=\cosh\lambda$ and $\nu=e^{i\theta}\sinh\lambda$ in (1.1) for which the unitary squeeze operator

$$\exp\{\tfrac{1}{2}[\xi^* a^2-\xi(a^\dagger)^2]\} ,$$

with $\xi=\lambda e^{i\theta}$, is well known. The squeeze operator corresponding to (1.1) has not previously been obtained.

In view of the well-known correspondence[10] between the unit-mass harmonic oscillator and the modes of the electromagnetic field we identify the operators $\hat{Q}=2^{-1/2}(a+a^\dagger)$ and $\hat{P}=-2^{-1/2}i(a-a^\dagger)$ with the position and momentum operators of the harmonic oscillator (we take $\hbar=\omega=1$).

The one-mode squeezing process with $\theta=0$ for the harmonic oscillator can be explicitly written as a compression of a phase-space coordinate accompanied by a corresponding dilation of the conjugate momentum,[11] $q'=\mu^{-1}q$ and $p'=\mu p$. In this light, squeezing is seen to be only a special case of the more general linear transformation in phase space which can be written $q'=Aq+Bp$, $p'=Cq+Dp$. Alternatively, we express the transformation in matrix form as

$$\begin{gathered}\begin{bmatrix} q' \\ p' \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}\equiv g\begin{bmatrix} q \\ p \end{bmatrix} , \\ \det g=1 , \\ g^{-1}=\begin{bmatrix} D & -B \\ -C & A \end{bmatrix} ,\end{gathered} \tag{1.2}$$

with $AD-BC=1$ and A, B, C, and D real.

Since (1.1) could equally be written in terms of $\hat{Q}'=A\hat{Q}+B\hat{P}$ and $\hat{P}'=C\hat{Q}+D\hat{P}$, constructing the unitary image of (1.2) that accomplishes this transformation will also provide us with the squeeze operator to produce the eigenstates of (1.1).

Several methods have been used to obtain the quantum-mechanical unitary operators corresponding to the phase-space scaling of the squeeze operator.[12]

In this paper we derive, in transparent fashion, the unitary-operator image of the classical transformation (1.2). In other words, letting $\hat{Q},\hat{P}$ and $\hat{Q}',\hat{P}'$ be the position and momentum operators we will construct the unitary operator $U^{(1)}(g)$ such that

$$\begin{aligned}\hat{Q}'&=[U^{(1)}(g)]^\dagger\hat{Q}U^{(1)}(g)=A\hat{Q}+B\hat{P} , \\ \hat{P}'&=[U^{(1)}(g)]^\dagger\hat{P}U^{(1)}(g)=C\hat{Q}+D\hat{P} .\end{aligned} \tag{1.3}$$

The method employed will make a clear connection between the classical transformation and the corresponding unitary operator in Hilbert space.

To illustrate the approach, we briefly review the development of the squeeze operator from the classical scaling transformation.[11] To begin, the squeeze operator is written in the physically appealing canonical coherent-state representation

$$U(g_s)=\frac{1}{2\pi}\left[\frac{\mu+\mu^{-1}}{2}\right]^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dq\,dp\left|\begin{bmatrix} 1/\mu & 0 \\ 0 & \mu \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle\left\langle\begin{bmatrix} q \\ p \end{bmatrix}\right| ,\quad \mu>0 \tag{1.4}$$

where

$$\left|\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle \equiv |p,q\rangle = \exp\left[-\tfrac{1}{4}(p^2+q^2)+\frac{1}{\sqrt{2}}(q+ip)a^\dagger\right]|0\rangle \tag{1.5}$$

is the canonical coherent state,[13] $|0\rangle$ is the harmonic oscillator's ground state, and

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{Q}-i\hat{P})$$

its creation operator. Setting

$$z = \frac{1}{\sqrt{2}}(q+ip) ,$$

the coherent state can also be written in terms of the complex arguments z and z^*, or simply in terms of z,

$$\left|\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle \equiv \left|\begin{bmatrix} z \\ z^* \end{bmatrix}\right\rangle \equiv |z\rangle = e^{-|z|^2/2} e^{za^\dagger}|0\rangle . \tag{1.6}$$

$U(g_s)$ as given by (1.4) may be integrated by means of the integration within ordered products (IWOP) technique[14,15] to obtain the normally ordered form

$$U(g_s) = \exp\left[-\frac{(a^\dagger)^2}{2}\tanh\tau\right] \times \exp[(-a^\dagger a + \tfrac{1}{2})\ln(\cosh\tau)] \times \exp\left[\frac{a^2}{2}\tanh\tau\right] , \tag{1.7}$$

where $\mu = e^\tau$. Equation (1.7) can be shown[16] to reduce to the more customary form

$$U^{(1)}(g) = \exp\{\tfrac{1}{2}\tau[a^2-(a^\dagger)^2]\} .$$

We will use similar methods to evaluate the unitary operator generated by the more general linear transformation (1.2).

In Sec. II, starting from the canonical coherent-state representation of $U^{(1)}(g)$ which provides the connection to the phase-space transformation, we integrate the expression by using the IWOP technique to obtain the normally ordered unitary operator. We thus obtain explicitly the quantum-mechanical unitary image of the classical phase-space transformation. We will show that for the particular transformation having $A=\mu^{-1}$, $B=C=0$, $D=\mu$ we recover (1.4) and (1.7).

In Sec. III, guided by the derivation of the two-mode squeeze operator,[11] we generalize the results of Sec. II to derive the quantum unitary operator $U^{(2)}(G)$ for the symplectic transformation G in two-mode phase space.

In Sec. IV we show that the normally ordered form of $U^{(1)}(g)$ can be directly used to construct eigenstates of (1.1), while the application of $U^{(2)}(G)$ to the coherent state produces a new three-parameter generalization of two-mode squeezed states of which the customary two-parameter squeezed states are a special case.

II. DERIVATION OF $U^{(1)}(g)$

In view of (1.4) we begin by postulating the following canonical coherent-state representation of the required operator:

$$U^{(1)}(g) = \frac{1}{2\pi} s^{-1/2}|s| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\, dp \left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle\left\langle\begin{bmatrix} q \\ p \end{bmatrix}\right| , \tag{2.1}$$

where

$$s = \tfrac{1}{2}[(A+D)+i(B-C)] . \tag{2.2}$$

The factor $s^{-1/2}|s|$ anticipates the normalization required to make U unitary as will be shown later. As a crucial first step, without which both the integration of (2.1) and the later generalization for two modes are much more difficult, we change the arguments p and q of the coherent state to the complex arguments z and z^* as in (1.6). Writing

$$\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle = \left|\begin{bmatrix} q' \\ p' \end{bmatrix}\right\rangle \equiv \left|\begin{bmatrix} z' \\ z'^* \end{bmatrix}\right\rangle \equiv |z'\rangle$$

we find

$$z' \equiv \frac{q'+ip'}{\sqrt{2}} = \tfrac{1}{2}[(A+D)+i(C-B)]\frac{(q+ip)}{\sqrt{2}} + \tfrac{1}{2}[(A-D)+i(B+C)]\frac{(q-ip)}{\sqrt{2}} = s^*z - rz^* ,$$

where

$$r = \tfrac{1}{2}[(D-A)-i(B+C)] \tag{2.3}$$

and the condition $AD-BC=1$ becomes $|s|^2-|r|^2=1$. The ket in (2.1) may then be rewritten

$$\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}\right\rangle = \left|\begin{bmatrix} s^* & -r \\ -r^* & s \end{bmatrix}\begin{bmatrix} z \\ z^* \end{bmatrix}\right\rangle \equiv |s^*z - rz^*\rangle \tag{2.4}$$

and (2.1) becomes

$$U^{(1)}(g)=s^{-1/2}|s|\int d^2z\frac{1}{\pi}|s^*z-rz^*\rangle\langle z| \quad \left[\int d^2z\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d[\mathrm{Re}(z)]d[\mathrm{Im}(z)]\right]$$

$$=s^{-1/2}|s|\int_{-\infty} d^2z\frac{1}{\pi}\exp\left[-\frac{|s^*z-rz^*|^2}{2}+(s^*z-rz^*)a^\dagger\right]|0\rangle\langle z| \ . \tag{2.5}$$

With the help of the operator identity $|0\rangle\langle 0|=$:exp$(-a^\dagger a)$: we perform the Gaussian integration (2.5) by means of the IWOP technique,

$$U^{(1)}(g)=s^{-1/2}|s|\int d^2z\frac{1}{\pi}:\exp\left[-|s|^2|z|^2+s^*za^\dagger+z^*(a-ra^\dagger)+\frac{r^*s^*}{2}z^2+\frac{rs(z^*)^2}{2}-a^\dagger a\right]: \tag{2.6}$$

$$=s^{-1/2}\exp\left[-\frac{r}{2s}(a^\dagger)^2\right]:\exp\left[\left[\frac{1}{s}-1\right]a^\dagger a\right]:\exp\left[\frac{r^*}{2s}a^2\right] . \tag{2.7}$$

Expression (2.7) can be further simplified by using the operator identity $e^{\lambda a^\dagger a}=$:exp$[(e^\lambda-1)a^\dagger a]$: to give finally

$$U^{(1)}(g)=s^{-1/2}|s|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\frac{1}{2\pi}\left|\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix}\right\rangle\left\langle\begin{matrix}p\\ q\end{matrix}\right|$$

$$=s^{-1/2}\exp\left[-\frac{r}{2s}(a^\dagger)^2\right]\exp\left[a^\dagger a\ln\left[\frac{1}{s}\right]\right]\exp\left[\frac{r^*}{2s}a^2\right] . \tag{2.8}$$

It remains to show that $U^{(1)}(g)$ is unitary. We let $U^{(1)}(g)$ act on coherent state $|\alpha\rangle$ to obtain

$$U^{(1)}(g)|\alpha\rangle=s^{-1/2}\exp\left[-\frac{r}{2s}(a^\dagger)^2\right]\exp\left[a^\dagger a\ln\left[\frac{1}{s}\right]\right]\exp\left[\frac{r^*}{2s}\alpha^2\right]|\alpha\rangle$$

$$=s^{-1/2}\exp\left[\frac{r^*}{2s}\alpha^2-\frac{|\alpha|^2}{2}\right]\exp\left[-\frac{r}{2s}a^{\dagger 2}\right]\exp\left[\frac{\alpha}{s}a^\dagger\right]|0\rangle \ . \tag{2.9}$$

Exploiting the overcompleteness relation $\int d^2z(1/\pi)|z\rangle\langle z|=1$ for the coherent state $|z\rangle$ and the identity $|z\rangle\langle z|\equiv$:exp$(-|z|^2+za^\dagger+z^*a-a^\dagger a)$: we calculate

$$\langle\alpha|[U^{(1)}(g)]^\dagger U^{(1)}(g)|\alpha\rangle=\frac{1}{|s|}\exp\left[\frac{r^*\alpha^2}{2s}+\frac{r(\alpha^*)^2}{2s^*}-|\alpha|^2\right]$$

$$\times\left\langle 0\left|\exp\left[-\frac{r^*}{2s^*}a^2+\frac{\alpha^*}{s^*}a\right]\int d^2z\frac{1}{\pi}\right|z\right\rangle\left\langle z\left|\exp\left[-\frac{r^2}{2s}(a^\dagger)^2+\frac{\alpha}{s}a^\dagger\right]\right|0\right\rangle$$

$$=\frac{1}{|s|}\exp\left[\frac{r^*\alpha^2}{2s}+\frac{r(\alpha^*)^2}{2s^*}-|\alpha|^2\right]$$

$$\times\left\langle 0\left|\int d^2z\frac{1}{\pi}:\exp\left[-|z|^2+z\left[a^\dagger+\frac{\alpha^*}{s^*}\right]+z^*\left[a+\frac{\alpha}{s}\right]^*-\frac{r}{2s^*}z^2-\frac{r}{2s}(z^*)^2-a^\dagger a\right]:\right|0\right\rangle$$

$$=\frac{1}{|s|}\exp\left[\frac{r^*\alpha^2}{2s}+\frac{r(\alpha^*)^2}{2s^*}-|\alpha|^2\right]|s|\exp\left[|\alpha|^2-\frac{r^*\alpha^2}{2s}-\frac{r(\alpha^*)^2}{2s^*}\right]=1 \ . \tag{2.10}$$

As the coherent states are overcomplete and nonorthogonal we conclude that

$$[U^{(1)}(g)]^\dagger U^{(1)}(g)=1 \ .$$

A similar procedure would show that $UU^\dagger=1$ proving U unitary.

The squeeze operator (1.4) or (1.7) is now easily obtained. For

$$A=\mu^{-1},\quad B=C=0,\quad D=\mu,\quad s=\frac{\mu+\mu^{-1}}{2},\quad r=\frac{\mu-\mu^{-1}}{2},\quad \frac{r^*}{s}=\frac{\mu-\mu^{-1}}{\mu+\mu^{-1}}=\tanh\tau \ .$$

Substituting these in (2.8) recovers (1.4) and (1.7).

The canonical coherent-state representation of $[U^{(1)}(g)]^\dagger$ is obtained by taking the Hermitian conjugate of (2.1), i.e.,

$$[U^{(1)}(g)]^\dagger=\frac{1}{2\pi}(s^*)^{-1/2}|s|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\left|\begin{pmatrix}q\\p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}\right|$$
$$=\frac{1}{2\pi}(s^*)^{-1/2}|s|\int_{-\infty}^{\infty}\int_{-\infty}^{-\infty}dp\,dq\left|\begin{pmatrix}D&-B\\-C&A\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\p\end{pmatrix}\right|\quad[=U^{(1)}(g^{-1})]$$
$$=(s^*)^{-1/2}|s|\int d^2z\frac{1}{\pi}\left|\begin{pmatrix}s&r\\r^*&s^*\end{pmatrix}\begin{pmatrix}z\\z^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z\\z^*\end{pmatrix}\right|$$
$$=(s^*)^{-1/2}\exp\left[\frac{r}{2s^*}a^{\dagger 2}\right]\exp\left[a^\dagger a\ln\left[\frac{1}{s^*}\right]\right]\exp\left[-\frac{r^*}{2s^*}a^2\right]. \tag{2.11}$$

Finally, transforming the annihilation and creation operators under $U^{(1)}$ with the aid of (2.8) and (2.11) we obtain an operator of form (1.1),

$$a'\equiv[U^{(1)}(g)]^\dagger aU^{(1)}(g)=s^*a-ra^\dagger\,,$$
$$a'^\dagger\equiv[U^{(1)}(g)]^\dagger a^\dagger U^{(1)}(g)=sa^\dagger-r^*a\,, \tag{2.12}$$

which satisfy the commutation relation $[a',a'^\dagger]=1$. Equation (2.12) may also be written as $\hat{Q}'=A\hat{Q}+B\hat{P}$ and $\hat{P}'=C\hat{Q}+D\hat{P}$.

III. GENERALIZATION TO THE TWO-MODE CASE

As we seek to generalize transformations such as the scaling transformation invoked by squeezing, we consider for guidance the two-mode squeeze operator $U^{(2)}(G_s)$.[11] In the canonical coherent-state representation

$$U^{(2)}(G_s)=\frac{\cosh\lambda}{(2\pi)^2}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_2|p_1'q_1';p_2'q_2'\rangle\langle p_1q_1;p_2q_2|\,, \tag{3.1}$$

where

$$|p_1'q_1';p_2'q_2'\rangle=\left|\begin{pmatrix}q_1'\\q_2'\\p_1'\\p_2'\end{pmatrix}\right\rangle=\left|\begin{pmatrix}\cosh\lambda&\sinh\lambda&0&0\\\sinh\lambda&\cosh\lambda&0&0\\0&0&\cosh\lambda&-\sinh\lambda\\0&0&-\sinh\lambda&\cosh\lambda\end{pmatrix}\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\rangle\equiv\left|G_s\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\rangle,\quad\lambda\text{ real}\,. \tag{3.2}$$

As in Sec. II, we express the two-mode coherent state in terms of $z_1=(q_1+ip_1)/\sqrt{2}$ and $z_2=(q_2+ip_2)/\sqrt{2}$, so that

$$\left|\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\rangle\equiv\left|\begin{pmatrix}z_1\\z_1^*\\z_2\\z_2^*\end{pmatrix}\right\rangle\equiv|z_1;z_2\rangle\,.$$

In the spirit of (2.4) we generalize (3.1) and consider the following operator:

$$U^{(2)}(G)=s\int\int d^2z_1d^2z_2\frac{1}{\pi^2}\left|W\begin{pmatrix}z_1\\z_1^*\\z_2\\z_2^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z_1\\z_1^*\\z_2\\z_2^*\end{pmatrix}\right|\,, \tag{3.3}$$

with

$$W=\begin{pmatrix}s&0&0&r\\0&s^*&r^*&0\\0&r&s&0\\r^*&0&0&s^*\end{pmatrix}\,,$$

where s and r are defined by (2.2) and (2.4) and $|s|^2-|r|^2=1$ as before. The corresponding matrix G operating in two-mode phase space will be given in (3.12) and includes the two-mode squeeze operator as will be shown. With this choice for W we recast (3.3) as

$$U^{(2)}(G)=s\int\int d^2z_1d^2z_2\frac{1}{\pi^2}|sz_1+rz_2^*;rz_1^*+sz_2\rangle\langle z_1;z_2| \ . \tag{3.4}$$

Using the identity $|00\rangle\langle 00|=:e^{-a^\dagger a-b^\dagger b}:$, where $b^\dagger$ (b) is the creation (annihilation) operator for mode 2, and the IWOP technique, we integrate (3.4) to obtain

$$\begin{aligned}U^{(2)}(G)&=s\int\int d^2z_1d^2z_2\frac{1}{\pi^2}:\exp[-|s|^2(|z_1|^2+|z_2|^2)-r^*sz_1z_2-rs^*z_1^*z_2^*+(sz_1+rz_2^*)a^\dagger\\&\qquad+(rz_1^*+sz_2)b^\dagger+z_1^*a+z_2^*b-a^\dagger a-b^\dagger b]:\\&=\frac{1}{s^*}:\exp\left[\frac{1}{s^*}(ra^\dagger b^\dagger-r^*ab)+\left[\frac{1}{s^*}-1\right](a^\dagger a+b^\dagger b)\right]:\\&=\exp\left[\frac{r}{s^*}a^\dagger b^\dagger\right]\exp\left[(a^\dagger a+b^\dagger b+1)\ln\left[\frac{1}{s^*}\right]\right]\exp\left[-\frac{r^*}{s^*}ab\right] \ .\end{aligned}\tag{3.5}$$

The annihilation operators a'' and b'' transform under $U^{(2)}$ to give

$$a''=U^{(2)}a(U^{(2)})^\dagger=s^*a-rb^\dagger,\quad b''=U^{(2)}b(U^{(2)})^\dagger=s^*b-ra^\dagger \ , \tag{3.6}$$

and satisfy the commutation relations

$$[a'',a''^\dagger]=1,\quad [b'',b''^\dagger]=1 \ .$$

To prove $U^{(2)}$ unitary we write its inverse as

$$\begin{aligned}[U^{(2)}(G)]^{-1}&=\exp\left[\frac{r^*}{s^*}ab\right]\exp\left[-(a^\dagger a+b^\dagger b+1)\ln\left[\frac{1}{s^*}\right]\right]\exp\left[-\frac{r}{s^*}a^\dagger b^\dagger\right]\\&=\exp\left[-(a^\dagger a+b^\dagger b+1)\ln\left[\frac{1}{s^*}\right]\right]\exp(r^*s^*ab)\exp\left[-\frac{r}{s^*}a^\dagger b^\dagger\right]\end{aligned}\tag{3.7}$$

and then employ the overcompleteness relation of the two-mode coherent state

$$\int\int d^2z_1d^2z_2\frac{1}{\pi^2}|z_1;z_2\rangle\langle z_1;z_2|=1 \tag{3.8}$$

to write

$$(U^{(2)})^{-1}=e^{(a^\dagger a+b^\dagger b+1)\ln s^*}e^{r^*s^*ab}\int\int d^2z_1d^2z_2\frac{1}{\pi}|z_1;z_2\rangle\langle z_1;z_2|\exp\left[-\frac{r}{s^*}a^\dagger b^\dagger\right] \ . \tag{3.9}$$

We integrate (3.9) with the IWOP technique to obtain

$$\begin{aligned}[U^{(2)}(G)]^{-1}&=s^*\int\int d^2z_1d^2z_2\frac{1}{\pi^2}:\exp\left[-|z_1|^2-|z_2|^2+z_1a^\dagger s^*+z_2b^*s^*+z_1^*a+z_2^*b+r^*s^*z_1z_2\right.\\&\qquad\left.-\frac{r}{s^*}z_1^*z_2^*-a^\dagger a-b^\dagger b\right]:\\&=\exp\left[-\frac{r}{s}a^\dagger b^\dagger\right]\exp\left[(a^\dagger a+b^\dagger b+1)\ln\left[\frac{1}{s}\right]\right]\exp\left[\frac{r^*}{s}ab\right]=[U^{(2)}(G)]^\dagger \ ,\end{aligned}\tag{3.10}$$

which completes the proof that $U^{(2)}$ is unitary. We can find the transformation G in phase space giving rise to $U^{(2)}(G)$ by writing

$$U^{(2)}(G)=s\int\int d^2z_1 d^2z_2 \frac{1}{\pi^2}|sz_1+rz_2^*;rz_1^*+sz_2\rangle\langle z_1;z_2|$$
$$=\frac{s}{(2\pi)^2}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_2 \left| G\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right| \tag{3.11}$$

and reversing the procedure leading to (2.4) to obtain

$$G=\frac{1}{2}\begin{bmatrix} A+D & D-A & C-B & -B-C\\ D-A & A+D & -B-C & C-B\\ B-C & -B-C & A+D & A-D\\ -B-C & B-C & A-D & A+D\end{bmatrix} . \tag{3.12}$$

It is easily verified that $\det G=1$ and that G is a symplectic matrix, implying that the transformation is canonical.[17]

In particular, when $A=e^{-\lambda}$, $B=C=0$, $D=e^{\lambda}$, then $s=\cosh\lambda=s^*$, $r=\sinh\lambda=r^*$, and (3.11) reduces to (3.1). We have thus shown that symplectic transformations of form (3.12) in two-mode phase space have a unitary-operator image in Hilbert space that generalizes the two-mode squeeze operator.

IV. GENERALIZED COHERENT STATES GENERATED BY $U^{(1)}(g)$ AND $U^{(2)}(G)$

The normally ordered unitary operator $U^{(1)}(g)$ and $U^{(2)}(G)$ provide us with a convenient way to generate a generalization (still within the framework of two-photon squeezing) of squeezed coherent states. Let us first consider the single-mode result.

Under the transformation $U^{\dagger}(g)$ of (2.11), the vacuum state $|0\rangle$ becomes

$$[U^{(1)}(g)]^{\dagger}|0\rangle=(s^*)^{-1/2}\exp\left[\frac{r}{2s^*}(a^{\dagger})^2\right]|0\rangle . \tag{4.1}$$

We now obtain a generalized coherent state by displacing the transformed vacuum state with the operator $D(\alpha)=\exp(\alpha a^{\dagger}-\alpha^* a)$,

$$|\alpha\rangle_g\equiv D(\alpha)[U^{(1)}(g)]^{\dagger}|0\rangle=(s^*)^{-1/2}\exp\left[-\frac{|\alpha|^2}{2}+\left[\alpha-\frac{r}{s^*}\alpha^*\right]a^{\dagger}-\frac{r}{2s^*}[(a^{\dagger})^2+(\alpha^*)^2]\right]|0\rangle . \tag{4.2}$$

The states so generated are overcomplete as can be easily seen by using the IWOP technique,

$$\int d^2\alpha\frac{1}{\pi}|\alpha\rangle_{gg}\langle\alpha|=|s|^{-1}\int d^2\alpha\frac{1}{\pi}{:}\exp\left[-|\alpha|^2+\alpha\left[a^{\dagger}-\frac{r^*}{s}a\right]+\alpha^*\left[a-\frac{r}{s^*}a^{\dagger}\right]+\frac{r}{2s^*}[(\alpha^*)^2+(a^{\dagger})^2] + \frac{r^*}{2s}(\alpha^2+a^2)-a^{\dagger}a\right]{:}=1 . \tag{4.3}$$

To show explicitly that $|\alpha\rangle_g$ is an eigenstate of the operator

$$a'\equiv[U^{(1)}(g)]^{\dagger}aU^{(1)}(g) , \tag{4.4}$$

we apply a to $|\alpha\rangle_g$, giving

$$a|\alpha\rangle_g=\left[\alpha-\frac{r}{s^*}\alpha^*+\frac{r}{s^*}a^{\dagger}\right]|\alpha\rangle_g . \tag{4.5}$$

It follows that

$$(s^*a-ra^{\dagger})|\alpha\rangle_g=a'|\alpha\rangle_g=(\alpha s^*-r\alpha^*)|\alpha\rangle_g . \tag{4.6}$$

For the two-mode case we can generalize in a similar fashion. We produce $|00\rangle_G$ by letting $U^{(2)}(G)$ act on $|00\rangle$, yielding

$$U^{(2)}(G)|00\rangle=(s^*)^{-1}\exp\left[\frac{r}{s^*}a^\dagger b^\dagger\right]|00\rangle \ . \tag{4.7}$$

The corresponding generalized coherent state is produced by displacing each mode, i.e.,

$$\begin{aligned}|\alpha,\beta\rangle_G=D(\alpha)D(\beta)|00\rangle_G&=(s^*)^{-1}\exp(\alpha a^\dagger-\alpha^* a)\exp(\beta b^\dagger-\beta^* b)\exp\left[\frac{r}{s^*}a^\dagger b^\dagger\right]|00\rangle\\&=(s^*)^{-1}\exp\left[\frac{r}{s^*}(a^\dagger-\alpha^*)(b^\dagger-\beta^*)-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha a^\dagger+\beta b^\dagger\right]|00\rangle \ .\end{aligned} \tag{4.8}$$

We are currently investigating the detailed nature of these states and speculate that the mixing of the phase-space coordinates engendered by the squeeze transformation, so clearly apparent in the classical image, will be reflected in the coordinate and momentum representation of these states. The generalized coherent states generated by canonical transformations in phase space may have applications in nonlinear optics. When g or G are time dependent (but still have unit determinant) the states generated should have application in dynamic systems. We will investigate potential applications of these states in the future.

V. CONCLUSIONS

We have obtained unitary operators corresponding to classical phase-space transformations in the one- and two-mode coherent-state representation. The method of derivation is direct, showing clearly the connection between the classical transformation and the corresponding quantum-mechanical unitary operator. The concise evaluation of the operators in coherent-state representation was greatly facilitated by the IWOP technique. The evaluation of the general linear transformation operators has provided a way to generalize the customary two-parameter squeezed state to a more general three-parameter squeezed state. The formalism employed displays clearly the mixing and rescaling of phase-space coordinates inherent in squeezing.

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[1]Bonny L. Schumaker, Phys. Rep. **135**, 317 (1985).

[2]R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[3]H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **IT-24**, 657 (1978); J. H. Shapiro, H. P. Yuen, and J. A. Machoda Mata, *ibid.* **IT-25**, 170 (1979).

[4]C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[5]See, for example, A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971); A. M. Perelomov, *Generalized Coherent States* (Springer, New York, 1986); G. D'Ariano and M. Rasetti, Phys. Rev. D **35**, 1239 (1987).

[6]D. Stoler, Phys. Rev. D **1**, 3217 (1972); **4**, 1925 (1971).

[7]E. Y. C. Lu, Lett. Nuovo Cimento **2**, 1241 (1971); **3**, 585 (1972).

[8]H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[9]See, for example, R. L. Robinson, Science **230**, 927 (1985); **233**, 280 (1986); R. E. Slusher, L. W. Holberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985); R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, D. F. Walls, *ibid.* **57**, 691 (1986).

[10]R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1983).

[11]Fan Hong-yi, H. R. Zaidi, and J. R. Klauder, Phys. Rev. D **35**, 1831 (1987).

[12]See, for example, Ref. 3; R. A. Fisher, M. M. Nieto, and V. D. Sandberg, Phys. Rev. D **29**, 1107 (1984); D. R. Truax, *ibid.* **31**, 1988 (1985).

[13]J. R. Klauder and Bo-Sture Shagerstam, *Coherent States* (World Scientific, Singapore, 1985).

[14]Fan Hong-Yi and Ruan Tu-nan, Sci. Sin., Ser. A **27**, 392 (1984).

[15]Fan Hong-Yi and H. R. Zaidi, Phys. Rev. A **37**, 2985 (1988).

[16]Fan Hong-Yi and J. R. Klauder, J. Phys. A **21**, L725 (1988).

[17]H. Goldstein, *Classical Mechanics,* 2nd ed. (Addison-Wesley, Reading, MA, 1980).