Dynamical interaction of an atomic oscillator with squeezed radiation inside a cavity

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The dynamical interaction of an oscillator with squeezed radiation inside a cavity is considered. Exact solution of the dynamical equation is presented using the Wigner function for the combined system of the atomic oscillator and the squeezed radiation. This solution enables us to study (i) relaxation of the oscillator for *arbitrary bandwidth* of the squeezed radiation, and (ii) effects of squeezed radiation on vacuum-field Rabi splittings. Explicit results for fluctuation spectra and the oscillator polarizabilities are given. New resonances in the fluctuation spectra appear as the radiation in the cavity becomes more and more squeezed. Strong narrowing of the vacuum-field Rabi splitting due to squeezed radiation is predicted. Very large enhancement of the oscillator polarizability at Rabi side peaks is also predicted. The modification of the cavity output due to the atomic oscillator is discussed.

I. INTRODUCTION

The relaxation of a quantum system in contact with a heat bath is well understood.[1] The normal heat baths are usually such that there are no phase correlations in the heat bath. The structure of the master equation for the quantum system is, however, very sensitive to the nature of such phase correlations.[2-4] Recently, it has been pointed out how the interaction with a broadband squeezed bath can modify in an important way the relaxation of a two-level system.[2] The modified relaxation equation can lead to important effects such as the inhibition of the decay of one of the components of the dipole moment,[2] line narrowing in the fluorescence[5] and absorption spectra[6] produced by the coherently driven atoms, and generation of the intelligent spin states[7] by a collective atomic system. Most of these studies are based on the assumption that the bandwidth of the squeezed bath is much larger than, say, the relaxation widths T_1^{-1} and T_2^{-1}. It is obviously important to understand the consequences of relaxing the above assumption. In a recent paper Parkins and Gardiner[8] have considered various approximations for treating the relaxation of a two-level system interacting with a squeezed bath. In the present work we treat an exactly soluble model so that one can understand various limiting cases. We discuss the exact dynamical evolution of an oscillator interacting with a squeezed bath of arbitrary bandwidth.

Note that in contrast to the existing works we treat the interaction of the atomic oscillator with squeezed radiation inside a cavity. We are thus able to answer questions which one studies in the context of cavity electrodynamics.[9] For example, how does the squeezed vacuum affect the decay of the atom inside the cavity? Our work can also answer questions such as how the vacuum-field Rabi splittings[10] are affected by the presence of the squeezed radiation.

The organization of this paper is as follows: In Sec. II we formulate the basic equation using the Wigner function for the combined system of the atomic oscillator and squeezed radiation. This equation can be solved exactly and various steady-state characteristics such as fluctuation spectra and polarizabilities can be calculated. The numerical results for such quantities are given in Secs. III and V. In Sec. IV we give the results obtained by using the adiabatic approximation. In the Appendix we point out the differences that arise in the oscillator relaxation if the oscillator interacts with the squeezed radiation outside the cavity. In Sec. VI we discuss how the output properties of the radiation are modified due to the atomic oscillator in the cavity. The squeezing spectrum is quite sensitive to the presence of the *long-lived atom* in the cavity.

II. BASIC EQUATION DESCRIBING THE DYNAMICAL INTERACTION BETWEEN SQUEEZED BATH AND OSCILLATOR

In this section we formulate basic equations describing the dynamical interaction between the oscillator and the bath. Let a and $a^\dagger$ be the annihilation and creation operators for the oscillator system of interest. Let ω_a be the frequency of the oscillator. The operators for the squeezed bath are denoted by b and $b^\dagger$ with ω_b representing the central frequency of the bath. The oscillator interacts with the bath through the interaction

$$H_{\mathrm{SB}} = g\hbar(a^\dagger b + ab^\dagger) . \tag{2.1}$$

A. Quantum dynamics of squeezed radiation

In order to study the interaction between atomic oscillator and squeezed radiation, we have to know the statistical dynamics of radiation. We thus consider the dynamical model for the squeezed bath. We rely heavily on the theories[3,11] that give the quantum statistics of the squeezed radiation produced by a nonlinear process. The dynamical models show that the fluctuations in the squeezed radiation can be characterized by a Fokker-Planck equation for the Wigner function[3] Φ_B associated with the density matrix of the squeezed radiation,

$$\frac{\partial\Phi_B}{\partial t}=\frac{\partial}{\partial\beta}[(\kappa\beta+iG\beta^*)\Phi_B]+\frac{\partial}{\partial\beta^*}[(\kappa\beta^*-iG^*\beta)\Phi_B]+2\frac{\partial^2}{\partial\beta\,\partial\beta^*}(D\Phi_B)+\frac{\partial^2}{\partial\beta^2}(D_0\Phi_B)+\frac{\partial^2}{\partial\beta^{*2}}(D_0^*\Phi_B)\ . \tag{2.2}$$

Here β is the c-number variable associated with the squeezed oscillator. The parameter κ essentially determines the bandwidth of the radiation and is related to the cavity losses and nonlinear absorption in the medium. The parameter G gives the magnitude of the parametric coupling and is proportional to $\chi^{(2)}$ ($\chi^{(3)}$) for the down conversion (nonlinear mixing) process and this depends on the microscopic dynamics of the nonlinear medium producing squeezed radiation. The D's give the fluctuation parameters. The fluctuations in b are given by a Gaussian Wigner function.[12] The phase-dependent correlations arise from the nonvanishing of G and D_0 terms.

As an example of how Eq. (2.2) arises, consider the model for the down conversion[13,14] which leads to the generation of the squeezed fields. This model can be characterized by an effective Hamiltonian

$$H=\hbar\omega_b b^\dagger b+2\hbar\omega_b c^\dagger c+\frac{i\hbar\tilde{G}}{2}(b^{\dagger 2}c-\text{H.c.})+i\hbar(\varepsilon_b b^\dagger e^{-i\omega_b t}-\text{H.c.})+i\hbar(\varepsilon_c c^\dagger e^{-2i\omega_b t}-\text{H.c.})+\mathcal{T}\ , \tag{2.3}$$

where $\mathcal{T}$ represents terms responsible for the decay of the modes b and c. Here c is the pump mode and ε_b and ε_c denote the coherent fields driving the modes b and c, respectively. Let γ_b and γ_c be the decay rates associated with the modes b and c. We will assume that $\gamma_c \gg \gamma_b$ so that the pump mode can be adiabatically eliminated. We also carry out the linearization around the steady state. The calculations show that the fluctuations of the b mode around steady state $b=\langle b\rangle+B$ are described by the density-matrix equation

$$\frac{\partial\rho_B}{\partial t}=-\kappa(B^\dagger B\rho_B-2B\rho_B B^\dagger+\rho_B B^\dagger B)-\frac{i}{\hbar}\left[\left[\frac{i\hbar}{2}\tilde{G}\langle c\rangle B^{\dagger 2}+\text{H.c.}\right],\rho_B\right]\ , \tag{2.4}$$

where $\langle b\rangle$, $\langle c\rangle$, and κ are defined by

$$\kappa=\gamma_b+\frac{\tilde{G}^2|\langle b\rangle|^2}{\gamma_c},\quad \gamma_b\langle b\rangle=\varepsilon_b+\tilde{G}\langle b\rangle^*\langle c\rangle\ ,\quad \gamma_c\langle c\rangle=\varepsilon_c-\frac{\tilde{G}}{2}\langle b\rangle^2\ . \tag{2.5}$$

The Wigner function Φ_B associated with the density matrix ρ_B can be shown to satisfy (2.2) with

$$D=\kappa,\quad D_0=0,\quad G=i\tilde{G}\langle c\rangle\ . \tag{2.6}$$

The steady-state values of the fluctuation parameters $\langle B^\dagger B\rangle$ and $\langle B^2\rangle$ are found from (2.4):

$$\langle B^\dagger B\rangle+\frac{1}{2}=\frac{1}{2}\left[1-\frac{|\tilde{G}\langle c\rangle|^2}{\kappa^2}\right]^{-1}, \tag{2.7}$$

$$\langle B^2\rangle=\frac{\tilde{G}\langle c\rangle}{2\kappa}\left[1-\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|^2\right]^{-1}. \tag{2.8}$$

On absorbing the phase of $G\langle c\rangle$ in the definition of $\langle B^2\rangle$, the variances in $X_1\equiv(B+B^\dagger)/2$ and $X_2\equiv(B-B^\dagger)/2i$ are found to be

$$\Delta X_{1,2}^2=\frac{1}{4}\frac{1}{\left[1\mp\left|\frac{\tilde{G}\langle c\rangle}{\kappa}\right|\right]}\ . \tag{2.9}$$

Thus the squeezing depends on the parameter $|G/\kappa|$. The squeezing in ΔX_2 increases as $|G/\kappa|$ increases. The time correlation function has the form

$$\lim_{t\to\infty}\langle B^\dagger(t+\tau)B(t)\rangle=e^{-(\kappa+|G|)\tau}x+ye^{-(\kappa-|G|)\tau}\ , \tag{2.10}$$

where the parameters x and y depend on the steady-state properties (2.7) and (2.8). Thus the spectrum of bath fluctuations consists of a sum of two Lorentzians with widths $(\kappa\pm|G|)$.

B. Combined dynamics of oscillator and squeezed radiation

We now turn to Eq. (2.2), which generally describes the generation of the squeezed radiation in most circumstances. We next consider the interaction of the oscillator system with the squeezed radiation characterized by (2.2). Let Φ be the Wigner function for the density matrix of the combined system of the oscillator and the squeezed radiation. For simplicity we will assume that $\langle b\rangle=0$ so that there is no need to distinguish between the operators b and B. Using (2.1) and (2.2) it is clear that Φ satisfies the differential equation

$$\frac{\partial\Phi}{\partial t}=\frac{\partial}{\partial\beta}[(\kappa\beta+ig^*\alpha+iG\beta^*)\Phi]+\frac{\partial}{\partial\beta^*}[(\kappa\beta^*-ig\alpha^*-iG^*\beta)\Phi]+2\frac{\partial^2}{\partial\beta\,\partial\beta^*}(D\Phi)+\frac{\partial^2}{\partial\beta^2}(D_0\Phi)+\frac{\partial^2}{\partial\beta^{*2}}(D_0^*\Phi)+\frac{\partial}{\partial\alpha}[(\Gamma\alpha+i\nu\alpha+ig\beta)\Phi]+\frac{\partial}{\partial\alpha^*}[(\Gamma\alpha^*-i\nu\alpha^*-ig^*\beta^*)\Phi]+\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}(\Gamma\Phi),\quad \nu=\omega_a-\omega_b \tag{2.11}$$

where α is the c-number variable associated with the mode a. The Γ-dependent terms correspond to any other source of damping for the system oscillator and these correspond to a term $-\Gamma(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)$ in the density-matrix equation. We will see that the interaction with the squeezed bath leads to additional damping terms under certain conditions. Equation (2.11) is our fundamental equation which describes the dynamics of the system oscillator interacting with squeezed radiation. It automatically takes into account the back reaction of the oscillator on the squeezed radiation. Although for calculation it is convenient to work with the differential equation for the Wigner function, we give for completeness the corresponding equation for the density matrix,

$$\frac{\partial\rho}{\partial t}=-i\left[\nu a^{\dagger}a+(ga^{\dagger}b+g^{*}ab^{\dagger})+\frac{G}{2}b^{\dagger 2}+\frac{G^{*}}{2}b^{2},\rho\right]-\Gamma(a^{\dagger}a\rho-2a\rho a^{\dagger}+\rho a^{\dagger}a)$$
$$-\left[D-\frac{\kappa}{2}\right](bb^{\dagger}\rho-2b^{\dagger}\rho b+\rho bb^{\dagger})-\left[D+\frac{\kappa}{2}\right](b^{\dagger}b\rho-2b\rho b^{\dagger}+\rho b^{\dagger}b)+D_0[b^{\dagger},[b^{\dagger},\rho]]+D_0^{*}[b,[b,\rho]] . \quad (2.12)$$

III. EXACT SOLUTIONS FOR THE DYNAMICAL PROPERTIES OF THE OSCILLATOR

In this section we discuss the exact solution of the basic equation (2.11). Let ψ be the column matrix formed by the mean values of a and b,

$$\psi^{\dagger}=(\langle a^{\dagger}\rangle,\langle b^{\dagger}\rangle,\langle a\rangle,\langle b\rangle) , \quad (3.1)$$

then it can be shown from (2.11) that ψ satisfies

$$\frac{\partial\psi}{\partial t}=-M\psi , \quad (3.2)$$

where M is the 4×4 relaxation matrix defined by

$$M=\begin{bmatrix} i\nu+\Gamma & ig & 0 & 0\\ ig^{*} & \kappa & 0 & iG\\ 0 & 0 & -i\nu+\Gamma & -ig^{*}\\ 0 & -iG^{*} & -ig & \kappa \end{bmatrix} . \quad (3.3)$$

The fluctuation matrix V defined by

$$V\equiv\begin{bmatrix} \langle a^{\dagger}a\rangle+\frac{1}{2} & \langle b^{\dagger}a\rangle & \langle a^{2}\rangle & \langle ab\rangle\\ \langle a^{\dagger}b\rangle & \langle b^{\dagger}b\rangle+\frac{1}{2} & \langle ba\rangle & \langle b^{2}\rangle\\ \langle a^{\dagger 2}\rangle & \langle a^{\dagger}b^{\dagger}\rangle & \langle a^{\dagger}a\rangle+\frac{1}{2} & \langle a^{\dagger}b\rangle\\ \langle b^{\dagger}a^{\dagger}\rangle & \langle b^{\dagger}b^{\dagger}\rangle & \langle b^{\dagger}a\rangle & \langle b^{\dagger}b\rangle+\frac{1}{2} \end{bmatrix}-\psi\psi^{\dagger} \quad (3.4)$$

satisfies

$$\dot{V}=-MV-VM^{\dagger}+2\mathcal{D} , \quad (3.5)$$

where $\mathcal{D}$ is the diffusion matrix

$$2\mathcal{D}=\begin{bmatrix} \Gamma & 0 & 0 & 0\\ 0 & 2D & 0 & 2D_0\\ 0 & 0 & \Gamma & 0\\ 0 & 2D_0^{*} & 0 & 2D \end{bmatrix} . \quad (3.6)$$

The steady-state value of V can obtained from

$$\lim_{t\to\infty} V=2\int_0^{\infty}d\tau\, e^{-M\tau}\mathcal{D}(e^{-M\tau})^{\dagger} . \quad (3.7)$$

These equations yield the mean values and the fluctuations in the mean values. The spectrum of the oscillator fluctuations can be obtained from (3.2) and the quantum regression theorem, i.e., from

$$\lim_{t\to\infty}\langle a^{\dagger}(t+\tau)a(t)\rangle\equiv(e^{-M\tau})_{31}\langle a^{2}\rangle+(e^{-M\tau})_{32}\langle ba\rangle+(e^{-M\tau})_{33}\langle a^{\dagger}a\rangle+(e^{-M\tau})_{34}\langle b^{\dagger}a\rangle . \quad (3.8)$$

Note that the basic dynamical equation (2.11) has the form of a linearized Fokker-Planck equation and hence its solution can be written down by inspection. The steady-state solution for the Wigner function $\Phi_a(\alpha,\alpha^{*})$ $[\equiv\int d^{2}\beta\,\Phi(\alpha,\beta)]$ associated with the oscillator system is Gaussian,

$$\Phi_a(\alpha,\alpha^{*})=\frac{1}{\pi(\tau^{2}-4|\mu|^{2})}\times\exp[-(\tau^{2}-4|\mu|^{2})^{-1}\times(\mu\alpha^{2}+\mu^{*}\alpha^{*2}+\tau|\alpha|^{2})] , \quad (3.9)$$

where

$$\tau=\langle a^{\dagger}a\rangle+\tfrac{1}{2},\quad \mu=-\tfrac{1}{2}\langle a^{\dagger}a^{\dagger}\rangle . \quad (3.10)$$

Note that the result for a normal heat bath will be obtained by setting $D_0=G=0$ whence one gets the distribution (3.9) with $\mu=0$. Thus all the dynamical properties of the oscillator interacting with a squeezed bath of arbitrary bandwidth can be obtained from (3.2), (3.5), and the Gaussian nature of the Wigner function. The number distribution associated with (3.9) shows interesting oscillatory character[15] for a certain range of the values of μ

and τ.

In Fig. 1 we show the behavior of the mean oscillator excitation $\langle a^\dagger a\rangle \equiv n_a$ and the squeezing parameter s, defined by ($\langle a\rangle=0$)

$$s=\left\langle \left[\frac{ae^{i\varphi}+e^{-i\varphi}a^\dagger}{2}\right]^2\right\rangle,\quad \varphi=\pi/4 \tag{3.11}$$

for various values of the parameter G (which is responsible for creating the squeezed radiation in the cavity). As G increases, the mean oscillator excitation increases. In fact, $\langle a^\dagger a\rangle$ tends to be equal to the mean excitation $\langle b^\dagger b\rangle$ [Eq. (2.7)] for the bath. The squeezing properties of the atomic oscillator are also quite interesting. The maximum squeezing (50%) occurs at the threshold, i.e., as $|G|$ approaches κ.

We next evaluate the spectrum of fluctuations $S(\delta)$ defined by

$$S(\delta)=\mathrm{Re}\int_0^\infty \lim_{\substack{t\to\infty\\ z\to i\delta}} \langle a^\dagger(t+\tau)a(t)\rangle e^{-z\tau}d\tau\ ,$$

$$\delta=\omega-\omega_b\ , \tag{3.12}$$

where the time correlation function $\langle a^\dagger(t+\tau)a(t)\rangle$ is given by (3.8). We calculate the spectrum of fluctuations for several values of the parameters κ and G. Note that large values of κ correspond to the large bandwidth of the squeezed bath. It turns out that the fluctuation spectrum changes remarkably by a change of G provided that Γ is small. We thus consider two cases: (a) κ is large ($\gg g$) and (b) κ is of the order of $g\sim 1$.

For large κ and small values of G the fluctuation spectrum (Fig. 2) consists of a resonant structure at the origin. The adiabatic approximation is expected to be good. As G increases, the adiabatic approximation breaks down as the bath spectrum now consists of a narrow resonance in addition to a broad resonance [Eq. (2.10)]. In such a case the spectrum (Fig. 2) develops new resonances (side peaks) which become more and more prominent as G increases. The emergence of the new resonances can be understood on the basis of the eigenvalues of the relaxation matrix M [Eq. (3.3)]. For $\Gamma=\nu=0$, the M matrix has eigenvalues

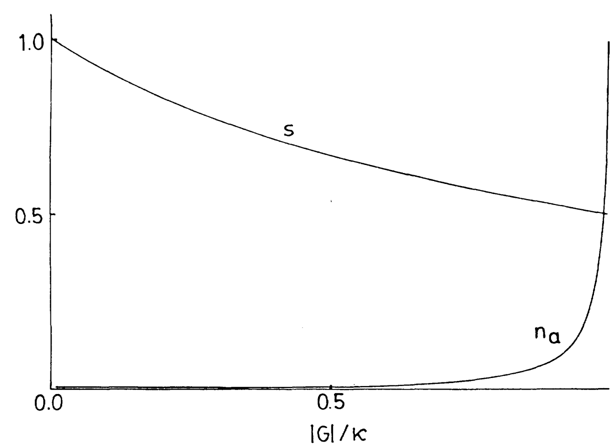

FIG. 1. Mean excitation n_a and the squeezing parameter s [Eq. (3.11)] for the oscillator as a function of $|G|/\kappa$ for $\Gamma=\nu=0$. Each curve is normalized to its maximum value.

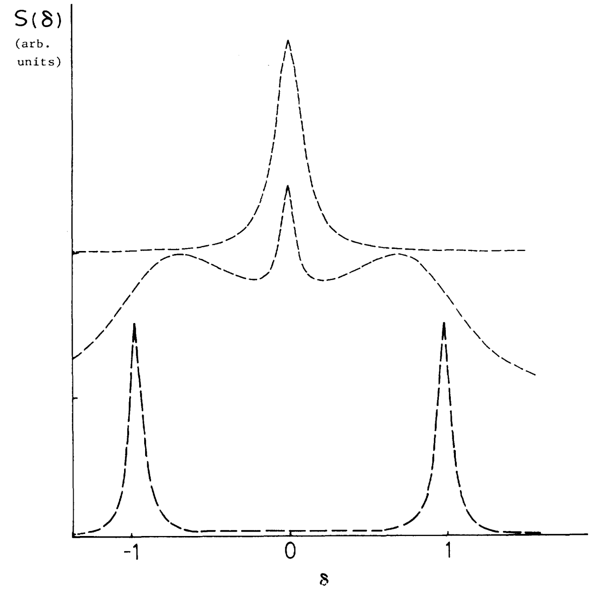

FIG. 2. Fluctuation spectrum $S(\delta)$ as a function of δ for $\kappa=10$, $g=1$, $\Gamma=\nu=0$ and for $|G|=0$, 9, and 9.9 (top to bottom). For clarity different curves are displaced. Each curve is normalized to its maximum value. The starting point on the y axis for each curve is marked.

$$\lambda=\frac{\kappa\pm|G|}{2}\pm\tfrac{1}{2}[(\kappa\pm|G|)^2-4|g|^2]^{1/2}\ . \tag{3.13}$$

Thus complex eigenvalues will result if

$$2|g|>(\kappa\pm|G|)\ . \tag{3.14}$$

Thus as $|G|$ increases the resonances can appear at $\pm[|g|^2-\frac{1}{4}(\kappa-|G|)^2]^{1/2}$ with a width $(\kappa-|G|)/2$. This also explains why the side peaks become narrower as $|G|$ becomes close to κ.

We next consider the case when κ is comparable to g. This really corresponds to a case when the adiabatic approximation is not possible even if $G=0$. The fluctuation spectrum is shown in Fig. 3. For $G=0$ the fluctuation spectrum leads to the usual vacuum-field Rabi splittings which are broadened as $\kappa\neq 0$. This figure exhibits the effect of squeezed radiation on the vacuum-field Rabi splittings. As G increases there is considerable narrowing of the vacuum-field Rabi peaks or, in other words, the Rabi peaks due to the squeezed vacuum can be extremely narrow and large when G is large, i.e., when the vacuum is strongly squeezed. For example, the ratio of the peak heights for $G=0.97$ and 0.5 is about 230. This behavior again can be understood on the basis of the eigenvalues (3.13). It should be noted that we have assumed that the atoms spend enough time in the cavity so that steady state is reached. If the steady state is not reached, then transient characteristics, for example, change in the energy of the atomic oscillator as a function of time and can be calculated.

IV. SOLUTIONS FOR LARGE BANDWIDTH OF SQUEEZED RADIATION

In this section we discuss the limiting case of a squeezed bath with large bandwidth and we compare our

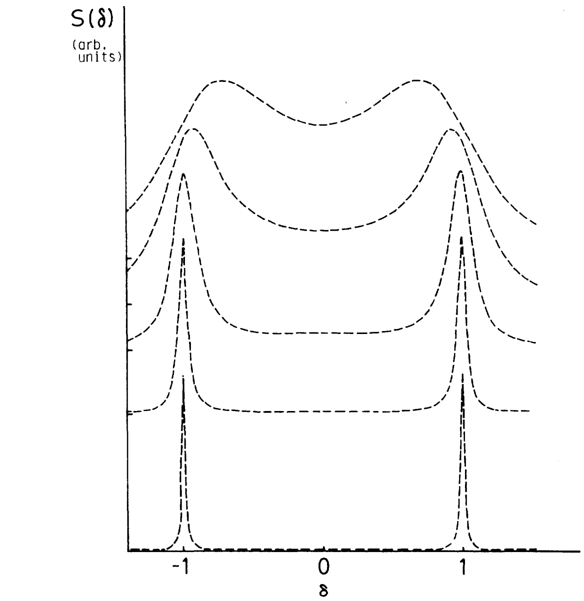

FIG. 3. Same as in Fig. 2 but now $\kappa=1$ and different curves from top to bottom are of $|G|=0$, 0.5, 0.8, 0.93, and 0.97.

results with the known results. For large bandwidth squeezed bath the dynamics (2.11) can be simplified by adiabatically eliminating the degrees of freedom associated with the bath. In order to keep the analysis simple we do adiabatic elimination for the special case $\nu=\Gamma=0$. The general case can also be treated by standard methods. We start with the Langevin equations associated with (2.11). These can be written as

$$\dot{\chi}=-M\chi+F(t) \ , \tag{4.1}$$

where χ is a column matrix with components α, β, α^*, and β^* and where $F(t)$ is the δ correlation Gaussian random force with

$$\langle F_i(t)F_j^*(t')\rangle=2\mathcal{D}_{ij}\delta(t-t') \ , \tag{4.2}$$

with $\mathcal{D}$ given by (3.6). We next solve equations for β and β^* by setting $\dot{\beta}=\dot{\beta}^*=0$ and substitute the resulting solution in the equations for α and α^*. This procedure leads to

$$\begin{bmatrix} \beta \\ \beta^* \end{bmatrix}=\begin{bmatrix} \kappa & iG \\ -iG^* & \kappa \end{bmatrix}^{-1}\begin{bmatrix} -ig^*\alpha+F_\beta \\ ig\alpha^*+F_\beta^* \end{bmatrix} \tag{4.3}$$

and

$$\dot{\alpha}=\frac{-|g^2|\kappa}{(\kappa^2-|G|^2)}\alpha-\frac{ig^2G}{(\kappa^2-|G|^2)}\alpha^*+f_\alpha \ . \tag{4.4}$$

The random force f_α is related to the old random force via

$$f_\alpha\equiv\frac{-ig}{(\kappa^2-|G|^2)}F_\beta-\frac{gG}{(\kappa^2-|G|^2)}F_\beta^* \ . \tag{4.5}$$

The diffusion coefficient for the random force f_α denoted by $d_{\alpha\alpha}$, etc., can be obtained in terms of the $\mathcal{D}$'s by using (4.5),

$$d_{\alpha\alpha}\equiv\frac{-g^2}{(\kappa^2-|G|^2)^2}\mathcal{D}_{\beta\beta}+\frac{g^2G^2}{(\kappa^2-|G|^2)^2}\mathcal{D}_{\beta^*\beta^*}+\frac{2ig^2G}{(\kappa^2-|G|^2)^2}\mathcal{D}_{\beta\beta^*} \ , \tag{4.6}$$

$$d_{\alpha^*\alpha}\equiv(\kappa^2-|G|^2)^{-2}|g|^2[\mathcal{D}_{\beta\beta^*}(1+|G|^2)-iG\mathcal{D}_{\beta^*\beta^*}+iG^*\mathcal{D}_{\beta\beta}] \ . \tag{4.7}$$

Note that $d_{\alpha\alpha}\neq0$ even if $\mathcal{D}_{\beta\beta}=0$. This is because of the nonvanishing squeezing parameter G. Note that one of the conditions for the validity of the adiabatic approximation is that the eigenvalues of the relaxation matrix for β variables, $\kappa\pm|G|$, must be large. The decay modes $\lambda_\pm$ associated with (4.4) are given by

$$\lambda_\pm=\frac{|g|^2}{\kappa\pm|G|} \ . \tag{4.8}$$

Thus the resonant interaction between the atomic oscillator and the squeezed bath leads to two modes of decay with increased and decreased decay constants relative to that for a normal cavity in which case the decay constant is $|g|^2/\kappa$. Note that the dipole near a phase conjugate mirror has a similar behavior.[16] The result (4.8) is similar to the result of Gardiner[2] on the inhibition of the decay of the one component of the atomic dipole moment. However, there are differences which we discuss in the Appendix. It should be borne in mind that the maximum value of $|G|$ in our model is κ and that for $|G|/\kappa\to1$, the adiabatic approximation breaks down.

V. EXACT SOLUTIONS FOR THE LINEAR RESPONSE—ABSORPTION AND MIXING PRODUCED BY THE OSCILLATOR

In this section we treat the effect of an external driving field on the oscillator system interacting with a squeezed bath. Let ω_l be the frequency of the external field driving the oscillator. The response equations are given now [cf. Eq. (3.2)],

$$\dot{\psi}=-M\psi+\begin{bmatrix} -ig_{\text{ext}}e^{-i\delta t} \\ 0 \\ ig^*_{\text{ext}}e^{i\delta t} \\ 0 \end{bmatrix}, \quad \delta=\omega_l-\omega_b \ . \tag{5.1}$$

Here g_{ext} is the coupling of the oscillator with the external field. Clearly the steady-state response is given by

$$\psi_\alpha=(-i\delta+M)^{-1}_{\alpha1}(-ig_{\text{ext}})e^{-i\delta t}+(i\delta+M)^{-1}_{\alpha3}(ig^*_{\text{ext}})e^{i\delta t} \ . \tag{5.2}$$

The first term gives the response at the applied frequency. The second term gives the response at $(2\omega_b-\omega_l)$. This latter response is produced because of the interaction of the oscillator with the squeezed bath. Thus the absorption spectra are determined by

$$S_a(\delta)=-\text{Re}(-i\delta+M)^{-1}_{11} \tag{5.3}$$

and the four-wave-mixing response is determined by

$$\chi_{\rm FWM}(\delta)=i(i\delta+M)^{-1}_{13} \ . \tag{5.4}$$

We present the numerical results for the effects of the squeezed vacuum on the absorption by an atomic oscillator in Figs. 4–6. The effects are more dramatic if the oscillator by itself has no relaxation width, i.e., it only has a width due to its interaction with the squeezed bath. If κ is large (Fig. 4) then there is no noticeable change in the absorption until G becomes of the same order as κ, i.e., until the radiation in the cavity is strongly squeezed, then the oscillator polarizability exhibits new resonances, i.e., side peaks which become more and more prominent as G becomes close to κ. This is a new feature which can only be predicted on the basis of a nonadiabatic theory, i.e., it would be missed if one had used the adiabatic approximation of Sec. IV. For $\kappa=1$, $G=0$, the absorption spectra (Fig. 5) exhibit the usual vacuum-field Rabi splittings with a width which is determined by κ. As the vacuum becomes squeezed, the Rabi peaks (Fig. 5) become narrower. For even larger values of G the absorption spectra exhibit a broad structure at the origin. The sidebands are extremely narrow here. Thus the polarizability of the oscillator at the sideband frequencies is enhanced over that in a normal cavity. For example, the ratio of the peak heights for $G=0.99$ and 0 is about 38. Finally Fig. 6 shows the effects of the squeezed bath on the oscillator polarizability if the atomic oscillator has a width Γ which is comparable to κ. Even here the squeezed radiation has a noticeable effect as is evident from Fig. 6. It may be added that the results for many atomic oscillators can be obtained from those for a single oscillator by using the replacement $g\rightarrow g\sqrt{N}$.

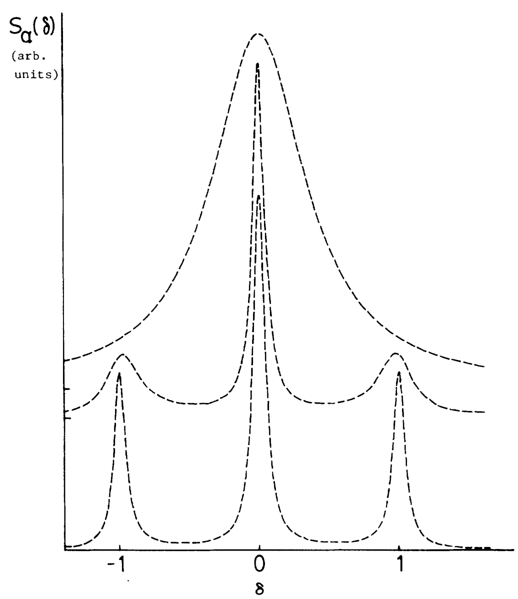

FIG. 4. Absorption spectrum $S_a(\delta)$ as a function of δ for $\kappa=10$, $g=1$, $\Gamma=\nu=0$ and for $|G|=0$, 9.7, and 9.9 (top to bottom). Each curve is normalized to its maximum value. The actual scale on the x axis for $G=0$ is one-fourth of that shown.

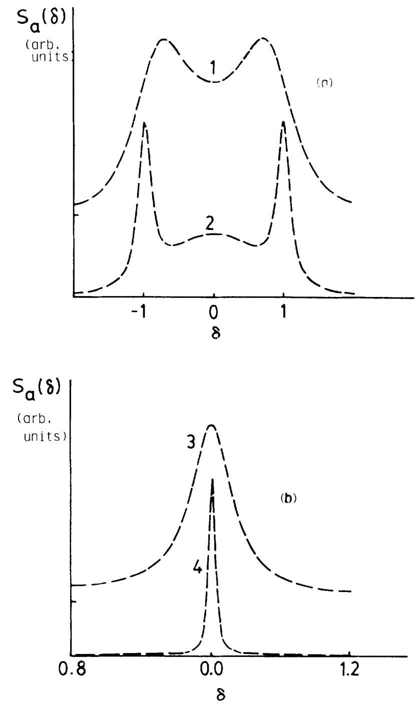

FIG. 5. Same as in Fig. 4 but now $\kappa=1$ and curves 1–4 are for the values of $|G|$ equal to (a) 0 and 0.8 (b) 0.93 and 0.99. For $|G|=0.93$ and 0.99 only one component of the doublet is shown.

VI. EFFECT OF THE ATOMIC OSCILLATOR ON THE PROPERTIES OF THE OUTPUT RADIATION

Finally in this section we investigate how the absorption by the atomic oscillator in the cavity changes the properties of the output radiation. In order to keep the analysis simple we discuss the special case of the oscillator on resonance with the cavity mode and with $\Gamma=0$. We also set in Eq. (2.12) $D=\kappa/2$ and $D_0=0$. The quantum Langevin equation corresponding to (2.12) can be written as

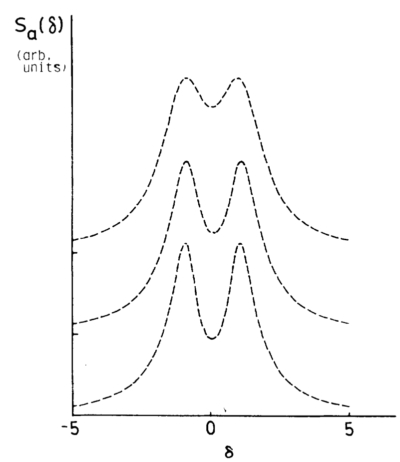

FIG. 6. Same as in Fig. 4 but now $\kappa=1$, $\Gamma=1$. Different curves from top to bottom are for $|G|=0$, 0.8, 0.99.

$$\dot{a} = -igb \;,$$
$$\dot{b} = -ig^* a - iGb^\dagger - \kappa b + F_b(t) \;, \tag{6.1}$$

where the operator force $F_b(t)$ has the properties

$$\langle F_b(t) \rangle = 0 \;,$$
$$\langle F_b^\dagger(t) F_b(t') \rangle = 0 \;,$$
$$\langle F_b(t) F_b^\dagger(t') \rangle = 2\kappa\delta(t-t') \;. \tag{6.2}$$

In order to obtain the properties of the output field we have to relate the output field to the input field. This can be done following the work of Collet and Gardiner.[14] They identify the random force $F_b(t)$ with the vacuum fields $b_{\text{in}\,L}$ and $b_{\text{in}\,R}$ entering the cavity from two sides. They thus set

$$F_b(t) = (\sqrt{\gamma_1} b_{\text{in}\,R} + \sqrt{\gamma_2} b_{\text{in}\,L}), \quad \kappa = \frac{\gamma_1 + \gamma_2}{2} \;, \tag{6.3a}$$

where the "in" field obeys the free-field commutation relations

$$[b_{\text{in}}(\omega), b_{\text{in}}^\dagger(\omega')] = \delta(\omega - \omega') \;. \tag{6.3b}$$

Here γ_1 and γ_2 represent the losses from two sides of the cavity. The output field is related to the cavity field b and the input field via

$$b_{\text{out}\,R}(\omega) = \sqrt{\gamma_1} b(\omega) - b_{\text{in}\,R}(\omega) \;. \tag{6.4}$$

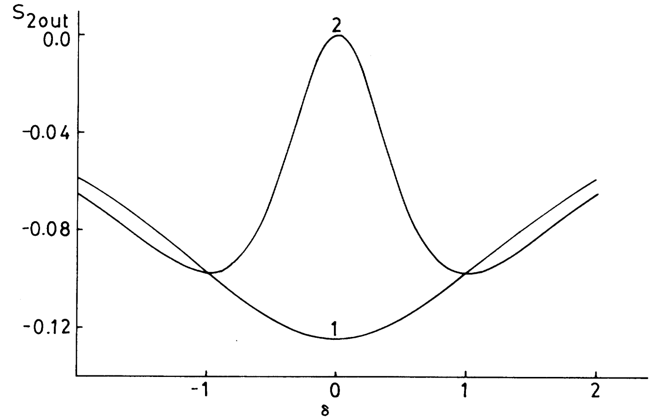

FIG. 7. Squeezing spectrum $S_{2\,\text{out}}(\delta)$ of the output as a function of δ for no atom in the cavity ($g=0$, curve 1) and for an atom in the cavity ($g=1$, curve 2). The parameters are $\Gamma = \nu = 0$, $\kappa = 1$, $|G| = 0.9$.

Thus the properties of the output mode can be studied using the solution of (6.1) and Eqs. (6.2)–(6.4). We quote the final results for the output field,

$$\langle b_{\text{out}\,R}^\dagger(\omega) b_{\text{out}\,R}(\omega') \rangle = \delta(\omega - \omega') S_{\text{out}}(\omega) \;, \tag{6.5}$$

where $S_{\text{out}}(\omega)$ is the spectrum of the output fluctuations,

$$S_{\text{out}}(\omega) = \frac{|G|}{2}\gamma_1 \left[\frac{\delta^2}{\{\delta^4 + [(\kappa - |G|)^2 - |g|^2]\delta^2 + |g|^4\}} - \mathcal{U} \right], \quad \delta = \omega - \omega_b \;. \tag{6.6}$$

where $\mathcal{U}$ is the term with $|G| \to -|G|$. The normally ordered spectra $S_{1\,\text{out}}(\omega)$ and $S_{2\,\text{out}}(\omega)$ of the output quadrature phase $x_1 = (b + b^\dagger)/2$, $x_2 = (b - b^\dagger)/2i$, can be computed similarly. We assume that the phase of G has been absorbed in the definition of the b's and thus x's are defined in terms of new b's. Calculations show that

$$S_{1\,\text{out}}(\omega) = \frac{\frac{\gamma_1}{2}|G|\delta^2}{\delta^4 + \delta^2[(\kappa - |G|)^2 - |g|^2] + |g|^4} \;, \tag{6.7}$$

$$S_{2\,\text{out}}(\omega) = \frac{-\frac{\gamma_1}{2}|G|\delta^2}{\delta^4 + \delta^2[(\kappa + |G|)^2 - |g|^2] + |g|^4} \;. \tag{6.8}$$

The nature of the various spectral profiles thus depends on the roots (3.13). The nature of the fluctuation spectrum (6.6) is similar to that discussed in Sec. III. The squeezing spectrum has some interesting features. It is clear from (6.8) that due to the absorption by the atomic oscillator the maximum squeezing does not occur at $\delta = 0$. In Fig. 7 we show the effect of coupling the cavity radiation with the oscillator on the squeezing spectra.

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APPENDIX: MASTER EQUATION FOR THE ATOMIC OSCILLATOR INTERACTING WITH BROADBAND SQUEEZED RADIATION INSIDE THE CAVITY

In this appendix we derive the master equation corresponding to the Langevin equation (4.4). This is easily converted into the equation for the Wigner function $\Phi_a(\alpha, \alpha^*)$ for the atomic oscillator,

$$\frac{\partial \Phi_a}{\partial t} = \frac{\partial}{\partial a}\left[\left[\frac{|g|^2\kappa}{\kappa^2 - |G|^2}\alpha + \frac{ig^2 G}{\kappa^2 - |G|^2}\alpha^* \right] \Phi_a \right] + \text{c.c.} + d_{\alpha\alpha}\frac{\partial^2}{\partial \alpha^2}\Phi_a + d_{\alpha^*\alpha^*}\frac{\partial^2 \Phi_a}{\partial \alpha^{*2}} + 2d_{\alpha^*\alpha}\frac{\partial^2 \Phi_a}{\partial \alpha^* \partial \alpha} \;. \tag{A1}$$

Using the correspondence rules between the operators and c numbers, Eq. (A1) can be transformed to the equation for the density matrix ρ_a for the atomic oscillator,

$$\frac{\partial\rho_a}{\partial t}=d_{\alpha^*\alpha^*}[a,[a,\rho_a]]+d_{\alpha\alpha}[a^\dagger,[a^\dagger,\rho_a]]-(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)\left[d_{\alpha^*\alpha}+\frac{1}{2}\frac{|g|^2\kappa}{\kappa^2-|G|^2}\right]$$
$$-(aa^\dagger\rho-2a^\dagger\rho a+\rho aa^\dagger)\left[d_{\alpha^*\alpha}-\frac{1}{2}\frac{|g|^2\kappa}{(\kappa^2-|G|^2)}\right]-i\left[\left[\frac{g^2G}{2(\kappa^2-|G|^2)}a^{\dagger 2}+\text{H.c.}\right],\rho_a\right]. \tag{A2}$$

The master equation (A2) resembles the master equation obtained by Gardiner[2] and others[4,17] for an oscillator interacting with a broadband squeezed radiation outside the cavity. In this case the parametric term [the coherent interaction term in (A2)], however, is missing. This term affects the decay modes of the oscillator. In the *absence* of such a term (A2) implies that

$$\langle\dot{a}\rangle=-\frac{|g|^2\kappa}{(\kappa^2-|G|^2)}\langle a\rangle\ . \tag{A3}$$

Thus it is clear that the dynamics of the atomic oscillator depends on whether one considers its interaction inside or outside the cavity producing squeezed radiation. This is because the spectral properties of the squeezed radiation inside and outside the cavity are different.[14]

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