Collapse and revival phenomenon in the evolution of a resonant field in a Kerr-like medium

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A quantum theory of the propagation of a single-mode resonant field through a Kerr-like medium is given. Numerical studies show the existence of the quantum collapses and revivals in the otherwise periodic exchange of energy between the field and the atomic oscillator. Approximate analytical methods are developed to explain the numerical results for the case of weak nonlinearity of the Kerr medium. For large nonlinearity, the correspondence with the Jaynes-Cummings model is discussed.

I. INTRODUCTION

This investigation concerns the study of nonclassical effects[1] in the propagation of a single-mode field through a Kerr-like medium[2–7] contained in a cavity. The nonclassical effects manifest themselves in several different ways. The well-known collapses and revivals[8–12] of Rabi oscillations in a two-level atom interacting with a single mode of the field are an example of the nonclassical effect and it arises due to discrete averaging over the initial photon-number distribution. The detailed nature of the collapses and revivals depends on the initial state of the field[9,10,12] and the nature of the atomic transition.[11] For example, the collapses and revivals in a coherent-state field are compact and regular for a two-photon transition, whereas these overlap somewhat for the case of a single-photon transition. Rempe *et al.*[9] have studied such a nonclassical effect in a Rydberg atom contained in a very-high-Q cavity.

In the present work we model the nonlinear medium as an anharmonic oscillator with quartic anharmonicity in the rotating-wave approximation. This system is quite rich in the kind of nonlinear phenomena it can exhibit and can describe a number of situations.

A Kerr-like medium can be described by an anharmonic oscillator.[2–7,13] Various versions of such models have been studied in the past.[4,13] For example, Kitagawa and Yamamoto[3] studied the behavior of a field b evolving dynamically under the influence of an effective interaction $b^{\dagger 2}b^2$ and demonstrated the generation of states with squeezed number fluctuations. Several others[5–7] emphasized the generation of macroscopic superposition of coherent states through an effective interaction of the form $(b^\dagger b)^k$, k an integer. These works dealt with an effective Hamiltonian involving only the photon operators. Such effective Hamiltonians are applicable only when the response time of the medium is so fast that the medium follows the field in an adiabatic manner. In general, one has to include both the medium and the field in the dynamics. Coupled oscillator models have also been studied in connection with problems such as optical bistability and four-wave mixing.[2,4]

In this paper we deal with a different regime and predict different kinds of nonclassical nonlinear phenomena. In Sec. II we present the details of our model. We discuss the solution under different conditions. In Sec. III and IV we develop approximate methods for obtaining an analytical solution for the system's density matrix at time t. The approximate solution enables one to understand qualitatively and quantitatively the numerical results of Sec. V.

II. QUANTUM-MECHANICAL MODEL FOR THE PROPAGATION OF LIGHT THROUGH A KERR MEDIUM

We consider the propagation of a single-mode field of frequency ω through a nonlinear medium. Let b and $b^\dagger$ be the annihilation and creation operators for the field mode. We model the nonlinear medium by an anharmonic oscillator with frequency ω_0. Let a and $a^\dagger$ be the annihilation and creation operators for the medium. One might imagine the nonlinear medium contained in a very good quality single-mode cavity. The total Hamiltonian for the system can be written as

$$H=\hbar\omega_0 a^\dagger a+\hbar\omega b^\dagger b+\hbar q a^{\dagger 2}a^2+\hbar g(a^\dagger b+b^\dagger a)\ , \quad (2.1)$$

where, in obtaining Eq. (2.1), we have made the rotating-wave approximation. Here q is the anharmonicity parameter and g is the coupling with the field mode. The Hamiltonian (2.1) is exactly solvable in the limiting cases $g=0$, $q\neq 0$; $g\neq 0$, $q=0$. In the *absence* of the anharmonicity there is periodic exchange of energy between the two oscillators. For example, if the oscillator b is in coherent state $|\beta\rangle$ at $t=0$ and the atomic oscillator is in the ground state then the energies of the two oscillators are given by $(\omega=\omega_0)$

$$\begin{aligned}\langle a^\dagger a\rangle &=\sin^2(gt|\beta|^2)\ ,\\ \langle b^\dagger b\rangle &=|\beta|^2-\langle a^\dagger a\rangle\ .\end{aligned} \quad (2.2)$$

Thus the periodic exchange is determined by the coupling parameter g. If $g=0$, then the wave function at time t can be written in the closed form as

$$|\psi(t)\rangle=\sum_{n_a,n_b} C(n_a,n_b)\exp\{-it[\omega_0 n_a+\omega n_b+q(n_a^2-n_a)]\}|n_a,n_b\rangle\ , \tag{2.3}$$

where the $C(n_a,n_b)$ determine the initial state

$$|\Psi(0)\rangle=\sum_{n_a,n_b} C(n_a,n_b)|n_a,n_b\rangle\ . \tag{2.4}$$

If the coupling between the two systems is nonzero, a closed-form solution can still be obtained in the adiabatic limit, i.e., when the two oscillators' frequencies are far from each other. This is the approximation most often used in nonlinear optics, for this approximation enables one to introduce the third-order nonlinear susceptibility and the effective Hamiltonian in terms of the field variables alone

$$H_{\rm eff}=\hbar\bar\omega b^\dagger b+\hbar\chi b^{\dagger 2}b^2\ , \tag{2.5}$$

where the new frequency $\bar\omega$ and the coupling constant χ are related to g and q by

$$\chi=qg^4/\Delta^4\ ,\quad \bar\omega=\omega-g^2/\Delta\ ,\quad \Delta=\omega_0-\omega\ . \tag{2.6}$$

Note that χ gives the dispersive part of the third-order nonlinearity of the medium. The effective Hamiltonian (2.5) is exactly soluble:

$$\begin{aligned}|\Psi(0)\rangle&=\sum_{n_b}C(n_b)|n_b\rangle\ ,\\ |\Psi(t)\rangle&=\sum_{n_b}C(n_b)\exp\{-it[(\bar\omega n_b-\chi(n_b^2-n_b)]\}|n_b\rangle\ .\end{aligned} \tag{2.7}$$

The effective Hamiltonian (2.5) has been used extensively in quantum optics.[2-7] However, in what follows, we do not work with (2.5) but use the full Hamiltonian (2.1).[14,15] We expect to see important quantum effects for the fields close to resonance and when the medium is contained in a very good cavity.

In order to solve the full Hamiltonian (2.1) we make use of the conservation law

$$a^\dagger a+b^\dagger b=N=\text{const}\ ,\quad [H,N]=0\ , \tag{2.8}$$

and therefore

$$\langle n,N-n|H|m,M-m\rangle=0\quad \text{if}\quad N\neq M\ . \tag{2.9}$$

Hence we diagonalize H in the space of the states $|n,N-n\rangle$, $n=0,1,2,\ldots,N$ for each value of N,

$$\begin{aligned}&H|\psi_{\alpha N}\rangle=\hbar\lambda_{\alpha N}|\psi_{\alpha N}\rangle\ ,\\ &\qquad N=0,1,2,\ldots,\infty\ ,\quad \alpha=0,1,2,\ldots,N\ .\end{aligned} \tag{2.10}$$

The diagonalized states can be written in terms of Fock states as

$$|\psi_{\alpha N}\rangle=\sum_{n=0}^{N} d_n^{\alpha N}|n,N-n\rangle,\quad d_n^{\alpha N}\equiv\langle n,N-n|\psi_{\alpha N}\rangle\ , \tag{2.11}$$

and the time evolution operator $U(t)=\exp(-iHt/\hbar)$ becomes

$$U(t)=\sum_{N=0}^{\infty}\sum_{\alpha=0}^{N}\exp(-i\lambda_{\alpha N}t)|\psi_{\alpha N}\rangle\langle\psi_{\alpha N}|\ . \tag{2.12}$$

Let the atom be initially in the ground state and the field in an arbitrary state so that the density matrix ρ of the system at time $t=0$ is given by

$$\rho(0)=\sum_{m,n=0}^{\infty}G_{n,m}|0,n\rangle\langle 0,m|. \tag{2.13}$$

The density matrix at time t then becomes

$$\begin{aligned}\rho(t)=\sum_{m,n=0}^{\infty}\sum_{\beta=0}^{m}\sum_{\alpha=0}^{n}&G_{n,m}|\psi_{\alpha n}\rangle\langle\psi_{\beta m}|\\ &\times\exp[-i(\lambda_{\alpha n}-\lambda_{\beta m})t]\\ &\times\langle\psi_{\alpha n}|0,n\rangle\langle 0,m|\psi_{\beta m}\rangle\ .\end{aligned} \tag{2.14}$$

The result (2.14) is especially simple if the field is initially in the Fock state $|N\rangle$; then

$$\begin{aligned}\rho(t)=\sum_{\alpha,\beta=0}^{N}&|\psi_{\alpha N}\rangle\langle\psi_{\beta N}|\exp[-it(\lambda_{\alpha N}-\lambda_{\beta N})]\\ &\times\langle\psi_{\alpha N}|0,N\rangle\langle 0,N|\psi_{\beta N}\rangle\ .\end{aligned} \tag{2.15}$$

If, on the other hand, the atom is in one of the excited states $|N\rangle$ and the field is in the ground state then the density matrix at time t is given by (2.15) with $|0,N\rangle\to|N,0\rangle$. The density matrix has a more complex form for the initial coherent state $|\beta\rangle$ of the field, in which case

$$G_{nm}=\beta^n\beta^{*m}\exp(-|\beta|^2)/\sqrt{n!m!}\ . \tag{2.16}$$

The statistical features of the field can be obtained from (2.14). The quantities of special interest are the mean intensity of the field, $I(t)=\langle b^\dagger b\rangle$, the second-order intensity correlation function

$$g^{(2)}(t)=(\langle b^{\dagger 2}b^2\rangle-\langle b^\dagger b\rangle^2)/\langle b^\dagger b\rangle^2\ , \tag{2.17}$$

and the parameter Q,

$$Q=(\langle b^{\dagger 2}b^2\rangle-\langle b^\dagger b\rangle^2)/\langle b^\dagger b\rangle\ . \tag{2.18}$$

The negative values of Q give a measure of the sub-Poissonian statistics of the field.

III. APPROXIMATE ANALYTICAL SOLUTIONS

Before we discuss exact numerical results, we present a method that enables one to obtain approximate results in closed form. These approximate results are valid for the case of weak nonlinearity and shed considerable light on the quantum behavior of the field in a nonlinear Kerr medium.

We introduce a transformation that diagonalizes the quadratic part of the interaction. Introducing the new Boson operators A and B defined by

$$A=(a+b)/\sqrt{2}, \quad B=i(a-b)/\sqrt{2}\ ,$$
$$[A,A^\dagger]=[B,B^\dagger]=[A,B]=0\ , \qquad (3.1)$$

the Hamiltonian (2.1) transforms into

$$H=\hbar\omega N+g\hbar(A^\dagger A-B^\dagger B)+\hbar q(3N^2-2N)/8$$
$$-\hbar q(A^\dagger A-B^\dagger B)^2/8+H'\ , \qquad (3.2)$$

where $N=a^\dagger a+b^\dagger b$ and the part H' contains nonresonant terms,

$$H'=-\hbar q[2i(A^{\dagger 2}AB+A^\dagger B^\dagger B^2-\text{H.c.})$$
$$+(A^{\dagger 2}B^2+\text{H.c.})]\ . \qquad (3.3)$$

The terms H' are nonresonant in the sense that these oscillate at least at the frequency $2g$. In what follows we make the secular approximation and drop the nonresonant terms H'. We thus work with the Hamiltonian H_s,

$$H_s=\hbar\omega N+\hbar q[(3N^2-2N)-(A^\dagger A-B^\dagger B)^2]/8$$
$$+\hbar g(A^\dagger A-B^\dagger B)\ . \qquad (3.4)$$

Introducing the Fock states $\|p,q\rangle\rangle$ of the operators $A^\dagger A$ and $B^\dagger B$, one clearly has

$$H_s\|p,N-p\rangle\rangle=\hbar\Omega(N,p)\|p,N-p\rangle\rangle\ , \qquad (3.5)$$

where

$$\Omega(N,p)=\omega N+q[(3N^2-2N)-(2p-N)^2]/8$$
$$+g(2p-N)\ . \qquad (3.6)$$

Thus the time evolution can be studied if we can relate the Fock states associated with $a^\dagger a$, $b^\dagger b$ and $A^\dagger A$, $B^\dagger B$. Such a relation can be obtained by the defining relations (3.1).

Consider the generator

$$G(\alpha,\beta)=\exp(\alpha a^\dagger+\beta b^\dagger)|0,0\rangle$$
$$=\sum_{m,n=0}^{\infty}(\alpha^n\beta^m/\sqrt{n!m!})|n,m\rangle\ , \qquad (3.7)$$

for the state $|n,m\rangle$. Similarly the generator for the states $\|n,m\rangle\rangle$ can be written as

$$\overline{G}(u,v)=\exp(uA^\dagger+vB^\dagger)\|0,0\rangle\rangle$$
$$=\sum_{m,n=0}^{\infty}(u^nv^m/\sqrt{n!m!})\|n,m\rangle\rangle\ . \qquad (3.8)$$

On using (3.1) and the fact that $\|0,0\rangle\rangle=|0,0\rangle$, (3.8) reduces to

$$\overline{G}(u,v)=\exp\{[u(a^\dagger+b^\dagger)-iv(a^\dagger-b^\dagger)]/\sqrt{2}\}|0,0\rangle$$
$$=G((u-iv)/\sqrt{2},(u+iv)/\sqrt{2})\ . \qquad (3.9)$$

Writing in full, we get from (3.9)

$$\sum_{m,n=0}^{\infty}(u^mv^n/\sqrt{n!m!})\|n,m\rangle\rangle=\sum_{p,q=0}^{\infty}\{[(u-iv)^p(u+iv)^q]/[(p!q!2^{p+q})^{1/2}]\}|p,q\rangle\ . \qquad (3.10)$$

Thus, on comparing the coefficients of the monomial u^mv^n on the two sides we get the desired relation

$$\|n,N-n\rangle\rangle=2^{-N/2}\sum_{m=n}^{N}\sum_{p=m-n}^{m}\{[(-i)^{N-n}\sqrt{(N-n)!n!}\sqrt{p!(N-p)!}]/[(m-p)!(N-m)!(p-m+n)!(m-n)!]\}$$
$$\times|p,N-p\rangle\ . \qquad (3.11)$$

The relation (3.9) shows that the coherent states $|\alpha,\beta\rangle$ of a and b and $\|\alpha,\beta\rangle\rangle$ of A and B are simply related

$$\|\alpha,\beta\rangle\rangle=\exp[-(|\alpha|^2+|\beta|^2/2]\overline{G}(\alpha,\beta)$$
$$=\exp[-(|\alpha|^2+|\beta|^2)/2]G((\alpha-i\beta)/\sqrt{2},(\alpha+i\beta)/\sqrt{2})$$
$$=\exp\{-[|\alpha|^2+|\beta|^2+|(\alpha-i\beta)/\sqrt{2}|^2+|(\alpha+i\beta)/\sqrt{2}|^2]/2\}|(\alpha-i\beta)/\sqrt{2},(\alpha+i\beta)/\sqrt{2}\rangle\ , \qquad (3.12)$$

i.e.,

$$\|\alpha,\beta\rangle\rangle=|(\alpha-i\beta)/\sqrt{2},(\alpha+i\beta)/\sqrt{2})\rangle\ . \qquad (3.13)$$

We next present an explicit expression for the wave function at time t. Let the system be initially in the coherent state $|\alpha,\beta\rangle$. The state at time t can be calculated approximately using the Hamiltonian H_s and the relation (3.13) between the coherent states of a,b and A,B:

$$|\psi(t)\rangle\simeq\exp(-iH_st/\hbar)|\alpha,\beta\rangle$$
$$=\exp(-iH_st/\hbar)\|(\alpha+\beta)/\sqrt{2},i(\alpha-\beta)/\sqrt{2}\rangle\rangle$$
$$=\exp[-(|\alpha|^2+|\beta|^2)/2]\sum_{N=0}^{\infty}\sum_{p=0}^{N}(\exp(-iH_st/\hbar)\{(\alpha+\beta)^p[i(\alpha-\beta)]^{N-p}\}/[2^{N/2}\sqrt{p!(N-p)!}])\|p,N-p\rangle\rangle\ ,$$
$$(3.14)$$

which, on using (3.5), reduces to

$$|\psi(t)\rangle=\exp[-(|\alpha|^2+|\beta|^2)/2]\sum_{N=0}^{\infty}\sum_{p=0}^{N}\exp[-i\Omega(N,p)t]\{(\alpha+\beta)^p[i(\alpha-\beta)]^{N-p}\}/[2^{N/2}\sqrt{p!}\sqrt{(N-p)!}]\|p,N-p\rangle\rangle \;. \tag{3.15}$$

This is our key result. All the quantum statistical features can be evaluated using (3.15). The mean number of photons at time t is given by the following compact expression:

$$\langle b^\dagger b\rangle=|\beta|^2\{1+\cos(2gt)\exp[-2|\beta|^2\sin^2(qt/4)]\}/2\;,\quad \alpha=0\;. \tag{3.16}$$

If initially the field is in a more general state characterized by the P function $P(\beta)$, i.e., if

$$\rho(0)=\int P(\beta)|\beta\rangle\langle\beta|d^2\beta\;, \tag{3.17}$$

then the mean number of photons can be obtained by averaging (3.16) with respect to $P(\beta)$,

$$\langle b^\dagger b\rangle=\tfrac{1}{2}\int d^2\beta\,P(\beta)|\beta|^2[1+\cos(2gt)\exp[-2|\beta|^2\sin^2(qt/4)]\}\;. \tag{3.18}$$

Thus, for the initial Fock state $|N\rangle$ of the field, (3.18) leads to

$$\langle b^\dagger b\rangle=\tfrac{1}{2}\{N+\cos(2gt)\langle N|b^\dagger{:}\exp[-2b^\dagger b\,\sin^2(qt/4)]{:}b|N\rangle\}\;, \tag{3.19}$$

where : : denotes the normal ordering of b and $b^\dagger$. On simplification, (3.19) reduces to

$$\langle b^\dagger b\rangle=(N/2)\{1+\cos(2gt)[\cos(qt/2)]^{N-1}\}\;. \tag{3.20}$$

For the chaotic state of the field $P(\beta)=\exp(-|\beta|^2/\bar{n})/\pi\bar{n}$. The mean number of photons is then found to be given by

$$\langle b^\dagger b\rangle=(\bar{n}/2)\{1+\cos(2gt)[1+2\bar{n}\,\sin^2(qt/4)]^{-2}\}\;. \tag{3.21}$$

The effect of the initial photon statistics on the dynamical evolution is obvious from Eqs. (3.16), (3.20), and (3.21).

Next, we examine the state (3.15) at certain special times. We assume the atom to be initially in the ground state, $\alpha=0$, so that Eq. (3.15) becomes

$$|\psi(t)\rangle=\exp(-|\beta|^2/2)\sum_{N=0}^{\infty}(\beta^2/2)^{N/2}\sum_{p=0}^{N}[(-i)^{N-p}/(\sqrt{p!}\sqrt{(N-p)!})]\exp[-i\Omega(N,p)t]\|p,N-p\rangle\rangle\;. \tag{3.22}$$

For $qt=4n\pi$, $n=$ integer, (3.22) reduces to

$$|\psi(t)\rangle=\|\pm\beta\exp(-igt)/\sqrt{2};\mp i\beta\exp(igt)/\sqrt{2})\rangle\rangle\;, \tag{3.23}$$

where the upper (lower) sign is for n even (odd). On using (3.13), (3.23) reduces to

$$|\psi(t)\rangle=|\mp i\sqrt{2}\beta\sin(gt),\pm\sqrt{2}\beta\cos(gt)\rangle\;. \tag{3.24}$$

We thus have the remarkable result that the state of the field at the times $qt=4n\pi$ is a coherent state with the amplitude depending on $\cos(4n\pi g/q)$.

The approximate analytical results of this section describe the system dynamics adequately for very weak nonlinearities. In Sec. IV we give an improved approximate analytical solution which has a wider range of applicability.

IV. AN IMPROVED APPROXIMATE ANALYTICAL SOLUTION

In this section we present an alternative way of obtaining a closed-form solution of the Schrödinger equation corresponding to the Hamiltonian (2.1) in the limit of weak anharmonicity. A closed-formed solution is obtained for an initial number state of the field and that of the atom. This solution has a wider range of applicability than that of Sec. III. However, the method of this section does not give a tractable closed-form solution for an initial superposed state of the field or that of the atom unless further approximations are made in which case we recover the results of Sec. III.

We define the angular momentum operators

$$J_+=a^\dagger b,\quad J_-=b^\dagger a,\quad J_z=(a^\dagger a-b^\dagger b)/2\;, \tag{4.1}$$

and rewrite the Hamiltonian (2.1) as

$$H=\hbar[2gJ_x+q(N-1)J_z+qJ_z^2]\;, \tag{4.2}$$

where we have dropped the unperturbed terms and set $\omega_0=\omega$. Note that $N=a^\dagger a+b^\dagger b$ is a constant of motion and

$$J^2=J_x^2+J_y^2+J_z^2=N(N/2+1)/2$$

is the total angular momentum operator. For the sake of convenience we define a new set of angular momentum operators obtained by rotation around the y axis:

$$\begin{aligned}R_z&=\cos(\theta)J_x+\sin(\theta)J_z\;,\\ R_+&=-i\,\sin(\theta)J_x+i\,\cos(\theta)J_z+J_y\;,\end{aligned} \tag{4.3}$$

where

$$\cos(\theta)=2g/[4g^2+q^2(N-1)^2]^{1/2} . \tag{4.4}$$

In terms of the R operators, the Hamiltonian (4.2) reads

$$H=H_0+H_1, \tag{4.5}$$

where

$$\begin{aligned}H_0&=\hbar\{2g\sec(\theta)R_z+q[\cos^2(\theta)(R_+R_-+R_-R_+)/4\\&\quad+\sin^2(\theta)R_z^2]\} ,\\H_1&=q\hbar[-\cos^2(\theta)(R_+^2+R_-^2)/4\\&\quad+\cos(\theta)\sin(\theta)(R_yR_z+R_zR_y)] .\end{aligned} \tag{4.6}$$

Clearly H_0 is diagonal in the representation consisting of the eigenvectors $|N,m)$ of R_z and

$$R^2\equiv R_x^2+R_y^2+R_z^2 ,$$

$$[R_z|N,m)=(N/2-m)|N,m),\ 0<m<N] ,$$

whereas H_1 connects the off-diagonal elements. In the secular approximation we neglect the off-diagonal terms in the Hamiltonian. If ν is the typical frequency scale associated with H_0, then the corrections due to the nonresonant terms are of the order of

$$[qN\cos^2(\theta)/\nu]^2 ,\quad [qN\cos(\theta)\sin(\theta)/\nu]^2 ,$$

etc., which are of the order of $(qN/\nu)^2$ if θ is small, i.e., if $2g \gg qN$. For small θ we recover the results of Sec. III. Clearly, the present approximate results have wider range of applicability. In secular approximation the wave function $|\psi(t)\rangle$ of the system is given by

$$|\psi(t)\rangle\approx\exp(-iH_0t/\hbar)|\psi(0)\rangle . \tag{4.7}$$

To evaluate $|\psi(t)\rangle$ by using Eq. (4.7), we expand the initial state in terms of the states $|N,m)$:

$$|\psi(0)\rangle=\sum_m C_m|N,m) , \tag{4.8}$$

so that

$$\begin{aligned}|\psi(t)\rangle&=\exp[-iqt\cos^2(\theta)N(N/2+1)/2]\\&\quad\times\sum_m C_m\exp(i\alpha_m t)|N,m) ,\\\alpha_m&=-2g\sec(\theta)(N/2-m)\\&\quad-q[1-3\cos^2(\theta)/2](N/2-m)^2 .\end{aligned} \tag{4.9}$$

If, initially, the atom is in its ground state and the field is in the Fock state of N photons, then $J_-|\psi(0)\rangle=0$. By operating with J_- on both sides of Eq. (4.8) and expressing J_- in terms of the R operators defined in Eq. (4.3), it is straightforward to show that the coefficients C_m are given by

$$C_m=[(N-m)!m!]^{-1/2}\{(-i)^m[1-\sin(\theta)]^{(N-m)/2}[1+\sin(\theta)]^{m/2}\} . \tag{4.10}$$

The mean number of photons in the field can be evaluated by using Eq. (4.9):

$$\langle b^\dagger b\rangle_N=(N/2)[-\sin(\theta)\langle R_z\rangle-\cos(\theta)\langle R_y\rangle] , \tag{4.11}$$

where

$$\langle R_z\rangle=\sum_{m=0}^{N}|C_m|^2[(N/2)-m)] , \tag{4.12}$$

$$\langle R_y\rangle=\sum_{m=0}^{N}C_{m-1}^*C_m\sqrt{m(N-m-1)}\exp[-i(\alpha_{m-1}-\alpha_m)t]-\sum_{m=0}^{N}C_{m+1}^*C_m\sqrt{m(N-m+1)}\exp[-i(\alpha_{m+1}-\alpha_m)t] . \tag{4.13}$$

Now, using Eqs. (4.10)–(4.13) it can be shown that

$$\langle b^\dagger b\rangle_N=N(1-\cos^2(\theta)\{1-r^{N-1}(t)\cos[(N-1)\phi(t)-\Gamma t]\}/2) , \tag{4.14}$$

where

$$\begin{aligned}r(t)&=[1-\cos^2(\theta)\sin^2(\lambda t)]^{1/2} ,\\\phi(t)&=\tan^{-1}[\sin(\theta)\tan(\lambda t)] ,\\\lambda&=q[1-3\cos^2(\theta)/2],\\\Gamma&=2g\sec\theta .\end{aligned} \tag{4.15}$$

The period of revival of the oscillations is π/λ. If, initially, the field is in an arbitrary state given by Eq. (2.13) then

$$\langle b^\dagger b\rangle=\sum_{N=0}^{\infty}G_{NN}\langle b^\dagger b\rangle_N , \tag{4.16}$$

where $\langle b^\dagger b\rangle_N$ is given by Eq. (4.14). Because of complicated N dependence of $\phi(t)$, etc. in Eq. (4.14) it is not easy to obtain a closed-form expression for the series in Eq. (4.16). For $(qN/2g)\ll 1$, $\phi(t)=0$, in which case Eqs. (4.16) and (4.14) reduce to Eqs. (3.16) and (3.20), respectively.

V. NUMERICAL RESULTS AND THE SPECIFIC QUANTUM FEATURES OF THE FIELD

In this section we present numerical results for the dynamical behavior of the photon field for a range of values of the anharmonicity parameter q.

A. Weak nonlinearity

The numerical results in the case of weak nonlinearity, $q/g=0.01$, are shown in Figs. 1 and 2. Figure 1 (2) gives the mean photon number $\langle b^\dagger b\rangle$ as a function of time gt when the field is initially in a Fock (coherent) state and the atomic oscillator is in the ground state. These results are to be compared with (2.2) obtained by setting $q=0$. From Figs. 1 and 2 we observe that there is a collapse and revival of the periodic exchange of energy between the atomic and the field oscillators. The collapse and revival of oscillations occurs even when the field is initially in a Fock state. This is in contrast with what happens in the Jaynes-Cummings model of a two-level atom interacting with the field in the Fock state. Thus the propagation of light through an optical fiber can result in new features. The collapses and revivals can be understood in terms of the approximate analytic results obtained in Sec. III. These are determined by the envelope functions in Eqs. (3.20) and (3.16). The envelope function for the initial Fock state is

$$[\cos(qt/2)]^{N-1}$$

and for the coherent state is

$$\exp[-2|\beta|^2\sin^2(qt/4)].$$

Thus it is clear that the time for the maximum of the first revival in the two cases is determined by $qt=2\pi$ and 4π, respectively. These times agree fairly well with the numerical results. For an input coherent field, the revivals occur on a longer time scale. It is also evident that the collapses will be sharper with increasing initial excitation in the field determined in the two cases by the parameters N and $|\beta|^2$. From the foregoing it is quite clear that the detailed behavior of the collapses and revivals is sensitive to the input photon statistics.

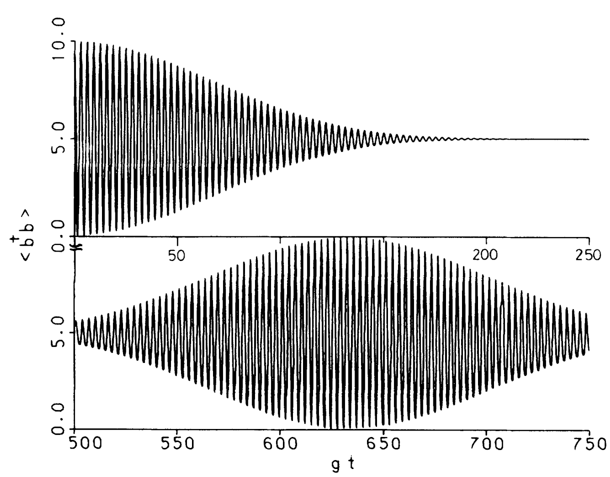

FIG. 1. Mean photon number $\langle b^\dagger b\rangle$ as a function of gt for the atom initially in the ground state and the field in the Fock state $|N=10\rangle$. The anharmonicity parameter is $q/g=0.01$. The lower curve is the first revival of oscillations.

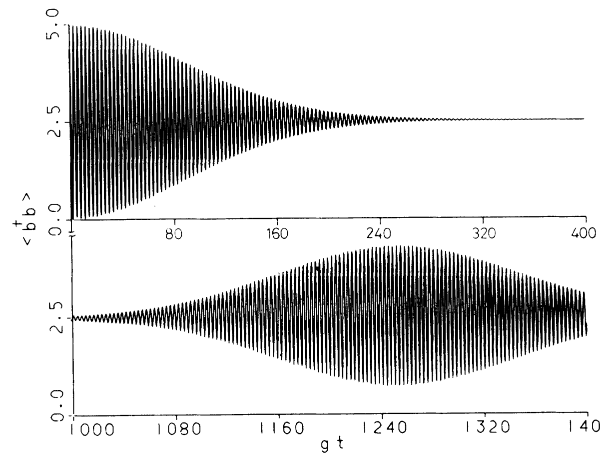

FIG. 2. Same as Fig. 1, but for the field initially in the coherent state $|\beta=\sqrt{5}\rangle$.

The numerical results plotted in Fig. 1 for an initial Fock state and weak nonlinearity, $q/g=0.01$. are in very good quantitative agreement with the approximate analytical results (3.20). However, for an initial coherent state of the field, although the time of revival is given quite accurately by the approximate analytic results of Sec. III [Eq. (3.16)], the detailed behavior of the oscillations in the revival in Fig. 2 is clearly at variance with Eq. (3.16). For instance, the maximum of the revival in Fig. 2 does not attain the value $|\beta|^2$ which is in contrast with Eq. (3.16). The numerically plotted Fig. 2 can, however, be reproduced very closely by working with the improved approximation scheme presented in Sec. IV. In the improved approximation it is possible to obtain a closed-form expression for $\langle b^\dagger b\rangle$ if the atom and the field are initially in number state [see Eq. (4.14)]. For an initial coherent state of the field, $\langle b^\dagger b\rangle$ is obtained by numerically summing the series in Eq. (4.16).

In Fig. 3 we have plotted the numerical results to show the collapse and revival phenomenon for a larger value of $q/g=0.05$ in the presence of an initial Fock state of the field. For this value of q the time of revival calculated from the approximate results of Sec. III [Eq. (3.20)] is evidently different from the numerical calculations. However, Fig. 3 agrees very well with the results of the improved approximation, Eq. (4.14). Similarly, in Fig. 4, we have presented the results of numerical calculations for an initial coherent-state field for $q/g=0.05$. In this case, the behavior of $\langle b^\dagger b\rangle$ is even qualitatively different from the approximate equation (3.16). This behavior is, however, in very good agreement with the improved approximation, Eq. (4.16).

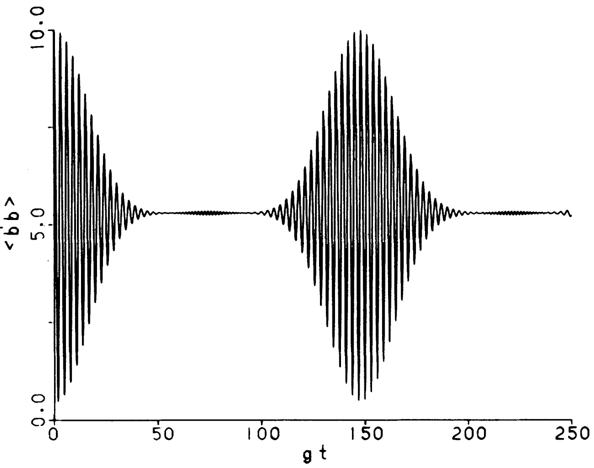

FIG. 3. Mean photon number $\langle b^\dagger b \rangle$ as a function of gt for the atom initially in the ground state and the field in the Fock state $|N=10\rangle$. The anharmonicity parameter is $q/g=0.05$.

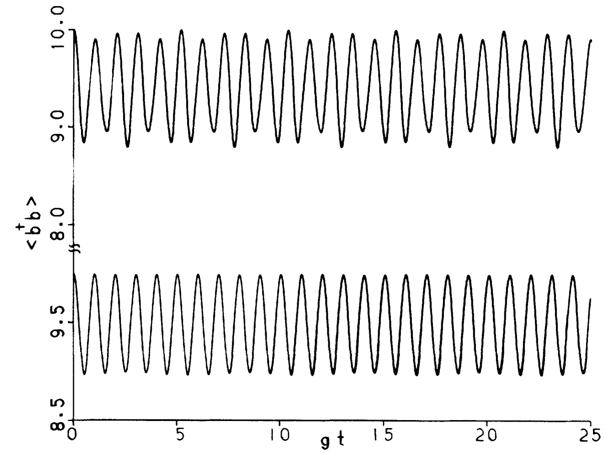

FIG. 5. Mean photon number $\langle b^\dagger b \rangle$ as a function of gt for the atom initially in the ground state and the field in the Fock state $|N=10\rangle$. The upper curve is for the anharmonicity parameter $q/g=5.0$, whereas the lower curve is for $q/g=10.0$.

B. Large nonlinearity

In Fig. 5 we exhibit the dynamical behavior of the mean number of photons for the field initially in the Fock state and for large values of q ($q/g=5,10$). Surprisingly, for $q/g=10$, there is no collapse and revival of the periodic exchange of energy between the two oscillators. This is in contrast with what we find in the case of weak nonlinearity. This behavior can, however, be understood by examining the eigenstates and eigenvalues of the unperturbed Hamiltonian

$$H_0=\hbar\omega(a^\dagger a+b^\dagger b)+\hbar q a^{\dagger 2}a^2 . \tag{5.1}$$

It should be borne in mind that $a^\dagger a+b^\dagger b$ is a constant of motion. Thus the transitions occur between the states

$$|\psi_0\rangle \equiv |0,N\rangle,|1,N-1\rangle,|2,N-2\rangle,\ldots,|N,0\rangle . \tag{5.2}$$

The energies of these states are

$$E_0 \equiv \hbar\omega N, \hbar\omega N, \hbar\omega N+2\hbar q, \ldots, \hbar\omega N+\hbar q N(N-1). \tag{5.3}$$

Note that for $q \gg g$ the resonant transitions with frequency ω are possible only between $|0,N\rangle$ and $|1,N-1\rangle$. Thus, for large nonlinearity and for the atom initially in the ground state, the system is equivalent to a two-level system with states $|0,N\rangle$ and $|1,N-1\rangle$ with an interaction matrix element $g/\sqrt{N}$. This explains the behavior for $q/g=10$ shown in Fig. 5. For $q/g=5$, the next level $|2,N-2\rangle$ also contributes to the dynamics of the system, giving rise to the appearance of another Rabi frequency. It should be noted that the collapse and revival phenomenon will occur for the initial states of the field other than the Fock state. In Fig. 6 we show the behav-

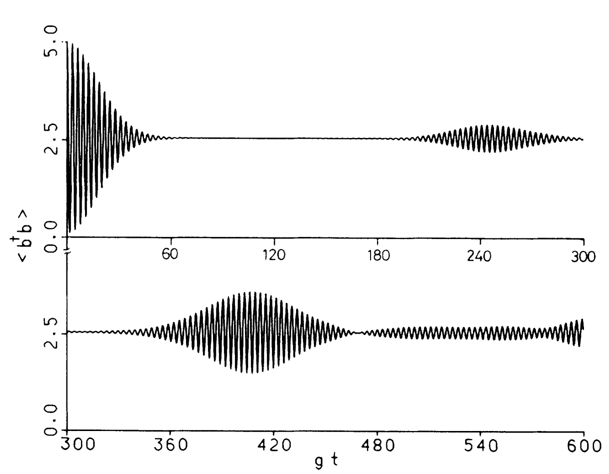

FIG. 4. Same as Fig. 3, but for the field initially in the coherent state $|\beta=\sqrt{5}\rangle$.

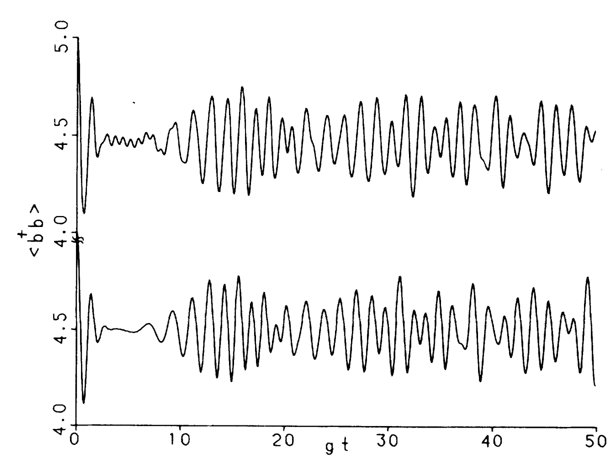

FIG. 6. Same as Fig. 5, but for the field initially in the coherent state $|\beta=\sqrt{5}\rangle$.

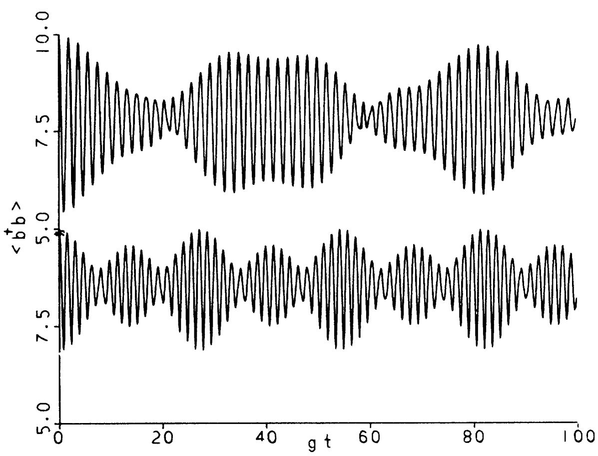

FIG. 7. Mean photon number $\langle b^{\dagger}b \rangle$ as a function of gt for the atom initially in the ground state and the field in the Fock state $|N=10\rangle$. The anharmonicity parameter is $q/g=0.5$ for the upper curve and 1.0 for the lower curve.

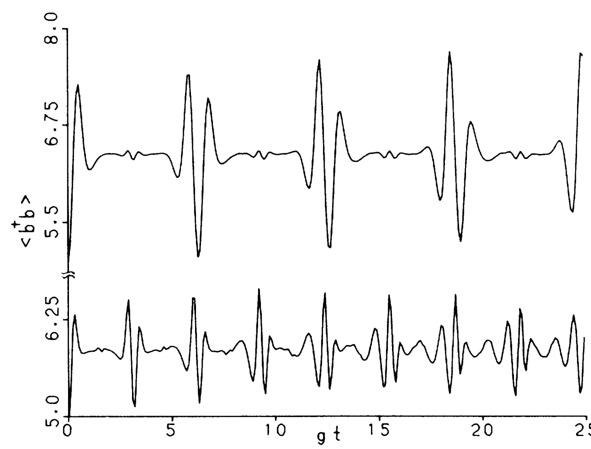

FIG. 9. Mean photon number $\langle b^{\dagger}b \rangle$ as a function of gt for the field in the coherent state $|\beta=\sqrt{5}\rangle$ and the atom also in the coherent state $|\alpha=\sqrt{5}\rangle$. The anharmonicity parameter is $q/g=0.5$ for the upper curve and 1.0 for the lower curve.

ior of $\langle b^{\dagger}b \rangle$ for $q/g=5,10$ in the presence of the coherent-state field. The evolution of $\langle b^{\dagger}b \rangle$ for $q/g=10$ in the presence of the coherent field appears to be different from that of a two-level atom (see Ref. 10). This is because in the case of an anharmonic oscillator there is some contribution to the dynamics of the system due to the transitions to the other levels in a coherent field. These additional transitions are caused by the states of smaller photon numbers occurring in the expansion of a coherent state in terms of the number states. Note that for the states of smaller number of photons the other levels cannot be considered as off resonant.

C. Intermediate values of q

In the case of an intermediate value of q, no analytic results are available. The numerical results are shown in Fig. 7 (for an initial Fock state) and Fig. 8 (for an initial coherent state). The comparison of these two figures shows that the initial photon statistics leads to important differences in the nature of the phenomenon of collapses and revivals. In the case of an initial coherent state of the field the collapses and revivals are much more irregular as compared with those in the case of a Fock-state field. An interesting case of both the oscillators initially in a coherent state is shown in Fig. 9 although it is not clear how to prepare the atomic nonlinear oscillator in a coherent-excitation mode.

We next discuss the changes in the statistical features of the field as it propagates through an optical fiber. In the dispersive limit, Kitagawa and Yamamoto[3] have shown how the field acquires a very strong sub-Poissonian character. In Fig. 10 we have plotted $Q(t)$ [Eq. (2.18)] as a function of time for two intermediate values of $q/g=0.5$ and 1. $Q(t)<0$ implies sub-

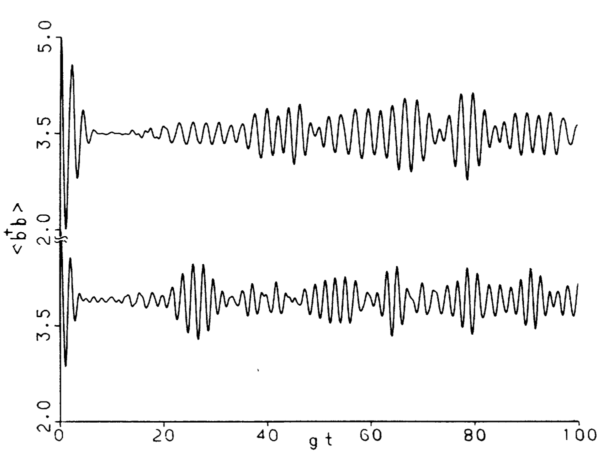

FIG. 8. Same as Fig. 7 but for the field initially in the coherent state $|\beta=\sqrt{5}\rangle$.

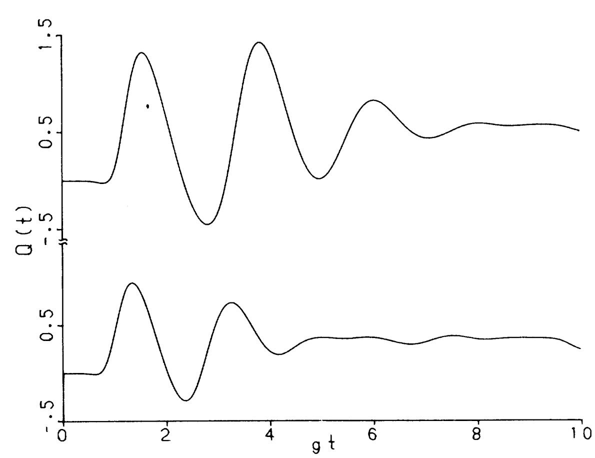

FIG. 10. $Q(t)$ as a function of gt for the atom initially in the ground state and the field in the coherent state $|\beta=\sqrt{5}\rangle$. The upper curve is for the anharmonicity parameters $q/g=0.5$, whereas the lower curve is for $q/g=1.0$.

Poissonian photon-number distribution or antibunching. From the figure it is seen that there is a small amount of antibunching for very early times. There is no significant antibunching even for weaker or stronger anharmonicities.

Thus, in conclusion, we have shown how the propagation characteristics of a resonant field through a Kerr-like medium are influenced by the quantum nature[16] of the field and by the photon statistics. The nonlinearities of the medium lead to collapses and revivals in the otherwise periodic behavior of the energy exchange between the medium and the field. We have also shown the sub-Poissonian nature of the field. Our results differ significantly from those obtained from the considerations involving adiabatic elimination of the atomic degrees of freedom.

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