Exact solutions of the cubic and quintic nonlinear Schrödinger equation for a cylindrical geometry

L. Gagnon* and P. Winternitz
Centre de Recherches Mathématiques, Université de Montréal, Case Postale 6128, Succursale A, Montréal, Québec, Canada H3C 3J7
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Exact solutions of the nonlinear Schrödinger equation $i\psi_t + \Delta\psi = a_0\psi + a_1\psi|\psi|^2 + a_2\psi|\psi|^4$, for which initial conditions can be imposed on a cylinder, are presented. A symmetry group of the equation is used to reduce it to an ordinary differential equation which is then solved with the help of a singularity analysis. Solutions are obtained in terms of elementary functions, Jacobi elliptic functions, and Painlevé transcendents.

I. INTRODUCTION

The purpose of this paper is to present exact solutions of the nonlinear Schrödinger equation (NLSE)

$$\begin{aligned} &i\psi_t + \Delta\psi = a_0\psi + a_1\psi|\psi|^2 + a_2\psi|\psi|^4 , \\ &\psi = \psi(x,y,z,t) \in \mathbb{C}, \quad a_i \in \mathbb{R}, \quad \{a_1,a_2\} \neq \{0,0\} \end{aligned} \tag{1.1}$$

for which the initial conditions correspond to a cylindrical geometry. More specifically, we wish to impose Cauchy conditions at some time $t = t_0$ on a cylinder $\rho = \rho_0$,

$$\begin{aligned} &\psi(x,y,z,t)|_{(t,\rho)=(t_0,\rho_0)} = f_1(\theta,z) , \\ &[\nabla\psi(x,y,z,t)]^2|_{(t,\rho)=(t_0,\rho_0)} = f_2(\theta,z) , \\ &x = \rho\cos\theta, \quad y = \rho\sin\theta . \end{aligned} \tag{1.2}$$

The NLSE with a quintic-cubic or simply cubic nonlinearity is physically relevant in many fields. It describes wave propagation in nonlinear and dispersive media. For instance, Eq. (1.1) may describe the nonlinear dynamics of superfluid films for which ψ is the condensate wave function related to the film thickness and to the superfluid velocity.[1] The NLSE (1.1) also figures in the time-dependent Landau-Ginzburg model of phase transitions; in this case the wave function ψ is a complex order parameter.[2-4] Another phenomenon governed by the same equation is the propagation of slowly varying electromagnetic wave envelopes in a plasma.[5] Other applications concern hydrodynamics and nonlinear optics. A similar equation in $1+1$ dimensions, containing some additional nonlinear terms, has been derived by multiple-scale techniques and describes the modulation of water waves in the neighborhood of the critical value $kh = 1.363$ (where k is the wave number and h the depth), for which the periodic wave trains are unstable.[6,7] Furthermore, in the $(1+1)$-dimensional case the NLSE (1.1) is particularly useful in the description of nonlinear wave propagation in optical fibers. In this application ψ is the electric field amplitude and a_1 and a_2 are related to the nonlinear refraction index.[8,9]

The cubic $(1+1)$-dimensional NLSE plays a privileged role in the theory of nonlinear wave propagation. Indeed, it belongs to the class of integrable nonlinear evolution equations that can be interpreted as completely integrable Hamiltonian systems.[10-12] Indeed, historically the equation $i\psi_t + \psi_{xx} = a_1\psi|\psi|^2$ played a role comparable only to that of the Korteweg–de Vries equation in the development of soliton theory.[10-12] In more than one space dimension the cubic NLSE is not integrable: no Lax pair exists, no linear solution techniques are available.

The quintic NLSE is not integrable even in $1+1$ dimensions, still less in higher dimensions. Equation (1.1) with $a_2 \neq 0$ in $1+1$ dimensions has been shown to have solitary wave solutions,[13] but they do not have the stability properties that would justify calling them solitons.[8] To our knowledge no systematic study of the NLSE (1.1) in more than one space dimension exists in the literature and the present article aims at filling in this gap.

Our techniques do not depend on the equation under study being integrable. They consist of a systematic application of group theory to reduce Eq. (1.1) to an ordinary differential equation (ODE) which we then solve analytically whenever possible. The method is called "symmetry reduction" and is simple, straightforward, and mathematically rigorous. It goes back to Lie[14] and is described in many contemporary books.[15-18] It has recently been applied to many different equations, in particular to the field equations of the classical relativistic ϕ^6 field theories,[19,20] the Kadomtsev-Petviashvili equation,[21,22] the Davey-Stewartson equations,[23] the three-wave equations,[24] and other multidimensional equations.[25]

The method consists of several steps. The first is to determine the symmetry group of the considered equation. For the NLSE (1.1) this is the extended Galilei group for $a_1 \neq 0$ and $a_2 \neq 0$. If either a_1 or a_2 vanish, the symmetry group is larger, namely, the Galilei-similitude group, including dilations.[26] The second step is to classify the subgroups of the symmetry group into conjugacy classes. For the Galilei and Galilei-similitude groups this was done in a separate article.[26] Here we are only interested in very specific subgroups, namely, those that have orbits of codimension one, compatible with a cylindrical geometry, in the space of independent variables. The requirement that a solution should be invariant under such a subgroup reduces Eq. (1.1) to an ODE. The final step is to solve the obtained ODE. This we do by one of two methods. If the ODE is of the Painlevé type (i.e., its solutions have no moving critical points) we reduce it to a standard form that can be solved in terms of elementary functions, elliptic functions, or Painlevé transcendents. If, on the other hand, the ODE itself still has a symmetry group, its order can be decreased by making use of that symmetry.

The procedure described above is implemented for the NLSE in Sec. II. In Secs. III and IV we present and discuss the exact solutions for the cubic and quintic-cubic NLSE, respectively. Section V is devoted to conclusions.

II. SYMMETRY GROUP, ITS SUBGROUPS, AND SYMMETRY REDUCTION

A. Symmetry group

The symmetry group of local point transformations of Eq. (1.1) consists of transformations of the form

$$\underline{x}'=\Lambda_g(\underline{x},\psi),\quad \psi'=\Omega_g(\underline{x},\psi)\ ,$$
$$\underline{x}=(\mathbf{r},t)=(x_1,x_2,x_3,t)=(x,y,z,t)\ , \tag{2.1}$$

such that $\psi'(\underline{x}')$ is a solution whenever Λ_g and Ω_g are defined and $\psi(\underline{x})$ is a solution (the subscript g denotes the group parameters). Thus the symmetry group leaves the equation invariant and transforms solutions amongst each other.

Point transformations can thus mix the dependent and independent variables, but by definition the new variables x' and ψ' do not depend on the derivatives of the original variables, e.g., on the ψ_x or ψ_t. A special case of point transformations are fiber-preserving point transformations, for which (2.1) reduces to

$$x'=\Lambda_g(x),\quad \psi'=\Omega_g(x,\psi)\ ,$$

i.e., the new independent variables depend only on the old independent ones (this, for instance, excludes hodograph transformations).

In order to determine the symmetry group, we use an algebraic approach, based on an infinitesimal version of (2.1). We look for the Lie algebra of the symmetry group, realized by vector fields of the form

$$V=\eta_1\partial_x+\eta_2\partial_y+\eta_3\partial_z+\eta_4\partial_t+\phi_1\partial_{u_1}+\phi_2\partial_{u_2}\ , \tag{2.2}$$

where η_i and ϕ_j are real functions of x, y, z, t, u_1, and u_2, where u_1 and u_2 are the real and imaginary parts of ψ, respectively. The functions η_i and ϕ_j satisfy a system of linear partial differential equations, called the "determining equations," obtained by requiring that the second prolongations pr^2V of V should annihilate Eq. (1.1) on the solution set of the equation. We have obtained the symmetry algebra of equation (1.1) in a computer-assisted manner using a MACSYMA program.[27]

The symmetry group for $a_1a_2\neq 0$ turns out to be the extended Galilei group $\tilde{G}$ (Ref. 26). If $a_1=0$, or $a_2=0$, but $(a_1,a_2)\neq(0,0)$, the symmetry group is larger, namely, it is the Galilei-similitude group $\tilde{G}^d$, containing dilations as an additional symmetry.[26]

A convenient basis for the Lie algebra $\tilde{g}$ of the extended Galilei group is given by the translations $\{T,P_i\}$, rotations J_i, proper Galilei boosts K_i, and the total mass operator M. For Eq. (1.1) the basis is realized as

$$T=\partial_t+a_0(u_2\partial_{u_1}-u_1\partial_{u_2}),\quad P_i=\partial_{x_i}\ ,$$
$$J_i=-\epsilon_{ikl}x_k\partial_{x_l},\quad K_i=t\partial_{x_i}-\tfrac{1}{2}x_i(u_2\partial_{u_1}-u_1\partial_{u_2})\ , \tag{2.3}$$
$$M=u_2\partial_{u_1}-u_1\partial_{u_2},\quad i=1,2,3\ .$$

The Lie algebra $\tilde{g}^d$ of the Galilei-similitude group contains an additional element, realized as

$$D=2t\partial_t+x\partial_x+y\partial_y+z\partial_z-\delta(u_1\partial_{u_1}+u_2\partial_{u_2})$$
$$+2a_0t(-u_1\partial_{u_2}+u_2\partial_{u_1})\ , \tag{2.4}$$
$$\delta=\begin{cases}1 & \text{if } a_2=0\\ \tfrac{1}{2} & \text{if } a_1=0\ .\end{cases}$$

Notice that the mass operator M in this case generates a constant change of phase of the function ψ and that time translations and Galilei boosts involve a change of phase of ψ, in addition to their natural action on space time. For further information on the Galilei group and its applications, see Ref. 28–30.

The actual group transformations that leave the NLSE (1.1) invariant are given in Ref. 26. They turn out to be fiber preserving and linear in the dependent variable, namely,

$$\mathbf{r}'=e^{\lambda/2}[R\mathbf{r}-\mathbf{a}+\mathbf{v}(t-t_0)]\ ,$$
$$t'=e^{\lambda}(t-t_0)\ , \tag{2.5}$$
$$\psi'(\mathbf{r}',t')=e^{-\lambda\delta/2}\exp\tfrac{1}{2}i[(\mathbf{v},R(\mathbf{x}-\mathbf{a}))+\tfrac{1}{2}\mathbf{v}^2(t-t_0)+\alpha$$
$$+2a_0(t-t_0)(1-e^{\lambda})]\psi(\mathbf{r},t)\ .$$

In (2.5), R is a rotation matrix, $\mathbf{a}$ represents space translations, t_0 a time translation, $\mathbf{v}$ Galilei boosts to a frame moving with constant velocity $\mathbf{v}$, λ a dilation, and α a constant change of phase.

The method of symmetry reduction for a partial differential equation involves the construction of solutions that are invariant under a subgroup of the symmetry group of the equation. When constructing the symmetry group, no mention was made of any boundary conditions. If these are added to the equation, they will cause a symmetry breaking, i.e., they will reduce the symmetry group to some subgroup (in the extreme case to the identity group, i.e., the symmetry can be completely destroyed).

A complete classification of all subgroups, of the Lie groups $\tilde{G}$ and $\tilde{G}^d$, was performed in an earlier article.[26] Here we are mainly interested in subgroups satisfying the following conditions.

(1) When their action is projected onto the space of in-

dependent variables, they have orbits of codimension 1. Their action on $(\underline{x},\psi)$ has orbits of codimension 3. Such subgroups will provide reductions of Eq. (1.1) to ODE's.

(2) They are compatible with Cauchy conditions imposed on a cylinder at a time $t=t_0$. Hence a basis for the corresponding Lie algebra will involve the basis elements

$$\{J_3+aM, P_3+cM\}, \quad a,c\in\mathbf{R} \tag{2.6a}$$

or

$$\{J_3+aM, K_3+cM\} \tag{2.6b}$$

(and at least one further operator).

It turns out that only three types of subgroups of $\tilde{G}$ and $\tilde{G}^d$ satisfy the preceding conditions. Their Lie algebras are represented by

$$L_1^{a,b}=\{J_3+aM, T+(b-a_0)M, P_3\}, \quad a\geq 0, \quad b\in\mathbf{R} \tag{2.7}$$

$$L_2^{a,b}=\{J_3+aM, P_3, D+bM\}, \quad a\geq 0, \quad b\geq 0 \tag{2.8}$$

$$L_3^{a,b}=\{J_3+aM, K_3, D+bM\}, \quad a\geq 0, \quad b\geq 0 \ . \tag{2.9}$$

For a and b fixed each of the preceding algebras represents a conjugacy class of Lie algebras with conjugacy considered under $\tilde{G}$ for (2.7) and $\tilde{G}^d$ for (2.8) and (2.9) (see Ref. 26). Algebras $L_i^{a,b}$ and $L_i^{a',b'}$ $(i=1,2,3)$ are mutually conjugated if and only if $(a,b)=(a',b')$.

Subgroups with orbits of dimension 4 in (x,y,z,t) space may also be of interest. They will reduce Eq. (1.1) to an algebraic equation that may or may not have nontrivial solutions. The only subgroup of this type that is of interest in the present contest is generated by

$$L_4^{a,b}=\{D+bM, T, P_3, J_3+aM\}, \quad a\geq 0, \quad b\geq 0 \tag{2.10}$$

with conjugacy considered under $\tilde{G}^d$.

B. Symmetry reduction

In order to perform symmetry reduction using some specific subgroup G_0 of the symmetry group of an equation, we must first find the invariants of G_0 and then rewrite the equation in terms of them. Let $\{X_i\}$ be some basis for the Lie algebra of G_0. Invariants are obtained by solving the equations

$$X_i Q(\mathbf{x},t,u_1,u_2)=0, \quad i=1,\ldots,l \ .$$

For the groups we are interested in and for Eq. (1.1) we shall always have three elementary functionally independent invariants that can be chosen in the form

$$I_1\equiv\xi(\mathbf{x},t) \ ,$$
$$I_2\equiv\Phi=\psi\alpha^{-1}(\mathbf{x},t) \ ,$$
$$I_3\equiv\Phi^*=\psi^*\alpha^{*-1}(\mathbf{x},t) \ .$$

These permit us to write the solution of the NLSE as

$$\psi(\mathbf{x},t)=\Phi(\xi)\alpha(\mathbf{x},t)=M(\xi)e^{i\chi(\xi)}\alpha(\mathbf{x},t) \ , \tag{2.11}$$

where $M(\xi)$ and $\chi(\xi)$ are the modulus and phase of $\Phi(\xi)$, respectively. Substituting (2.11) into (1.1) we obtain a complex second-order ODE for $\Phi(\xi)$ that can be rewritten as a pair of real equations for $M(\xi)$ and $\chi(\xi)$. Running through this procedure for the algebra (2.7), we find that (2.11) specializes to

$$\begin{gathered}\psi(\mathbf{r},t)=M(\rho)e^{\chi(\rho)}e^{ai\theta}e^{-ibt} \ , \\ \rho=(x^2+y^2)^{1/2}, \quad \theta=\arctan y/x \ .\end{gathered} \tag{2.12}$$

The equation for $\chi(\rho)$ can be solved and we obtain

$$\chi=\int\frac{S_0}{\rho M^2}d\rho+\chi_0 \ , \tag{2.13}$$

$$\ddot{M}-\frac{S_0^2}{M^3\rho^2}+\frac{\dot{M}}{\rho}-\frac{a^2}{\rho^2}M=(a_0-b)M+a_1M^3+a_2M^5 \ , \tag{2.14}$$

where S_0 and χ_0 are real integration constants.

The algebras (2.8) and (2.9) both involve dilations and can hence only be used when either $a_1=0$, or $a_2=0$, in (1.1). If this is the case, then the algebra $L_2^{a,b}$ of (2.8) and $L_3^{a,b}$ of (2.9) lead to the expressions

$$\psi(\mathbf{r},t)=M(\xi)e^{i\chi(\xi)}t^{-\delta/2}e^{i[-a_0t+a\theta-(b/2)\ln t]}, \quad \xi=\frac{t}{\rho^2} \tag{2.15}$$

and

$$\psi(\mathbf{r},t)=M(\xi)e^{i\chi(\xi)}t^{-\delta/2}e^{i[-a_0t+z^2/4t+a\theta-(b/2)\ln t]} \ , \quad \xi=\frac{t}{\rho^2} \tag{2.16}$$

respectively, with δ as in (2.4).

The functions $M(\xi)$ and $\chi(\xi)$ satisfy a coupled system of ODE's which we write as

$$\ddot{\chi}+\left[\frac{2\dot{M}}{M}+\frac{1}{\xi}\right]\dot{\chi}+\frac{\dot{M}}{4M\xi^2}+\frac{\mu}{8\xi^3}=0 \ , \tag{2.17a}$$

$$\begin{gathered}4\xi^3(\ddot{M}-M\dot{\chi}^2)+4\xi^2\dot{M}-\xi M\dot{\chi}+\left[\frac{b}{2}-a^2\xi\right]M =a_1M^3+a_2M^5 \ , \\ \mu=\begin{cases}-\delta & \text{for } (2.14) \\ 1-\delta & \text{for } (2.15) \ ,\end{cases} \\ \delta=\begin{cases}1 & \text{for } a_2=0 \\ \frac{1}{2} & \text{for } a_1=0 \ ,\end{cases} \\ a_1a_2=0 \ .\end{gathered} \tag{2.17b}$$

The system (2.17) can be decoupled and we obtain an expression for $\chi(\xi)$ in terms of M together with a nonlinear ODE for $M(\xi)$. This equation is of second order for the cubic NLS equation and the reduction (2.15). It is of third order in all other cases.

The four-dimensional Lie algebra (2.10) leads to the expression

$$\psi(\mathbf{r},t)=C\rho^{-\delta}e^{i(a\theta-a_0t-b\ln\rho)} \ , \tag{2.18a}$$

where C is a constant. Substituting into the NLSE (1.1) we find

$$b=0,\quad |C|^4=\frac{1-4a^2}{4a_2},\quad \delta=\tfrac{1}{2}\quad \text{for } a_2\neq 0,\quad a_1=0 \tag{2.18b}$$

$$b=0,\quad |C|^2=\frac{1-a^2}{a_1},\quad \delta=1\quad \text{for } a_2=0,\quad a_1\neq 0\ . \tag{2.18c}$$

C. Singularity analysis

Our next task is to solve the ODE's (2.14) and (2.17), obtained by symmetry reduction. In general this is a formidable task, since the equations are nonlinear and complicated. Two systematic approaches are available and we make use of both of them. One consists of finding the symmetry group of the obtained ODE, if one exists, and then using it to decrease the order of the equation. The second method, which we found to be more fruitful, is to determine whether the ODE happens to belong to a class of integrable nonlinear ODE's, namely, the class of Painlevé-type equations.[31,32]

An ODE is said to have the Painlevé property if none of its solutions has movable critical points, i.e., singularities, other than poles, the position of which depends on the initial conditions. Equations with the Painlevé property are in general much easier to solve than other ones. In particular, Painlevé and Gambier have classified all equations of the form

$$\ddot{y}=F(x,y,\dot{y})\ , \tag{2.19a}$$

where F is rational in y and $\dot{y}$ and analytic in x.[31,32]

Our procedure is the following.

(1) First submit the equation to the "Painlevé test,"[33] in order to determine whether it satisfies certain necessary conditions for having the Painlevé property. This test is algorithmic and has been implemented as a MACSYMA program.[34]

(2) If the equation passes this test then we look for a transformation of the form

$$y(x)=\frac{\alpha w(\zeta)+\beta}{\gamma w(\zeta)+\delta},\quad \zeta=\zeta(x) \tag{2.19b}$$

where α, β, γ, δ, and ζ are functions of x that takes the equation into one of 50 standard forms.[31,32] Of these 50, 44 can be integrated in terms of known transcendents or at least be reduced to first-order equations. The remaining six define the so-called Painlevé transcendents $P_I,\ldots,P_V$.

We concentrate here on the second-order ODE's obtained by symmetry reduction. The one obtained from the reductions (2.15) does not pass the Painlevé test for any values of the parameters involved (except $a_1=a_2=0$, which is of no interest).

Equation (2.14) itself does not directly have the Painlevé property. The test, however, indicates that if we set

$$M(\rho)=[H(\rho)]^{1/2},\quad H\geq 0 \tag{2.20}$$

then the equation for $H(\rho)$ may have the Painlevé property for certain values of the parameters. More specifically, the Painlevé test is passed for

$$a_2\neq 0,\quad a^2=\tfrac{1}{16},\quad a_0-b=\frac{3}{16}\frac{a_1^2}{a_2} \tag{2.21}$$

and for

$$a_2=0,\quad a_1\neq 0,\quad a^2=\tfrac{1}{9}\ . \tag{2.22}$$

We further note that Eq. (2.13) is invariant under dilations if

$$S_0=0,\quad a_0=b,\quad a_1a_2=0\ . \tag{2.23}$$

The symmetry generator has the form

$$\chi=\xi\,\partial_\xi-\delta M\,\partial_M \tag{2.24}$$

and will be used below.

III. SOLUTIONS OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION

We now consider Eq. (1.1) with $a_1\neq 0$, $a_2=0$, and the corresponding reduced equation (2.14). We shall obtain solutions either when (2.14) has the Painlevé property, i.e., $a^2=\frac{1}{9}$, or when it has a nontrivial symmetry, i.e., $S_0=0$, $a_0=b$.

Consider first the case $a^2=\frac{1}{9}$ and put $M(\rho)=[H(\rho)]^{1/2}$ as in (2.20). The function H satisfies

$$\ddot{H}=\frac{\dot{H}^2}{2H}-\frac{1}{\rho}\dot{H}+\frac{2}{9\rho^2}H+2(a_0-b)H+2a_1H^2+\frac{2S_0^2}{\rho^2 H}\ . \tag{3.1}$$

Since (3.1) passes the Painlevé test we transform it to standard form, following the procedure described, e.g., by Ince.[31] The result is that we put

$$H(\rho)=\lambda(\rho)W(\eta),\quad \eta=\eta(\rho) \tag{3.2}$$

and choose

$$\eta=k_0\rho^{2/3},\quad \lambda=\frac{8}{9a_1}k_0^2\rho^{-2/3}\ , \tag{3.3}$$

where k_0 is a real constant, we obtain a "standard" equation for $W(\eta)$, namely,

$$\ddot{W}=\frac{1}{2W}\dot{W}^2+4W^2+\frac{9^3(S_0a_1)^2}{2^7k_0^6}\frac{1}{W}+\frac{9}{2k_0^3}(a_0-b)\eta W\ . \tag{3.4}$$

Equation (3.4) represents four different cases.

(1) $a_0=b$, $S_0=0$. Equation (3.4) then reduces to the PXVIII equation[31] and its first integral is

$$\dot{W}^2=4W(C+W^2)\ . \tag{3.5}$$

(2) $a_0=b$, $S_0\neq 0$. We choose

$$k_0=i\tfrac{3}{2}(a_1S_0)^{1/3} \tag{3.6}$$

and obtain a special case of PXXXIII. Its first integral is

$$\dot{W}^2=4W^3+4CW+1 \ . \tag{3.7}$$

In (3.5) and (3.7), C is an integration constant and depending on its value we get solutions in terms of either elementary functions or Jacobi elliptic ones. The physical meaning of C depends on the model under consideration; usually it is related to the energy of the system.

(3) $a_0 \neq b$, $S_0=0$. We choose

$$k_0^3=\frac{9(a_0-b)}{4} \tag{3.8}$$

and obtain the equation PXX

$$\ddot{W}=\frac{1}{2W}\dot{W}^2+4W^2+2\eta W \ . \tag{3.9}$$

Putting

$$W=u^2 \tag{3.10}$$

we obtain a special case of one of the six irreducible Painlevé equations

$$\ddot{u}=2u^3+\eta u \tag{3.11}$$

that is solved in terms of the second Painlevé transcendent $u(\eta)=P_{\rm II}(\eta)$.

(4) $a_0 \neq b$, $S_0 \neq 0$. We put

$$k_0^3=-\frac{9(a_0-b)}{2}, \quad \alpha=i\epsilon\frac{3S_0 a_1}{4(a_0-b)}, \quad \epsilon=\pm 1 \tag{3.12}$$

and obtain the equation PXXXIV

$$\ddot{W}=\frac{1}{2W}\dot{W}^2+4\alpha W-\eta W-\frac{1}{2W} \ . \tag{3.13}$$

The general solution of (3.13) is expressed in terms of the second Painlevé transcendent. Indeed, put

$$W=\frac{1}{2\alpha}(\dot{V}+V^2+\tfrac{1}{2}\eta) \ ; \tag{3.14}$$

then V satisfies

$$\ddot{V}=2V^3+\eta V-2\alpha-\tfrac{1}{2} \ , \tag{3.15}$$

which is solved by $V=P_{\rm II}(\eta)$. Let us now turn to explicit solutions of the cubic NLSE.

We first notice that Eq. (2.14) with $a_2=0$ allows constant solutions. Substituted into (2.12) these provide the solutions $\psi(\mathbf{r},t)=0$ and

$$\psi(\mathbf{r},t)=\left[\frac{b-a_0}{a_1}\right]^{1/2} e^{i\chi_0}e^{-ibt}, \quad \frac{b-a_0}{a_1}>0 \ . \tag{3.16}$$

Equation (3.4) also allows constant solutions for $a_0=b$. Returning to the wave function ψ we obtain

$$\psi(\mathbf{r},t)=S_0^{1/3}\left[-\frac{1}{a_1}\right]^{1/6}\left[\frac{1}{\rho}\right]^{1/3} \times \exp i[\tfrac{3}{2}S_0^{1/3}(-a_1)^{1/3}\rho^{2/3} +\tfrac{1}{3}\theta-a_0 t+\chi_0], \quad a_1<0 \ . \tag{3.17}$$

Notice that the norm of solution (3.16) is constant whereas that of (3.17) varies with ρ and has a branch point singularity at $\rho=0$. The phase of (3.17) depends on ρ, θ, and t.

Further solutions are obtained from the Painlevé-type ODE's. Consider first Eq. (3.5). It will yield solutions of the form

$$\psi(\mathbf{r},t)=M(\rho)e^{i(\chi_0+\frac{1}{3}\theta-a_0 t)} \ ,$$
$$M=\frac{2\sqrt{2}}{3}k_0\left[\frac{1}{\rho}\right]^{1/3}\left[\frac{W(\eta)}{a_1}\right]^{1/2}, \quad \eta=k_0\rho^{2/3}+\eta_0 \ . \tag{3.18}$$

The relation between $\dot{W}$ and W given by Eq. (3.5) can be illustrated by the phase diagrams of Figs. 1(a), 1(b), and 1(c) for $C=0$, $C<0$, and $C>0$, respectively. Since M must be real $W(\eta)/a_1$ must be positive (for k_0 real). Since $\dot{W}^2$ must be positive, we see from Fig. 1 that we have $0 \le W < \infty$ for $C=0$ or $C=p^2>0$ and $-p \le W \le 0$ or $p \le W < \infty$ for $C=-p^2<0$. Thus only the case $C<0$ will provide finite solutions for W.

Solving (3.5), we obtain solutions (3.18) with

$$M=\frac{2\sqrt{2}}{3}\frac{1}{\sqrt{a_1}}\left[\frac{1}{\rho}\right]^{1/3}\frac{1}{\rho^{2/3}+c_2}, \quad a_1>0, \quad C=0 \tag{3.19}$$

$$M=\frac{2c_1}{3}\frac{1}{\sqrt{-a_1}}\left[\frac{1}{\rho}\right]^{1/3} cn\left[c_1\rho^{2/3}+c_2, \frac{1}{\sqrt{2}}\right], \quad a_1<0, \quad C<0 \tag{3.20}$$

$$M=\frac{2c_1}{3}\frac{1}{\sqrt{a_1}}\left[\frac{1}{\rho}\right]^{1/3}\left[cn\left[c_1\rho^{2/3}+c_2, \frac{1}{\sqrt{2}}\right]\right]^{-1}, \quad a_1>0, \quad C<0 \tag{3.21}$$

$$M=\frac{2\sqrt{2}c_1}{3}\frac{1}{\sqrt{a_1}}\left[\frac{1}{\rho}\right]^{1/3} tn\left[c_1\rho^{2/3}+c_2, \frac{1}{\sqrt{2}}\right] \times dn\left[c_1\rho^{2/3}+c_2, \frac{1}{\sqrt{2}}\right], \quad a_1>0, \quad C>0 \tag{3.22}$$

respectively, where c_1 and c_2 are real constants.

The only nonperiodic solution is (3.19) [when the curve on Fig. 1(a) has an inflection point at $W=0$]; it is singular for $\rho=0$ and $\rho^{2/3}=-c$ (which is a physical point if $c<0$). Solution (3.20) is periodic and finite everywhere, except for $\rho=0$. The periodic solutions (3.21) and (3.22) both have poles for

$$\tau=c_1\rho^{2/3}+c_2=(2n+1)K, \quad n\in\mathbb{Z} \tag{3.23}$$

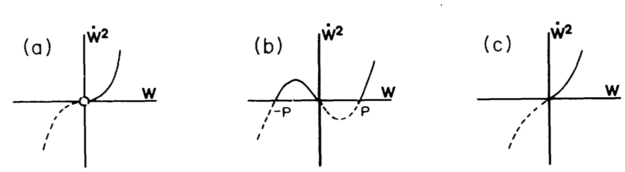

FIG. 1. Phase diagram for Eq. (3.5). (a) $C=0$, $\dot{W}^2=W=0$ is an inflection point. (b) $C=-p^2<0$. (c) $C=p^2>0$.

where $4K$ is the period of the Jacobi elliptic functions.[35]

Further solutions are obtained from Eq. (3.7). The solution of the cubic NLSE will be

$$\psi(\mathbf{r},t)=M(\rho)e^{i\chi(\rho)}e^{i(\theta/3-a_0t)} ,$$

$$M=\sqrt{2}S_0^{1/3}\left[\frac{1}{-a_1}\right]^{1/6}\left[\frac{1}{\rho}\right]^{1/3}[W(\eta)]^{1/2} , \tag{3.24}$$

$$\eta=i\tfrac{3}{2}(a_1S_0)^{1/3}\rho^{2/3}+\eta_0 ,$$

with $W(\eta)$ satisfying (3.7) and $\chi(\rho)$ given by (2.12).

Since η is pure imaginary, we shall put

$$W(\eta)=\Omega(\zeta), \quad \eta=i\zeta \tag{3.25}$$

where Ω satisfies the equation

$$\dot{\Omega}^2=-(4\Omega^3+4C\Omega+1) . \tag{3.26}$$

In Fig. 2 we plot $\dot{\Omega}$ as a function of Ω.

Figure 2(a) corresponds to the case when the polynomial on the right-hand side of (3.26) has a double root, i.e.,

$$W_1=W_2=\tfrac{1}{2}, \quad W_3=-1, \quad C=-\tfrac{3}{4} . \tag{3.27a}$$

Figure 2(b) corresponds to three different real roots,

$$W_3<0<W_2<\tfrac{1}{2}<W_1 ,$$

$$W_{2,3}=-\frac{1}{2W_1}[W_1^2\mp(W_1^4+W_1)^{1/2}] ,$$

$$-\infty<C=-\frac{4W_1^2+1}{4W_1}<-\tfrac{3}{4} . \tag{3.27b}$$

Figure 2(c) corresponds to one real and two complex roots, i.e.,

$$W_1=-2r, \quad W_{2,3}=r\pm is, \quad 0<r<\tfrac{1}{2}$$
$$C=-4r^2+\frac{1}{8r}, \quad s=[(1-8r^3)/8r]^{1/2} . \tag{3.27c}$$

From Fig. 2, we see that the solutions will always be periodic and that a nonsingular solution will exist in the case $C<-\frac{3}{4}$ only.

Solving Eq. (3.7) in the three different cases and in case of $C<-\frac{3}{4}$ for two different regions, we obtain the following results:

$$M(\rho)=S_0^{1/3}\left[\frac{1}{a_1}\right]^{1/6}\left[\frac{1}{\rho}\right]^{1/3}\left[\frac{5+\sin\tau}{1-\sin\tau}\right]^{1/2} ,$$

$$a_1>0 \tag{3.28}$$

$$\chi(\rho)=\frac{1}{\sqrt{6}}\left[-\tau+\frac{1}{\sqrt{6}}\arctan\frac{5\tan\tau/2+1}{2\sqrt{6}}\right]+\chi_0 ,$$

$$C=-\tfrac{3}{4}$$

$$\tau=3\sqrt{3/2}(a_1S_0)^{1/3}\rho^{2/3}+c_2 .$$

The solution is clearly periodic and singular for $\tau=\pi/2+2n\pi$, as well as for $\rho=0$.

For $C<-\frac{3}{4}$, we obtain a finite solution for Ω satisfying $W_2\le\Omega\le W_1$. It is given by

$$M=\sqrt{2}\left[\frac{1}{-a_1}\right]^{1/6}\left[\frac{S_0}{\rho}\right]^{1/3}$$
$$\times[W_1-(W_1-W_2)sn^2(\tau,k)]^{1/2}, \quad a_1<0$$

$$\chi=-\tfrac{1}{2}(2W_1+W_2)^{-1/2}\int\frac{d\tau}{W_1-(W_1-W_2)sn^2(\tau,k)}+\chi_0 , \tag{3.29}$$

$$\tau=(2W_1+W_2)^{1/2}\tfrac{3}{2}(a_1S_0)^{1/3}\rho^{2/3}+c_2 ,$$

$$k=\left[\frac{W_1-W_2}{2W_1+W_2}\right]^{1/2} .$$

The integral in the expression for χ in (3.29) is a standard elliptic integral.[35]

The other solution for $C<-\frac{3}{4}$ is singular and corresponds to $-\infty<\Omega<W_3<0$. In this case, we have

$$M=\sqrt{2}S_0^{1/3}\left[\frac{1}{a_1}\right]^{1/6}\left[\frac{1}{\rho}\right]^{1/3}$$
$$\times\frac{[W_1+W_2+W_2sn^2(\tau,k)]^{1/2}}{cn(\tau,k)}, \quad a_1>0$$

$$\chi=\tfrac{1}{2}(2W_1+W_2)^{-1/2}\int\frac{cn^2(\tau,k)d\tau}{W_1+W_2+W_2sn^2(\tau,k)}+\chi_0 , \tag{3.30}$$

$$\tau=(2W_1+W_2)^{1/2}\tfrac{3}{2}(a_1S_0)^{1/3}\rho^{2/3}+c_2 ,$$

$$k=\left[\frac{W_1-W_2}{2W_1+W_2}\right]^{1/2} .$$

The amplitude M in (3.30) has poles at $\tau=(2n+1)K$, where $4K$ is the real period.

For $C>-3/4$, we get a further singular solution,

$$M(\rho)=\sqrt{2}S_0^{1/3}\left[\frac{1}{a_1}\right]^{1/6}\left[\frac{1}{\rho}\right]^{1/3}$$
$$\times\left[\frac{A+2r-(A-2r)cn(\tau,k)}{1+cn(\tau,k)}\right]^{1/2}, \quad a_1>0$$

$$\chi(\rho)=\frac{1}{4\sqrt{A}}\int d\tau\frac{1+cn(\tau,k)}{A+2r-(A-2r)cn(\tau,k)}$$
$$+\chi_0 , \tag{3.31}$$

$$k=\left[\frac{A-3r}{2A}\right]^{1/2}, \quad A=8r^2+\frac{1}{8r}, \quad 0<r<\tfrac{1}{2} .$$

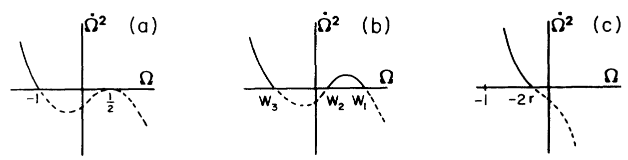

FIG. 2. Phase diagram for Eq. (3.26). (a) $C=-\frac{3}{4}$. (b) $C<-\frac{3}{4}$. (c) $C>-\frac{3}{4}$; for $-\frac{3}{4}<C\le 0$ the curve will have one more critical point at $\Omega^2=-C/3$ (not shown).

The singularities of $M(\rho)$ are at

$$\rho=0, \quad \tau=2K+4nK \ . \tag{3.32}$$

For $a_0-b\neq 0$, we obtain solutions in terms of the Painlevé transcendent $P_{\rm II}$. In the case of Eq. (3.9), they will take the form

$$\begin{aligned}\psi(\mathbf{r},t)&=\frac{2\sqrt{2}}{3\sqrt{a_1}}\left[\frac{9(a_0-b)}{4\rho}\right]^{1/3}\\&\quad\times P_{\rm II}(\eta)e^{i\theta/3}e^{-ibt}e^{i\chi_0}\ ,\\ \eta&=\left[\frac{9(a_0-b)}{4}\right]^{1/3}\rho^{2/3}\ .\end{aligned}\tag{3.33}$$

For $a_1>0$, the parameters in $P_{\rm II}(\eta)$ must be so chosen that $P_{\rm II}$ is real; for $a_1<0$, on the contrary, $P_{\rm II}$ must be chosen to be pure imaginary.

Finally, Eq. (3.13) also provides solutions in terms of $P_{\rm II}(\eta)$. They are more difficult to analyze and we shall not go into them here.

The case when Eq. (2.14) does not have the Painlevé property, but does have a symmetry group is $S_0=0$, $a_0=b$, and $a^2\neq\frac{1}{9}$. It provides us with implicit solutions, i.e., replaces a differential equation by a functional one.

Indeed, putting

$$y=\rho M, \quad w(y)=\ln M, \quad z=\frac{dw}{dy}\ ,$$

we obtain a Riccati equation for z which we transform into a Bessel equation. Returning to M, we obtain the functional equation

$$\begin{aligned}M=e^{w_0}[&c_1J_{a\sqrt{2}}(\sqrt{-2a_1}\rho M)\\&+c_2Y_{a\sqrt{2}}(\sqrt{-2a_1}\rho M)]^{1/2}\ ,\end{aligned}\tag{3.34}$$

where w_0, c_1, and c_2 are constants and J and Y are two independent solutions of the Bessel equation.

As stated above, Eqs. (2.17) corresponding to the solution (2.15) and (2.16) do not have the Painlevé property, nor any nontrivial symmetry group. We can, however, look for solutions with a constant value of M. For $a_2=0$, $s=1$, we find that $M=0$ implies $\mu=0$, $a=0$, and $\dot{\chi}=0$. In this case, we obtain the solution

$$\psi(\mathbf{r},t)=\left[\frac{b}{2a_1}\right]^{1/2}t^{-1/2}e^{i[-a_0t+z^2/4t-(b/2)\ln t+\chi_0]}, \quad \frac{b}{a_1}>0\ . \tag{3.35}$$

We sum up this section by noticing that for $a_1<0$ we have obtained solutions (3.17), (3.20), and (3.29). All of these are finite, except for a $\rho^{-1/3}$-type point singularity at $\rho=0$. For $a_1>0$, we have solutions (3.19), (3.21), (3.22), (3.28), (3.30), and (3.31). They all have $\rho^{-1/3}$-type singularities at $\rho=0$. Solution (3.19) is otherwise regular for $\rho>0$ if we take $c>0$. All the other solutions are periodic and have either one or two singularities per period. Solutions (3.16) and (3.35) exist for both signs of a_1. As for the implicit solution (3.34) and the solutions in terms of the Painlevé transcendent $P_{\rm II}$, we have not analyzed their reality properties. Finally, (2.18a) and (2.18c) provides a solution for $a_1<0$, $a^2>1$ or $a_1>0$, $a^2<1$.

IV. SOLUTIONS OF THE QUINTIC NONLINEAR SCHRÖDINGER EQUATION

We now have $a_2\neq 0$ in the NLSE (1.1) and in all reduced equations. Let us first look at the reduction (2.12) and the ODE (2.14). The results of the Painlevé test of Sec. II indicate that we should perform the transformation (2.20), i.e., $M(\rho)=[H(\rho)]^{1/2}$. We obtain the equation

$$\begin{aligned}\ddot{H}=\frac{\dot{H}^2}{2H}-\frac{1}{\rho}\dot{H}+\frac{2a^2}{\rho^2}H+\frac{2S_0^2}{\rho^2H}+2(a_0-b)H+2a_1H^2\\+2a_2H^3\ .\end{aligned}\tag{4.1}$$

This equation, as indicated above, passes the Painlevé test if a and b satisfy (2.21). Moreover, for $S_0=a=0$ (4.1) has constant solutions. They provide solutions of the quintic NLSE of the form

$$\psi(\mathbf{r},t)=M_0e^{i\chi_0}e^{-ibt}\ , \tag{4.2}$$

with χ_0 and b constants. The norm M_0 in (4.2) satisfies $M_0=0$ or

$$M_0^2=\frac{-a_1+\epsilon[a_1^2-4a_2(a_0-b)]^{1/2}}{2a_2}, \quad \epsilon=\pm1\ . \tag{4.3}$$

Since we have $M^2\geq 0$, the constants in (4.3) must satisfy one of the following relations:

$$\begin{aligned}&a_2>0,\quad a_1>0,\quad b\geq a_0,\quad \epsilon=+1\ ;\\&a_2<0,\quad a_1\leq 0,\quad b\leq a_0,\quad \epsilon=-1\ ;\\&a_2>0,\quad a_1<0,\quad a_0-\frac{a_1^2}{4a_2}\leq b\leq a_0,\quad \epsilon=\pm1\ ;\\&a_2>0,\quad a_1\leq 0,\quad a_0<b,\quad \epsilon=+1\ ;\\&a_2<0,\quad a_1>0,\quad a_0\leq b\leq a_0-\frac{a_1^2}{4a_2},\quad \epsilon=\pm1\ ;\\&a_2<0,\quad a_1>0,\quad b<a_0,\quad \epsilon=-1\ .\end{aligned}\tag{4.4}$$

We see that, in two cases, we have the possibility of two signs: $\epsilon=\pm1$. Physically, this will correspond to a degenerate ground state of the system (in addition to the fact that the zero solution $M_0=0$ also exists and that the phase χ_0 is arbitrary). We now proceed to transform (4.1) to its standard form, which turns out to be different for $a_1=0$ and $a_1\neq 0$.

As in the case of the cubic NLSE, we perform the transformation (3.2). For $a_1\neq 0$, we choose

$$\lambda=\tfrac{1}{2}\epsilon\sqrt{\epsilon a_1}\left[\frac{3}{4a_2}\right]^{3/4}\frac{1}{\sqrt{\rho}}, \quad \eta=\sqrt{\epsilon a_1}\left[\frac{3}{4a_2}\right]^{1/4}\sqrt{\rho}\ , \tag{4.5}$$

and find that $W(\eta)$ satisfies the equation PXXXI, i.e., one of the six irreducible Painlevé-type equations,

$$\ddot{W}=\frac{1}{2W}\dot{W}^2+\tfrac{3}{2}W^3+4\eta W^2+2\eta^2 W+\frac{2^9}{9}\left[\frac{S_0 a_2}{a_1}\right]^2\frac{1}{W} . \tag{4.6}$$

We thus obtain a solution of the NLSE in the form (2.12) with

$$M(\rho)=\lambda^{1/2}(\rho)[P_{\rm IV}(\eta)]^{1/2} , \tag{4.7}$$

where λ and η are given in (4.5) and $P_{\rm IV}(\eta)$ is the fourth Painlevé transcendent.[31] The phase $\chi(\rho)$ is then given by the integral (2.13).

For $a_1=0$ (and $a_2\neq 0$), the situation is somewhat simpler, in that we obtain an equation for elliptic functions. Indeed, putting

$$\begin{aligned} &M(\rho)=\lambda_0^{1/2}\rho^{-1/4}W(\eta)^{1/2} ,\\ &\eta=4\epsilon\left[\frac{a_2}{3}\right]^{1/2}\lambda_0\rho^{1/2}+\eta_0 ,\\ &\epsilon=\pm 1 ,\\ &a^2=\tfrac{1}{16} ,\\ &a_0=b ,\\ &\lambda_0\in\mathbb{R} , \end{aligned} \tag{4.8}$$

where λ_0 and η_0 are constants, we obtain the equation PXXX

$$\ddot{W}=\frac{\dot{W}^2}{2W}+\tfrac{3}{2}W^3+\frac{3S_0^2}{2a_2\lambda_0^4 W} . \tag{4.9}$$

Equation (4.9) can be integrated once to yield

$$\dot{W}^2=W^4+4CW-3\frac{S_0^2}{a_2\lambda_0^4}\equiv P(W) , \tag{4.10}$$

where C is an integration constant.

First of all, Eq. (4.9) allows constant solutions which provide the following solution of the quintic NLSE

$$\begin{aligned} &\psi(\mathbf{r},t)=\left[\frac{S_0^2}{-a_2\rho^2}\right]^{1/8}\exp i\left[2(-a_2S_0^2\rho^2)^{1/4}+\frac{\theta}{4}-a_0t+\chi_0\right] ,\\ &a_1=0,\quad a_2<0 . \end{aligned} \tag{4.11}$$

Further solutions are obtained by solving Eq. (4.10). The solutions are very different for $a_2>0$, or $a_2<0$, and we treat the two cases separately.

Consider first the case $a_2>0$. The quartic polynomial $P(W)$ on the right-hand side of (4.10) has four real roots only if $C=0$ and $S_0=0$. In this case, we obtain

$$\begin{aligned} &\psi(\mathbf{r},t)=Me^{i(\theta/4-a_0t+\chi_0)} ,\\ &M=\frac{1}{2}\left[\frac{3}{a_2\rho}\right]^{1/4}\left[\frac{1}{\epsilon\sqrt{\rho}+c_0}\right]^{1/2} ,\quad \epsilon=\pm 1\\ &\epsilon\sqrt{\rho}+c_0>0,\quad a_2>0,\quad a_1=0 . \end{aligned} \tag{4.12}$$

The norm M is singular at $\rho=0$ and also for $\sqrt{\rho}=-\epsilon c_0$, which is in the physical region of ρ if we have $\epsilon c_0<0$. The corresponding phase diagram $(\dot{W}^2, W)$ is given in Fig. 3(a).

The polynomial $P(W)$ may have two real roots and two complex ones. They satisfy

$$\begin{aligned} &W_1\equiv r,\quad W_2\equiv s,\quad W_{3,4}=-\frac{r+s}{2}\pm iq ,\\ &q=\tfrac{1}{2}(3r^2+3s^2+2rs)^{1/2},\quad r\le 0\le s,\quad r<s\\ &C=-\tfrac{1}{4}(r+s)(r^2+s^2) ,\\ &S_0^2=-\frac{a_2\lambda_0^4}{3}rs(r^2+s^2+rs)>0 . \end{aligned} \tag{4.13}$$

The corresponding phase diagram is in Fig. 3(b), where, in particular, we may have $0=r<s$ or $r<s=0$. The solution of the NLSE, in this case, is

$$\psi(\mathbf{r},t)=\left[\frac{1}{\rho}\right]^{1/4}[\lambda_0 W(\xi)]^{1/2}e^{i\chi(\xi)}e^{i(\theta/4-a_0t+\chi_0)} , \tag{4.14}$$

with

$$W(\xi)=\frac{-rA+sB+(rA+sB)cn(\xi,k)}{(A+B)cn(\xi,k)-A+B} , \tag{4.15a}$$

$$\begin{aligned} &\chi(\xi)=\left[\frac{-rs(r^2+s^2+rs)}{AB}\right]^{1/2}\times\frac{1}{2\epsilon}\int\frac{(A+B)cn(\xi,k)-A+B}{-rA+sB+(rA+sB)cn(\xi,k)}d\xi ,\\ &k^2=\frac{(A+B)^2-(s-r)^2}{4AB} ,\\ &A=(r^2+2rs+3s^2)^{1/2},\quad B=(3r^2+2rs+s^2)^{1/2} ,\\ &\xi=\sqrt{AB}\,4\epsilon\left[\frac{a_2}{3}\right]^{1/2}\lambda_0\rho^{1/2}+\xi_0 . \end{aligned} \tag{4.15b}$$

The integral in the expression for $\chi(\xi)$ in (4.15) is a standard one, reducible to elliptic integrals (see formula

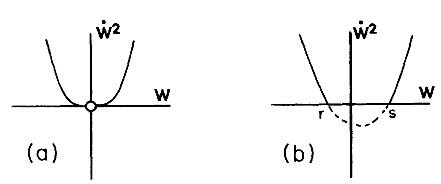

FIG. 3. Phase diagram for Eq. (4.10). (a) $C=S_0=0$. (b) $(C,S_0)\neq(0,0)$; the curve also has a critical point at $W^3=\frac{1}{4}(r+s)(r^2+s^2)$ (not shown).

361.62 in Ref. 35). Since λ_0 in (4.14) is a real constant, solution (4.14) actually represents two different solutions, each of them defined in certain complementary bands in ρ space. Indeed, expression (4.15a) is singular and changes sign at the singularities

$$\xi_a=\xi_a+4nK,\quad cn(\xi_a,k)=\frac{A-B}{A+B},\quad a=1,2\ , \tag{4.16}$$

i.e., twice within each period. In the bands where we have $W>0$, we choose $\lambda_0>0$ and vice versa. A simple analysis shows the $P(W)$ cannot have four complex roots. Turning to the case when we have $a_2<0$, we see that for $\lambda_0\in\mathbb{R}$ expression (4.8) yields η pure imaginary.

We put

$$W(\eta)=R(\zeta),\quad \eta=i\zeta \tag{4.17}$$

where $R(\zeta)$ satisfies

$$\dot{R}^2=-\left[R^4+4CR-\frac{3S_0^2}{a_2\lambda^2}\right],\quad a_2<0\ . \tag{4.18}$$

The polynomial $P(R)$ on the right-hand side of (4.18) can have four real roots only for $C=S_0=0$. This provides no real solutions [see Fig. 4(a)]. Four complex roots cannot occur, so the only remaining case is that of two real and two complex conjugate roots. We denote the roots

$$\begin{aligned}
&R_1=r\ ,\\
&R_2=s\ ,\\
&R_{3,4}=-\frac{r+s}{2}\pm iq\ ,\\
&q=\tfrac{1}{2}(3r^2+2s^2+2rs)^{1/2},\quad r<s\leq 0\ \text{ or }\ 0\leq r<s\\
&C=-\tfrac{1}{4}(r+s)(r^2+s^2)\ ,\\
&S_0^2=-\frac{a_2\lambda_0^4}{3}rs(r^2+s^2+rs)>0\ .
\end{aligned} \tag{4.19}$$

The corresponding phase diagram (shown only for the case $0\leq r<s$) is given on Fig. 4(b). From the figure, we see that the solution for R will be finite and will satisfy $r\leq R\leq s$.

Transforming (4.18) to its standard form,[35] we find a solution of the NLSE in the form (4.14) with

$$\begin{aligned}
&W(\xi)=\frac{(rA-sB)cn(\xi,k)+sB+rA}{(A-B)cn(\xi,k)+A+B}\ ,\\
&\chi(\xi)=\left[\frac{rs(r^2+s^2+rs)}{AB}\right]^{1/2}\\
&\qquad\times\frac{1}{2\epsilon}\int\frac{(A-B)cn(\xi,k)+A+B}{(rA-sB)cn(\xi,k)+rA+sB}d\xi\ ,\\
&\xi=4\epsilon\left[\frac{-ABa_2}{3}\right]^{1/2}\lambda_0\rho^{1/2}+\xi_0\ ,\\
&k^2=\frac{(r-s)^2-(A-B)^2}{4AB}\ ,\\
&A=(r^2+2rs+3s^2)^{1/2}\ ,\\
&B=(3r^2+2rs+s^2)^{1/2}\ ,\\
&0\leq r<s,\lambda_0>0\ .
\end{aligned} \tag{4.20}$$

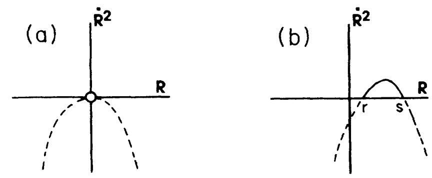

FIG. 4. Phase diagram for Eq. (4.18). (a) $C=S_0=0$. (b) $(C,S_0)\neq(0,0)$; the curve also has a critical point at $W^3=\frac{1}{4}(r+s)(r^2+s^2)$.

The only singularity of this solution is at $\rho=0$. The case $r<s\leq 0$ requires $\lambda_0<0$ and gives the same solution for M.

Equation (2.14) for $a_2\neq 0$, $a_1=0$, $a_0=b$, and $S_0=0$ is invariant under dilations. We use this to obtain an implicit solution. Indeed, putting

$$y=\rho M^2,\quad w(y)=\ln M,\quad z=w_y\ ,$$

we obtain a Riccati equation for z, which we transform into a Bessel equation. The final result is that (2.12) with $\chi=\chi_0$, $b=a_0$, and $a_1=0$ will yield a solution of (1.1) if M satisfies the transcendental equation

$$\begin{aligned}
M=e^{w_0}[&c_1J_{a\sqrt{3}}(\sqrt{-3a_2}\rho M^2)\\
&+c_2Y_{a\sqrt{3}}(\sqrt{-3a_2}\rho M^2)]^{1/3}\ ,
\end{aligned} \tag{4.21}$$

where w_0, c_1, and c_2 are constants and J and Y are solutions of the Bessel equation.

To sum up the solutions obtained in this Section, we notice that they all have the form (2.12), with $\chi(\rho)$ given by (2.13). The amplitude $M(\rho)$ is given by (4.3) or (4.7) for $a_1\neq 0$. For $a_1=0$ and $a_2<0$, we obtain the solution (4.14) with $W(\xi)$ as in (4.20). The amplitude $M(\rho)$ has a $\rho^{-1/4}$ point singularity for $\rho=0$ and is otherwise regular and periodic for $0<\rho<\infty$. A further solution is given by (4.11). It also has a $\rho^{-1/4}$ singularity for $\rho=0$ and is otherwise regular and nonperiodic. For $a_2>0$ and $a_1=0$, we obtain two types of solutions. They both have $\rho^{-1/4}$ singularities for $\rho=0$. Solution (4.12) has a nonperiodic amplitude $M(\rho)$, singular for $\sqrt{\rho}=-\epsilon c_0$, which is in the physical region only if we have $\epsilon c_0<0$. The other solution is (4.14) with $W(\xi)$ and $\chi(\xi)$ as in (4.15). The function $W(\xi)$ has two poles per period and the amplitude $M(\rho)$ exists (i.e., is real) in bands in ρ space. Implicit solutions are given by (4.21). Notice also that (2.18a) and (2.18b) provides a further nonperiodic solution, singular at $\rho=0$ only (for $a_2>0$, we choose $a^2<\frac{1}{4}$, for $a_2<0$, $a^2>\frac{1}{4}$).

V. CONCLUSIONS

In Secs. III and IV we have obtained numerous explicit solutions of the NLSE (1.1), satisfying initial conditions of the type (1.2) for $a_2=0$ and $a_2\neq 0$, respectively. The main properties of these solutions were summed up in the last paragraphs of the corresponding sections. Here we shall only add a few comments.

(1) Our solutions are adapted to a cylindrical geometry but they are not solutions of the "cylindrical nonlinear Schrödinger equation"

$$i\psi_t+\psi_{\rho\rho}+\frac{1}{\rho}\psi_\rho=a_0\psi+a_1\psi|\psi|^2+a_2\psi|\psi|^4 \ . \tag{5.1}$$

Indeed, the Lie algebra of the symmetry group of this equation is given by

$$\{T,M\}, \quad a_1\neq 0, \quad a_2\neq 0 \tag{5.2a}$$

$$\{T,M,D\}, \quad (a_1,a_2)\neq(0,0), \quad a_1a_2=0 \ . \tag{5.2b}$$

None of the subalgebras of (5.2) leads to a Painlevé-type ODE.

Thus a careful analysis of the subgroup structure of the symmetry group of the equation was crucial. If we set $a=0$ in (2.7), as suggested by the simplest interpretation of cylindrical symmetry, we would loose virtually all the solutions presented in this article. In terms of the Cauchy condition (1.2), this means that the solutions $\psi(\underline{x})$ have a prescribed θ dependence at $t=t_0$ on the cylinder $\rho=\rho_0$ and that we cannot choose $f_i(\theta,z)$ in (1.2) to be constant. The emphasis in this article was on the Painlevé-type ODE's. Solutions are, however, sometimes obtained for other ODE's [see, e.g., (2.18), (3.34), and (4.21)].

(2) Most of the obtained solutions involve periodic functions of the radius ρ, and $|\psi|^2$ plays the role of a periodic potential in (1.1). It is hence not surprising that in some cases, in particular for $a_2>0$, we have solutions defined in bands, i.e., for ρ satisfying

$$\rho_1+nK_0<\rho<\rho_2+nK_0 \ , \tag{5.3}$$

where K_0 is related to the period of the Jacobi elliptic function.

(3) Each solution presented in this article will provide a class of solutions when acted upon by the Galilei group, dilations (whenever applicable, i.e., if $a_1=0$ or $a_2=0$), and reflections of any of the coordinates x, y, and z (e.g., the replacement $\theta\to-\theta$). The corresponding group transformations are given in Ref. 26 and we shall not repeat them here.

The physical interpretation of the obtained solutions depends on the model that leads to Eq. (1.1). The same goes for the application of the solutions. If Eq. (1.1) is interpreted as arising in a Hamiltonian or Lagrangian theory, the solutions can be used to calculate the energies of elementary excitations. They can serve as classical limits of quantum solutions and be used in a quantization of Eq. (1.1). The obtained exact solutions can also serve as the basis of a perturbation scheme and will then induce further classes of approximate solutions.

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*Present address: Laboratoire de Recherches en Optique et Laser, Département de Physique, Université Laval, Cité Universataire, Ste. Foy (Québec), Canada G1K7P4.

[1]S. Kurihara, J. Phys. Soc. Jpn. **50**, 3262 (1981).

[2]V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).

[3]J. A. Tuszynski, R. Paul, and R. Chatterjee, Phys. Rev. B **29**, 380 (1984).

[4]V. L. Ginzburg and A. A. Sobyanin, J. Low Temp. Phys. **49**, 507 (1982).

[5]M. V. Goldman, Rev. Mod. Phys. **56**, 709 (1984).

[6]R. S. Johnson, Proc. R. Soc. London, Ser. A **357**, 131 (1977).

[7]T. Kakutani and K. Michihiro, J. Phys. Soc. Jpn. **52**, 4129 (1983).

[8]S. Cowan, R. H. Enns, S. S. Rangnekar, and S. S. Sangnera, Can. J. Phys. **64**, 311 (1986).

[9]A. Kumar, S. N. Sarkar, and A. K. Ghatak, Opt. Lett. **11**, 321 (1986).

[10]V. E. Zakharov and P. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys.—JETP **34**, 62 (1972)]; **64**, 1627 (1973) [**37**, 823 (1973)]; Funct. Anal. Appl. **8**, 226 (1974); **13**, 166 (1979).

[11]M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[12]L. D. Faddeev and L. A. Takhtadzhyan, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, 1987).

[13]Kh. I. Pushkarov, D. I. Pushkarov, and I. V. Tomov, Opt. Quantum Electron. **11**, 471 (1979).

[14]S. Lie, *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen* (Teubner, Leipzig, 1891).

[15]P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986).

[16]G. W. Bluman and J. D. Cole, *Similarity Method for Differential Equations* (Springer, New York, 1974).

[17]L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).

[18]P. Winternitz, in *Nonlinear Phenomena,* Vol. 189 of *Lecture Notes in Physics,* edited by J. Ehlers, K. Hepp, R. Kippenhahn, H. A. Weidenmüller, and J. Ziffartz (Springer, New York, 1983), p. 263.

[19]P. Winternitz, A. M. Grundland, and J. Tuszynski, J. Math. Phys. **28**, 2194 (1987).

[20]A. M. Grundland, J. Tuszynski, and P. Winternitz, Phys. Lett. A **119**, 340 (1987).

[21]D. David, N. Kamran, D. Levi, and P. Winternitz, Phys. Rev. Lett. **55**, 2111 (1985); J. Math. Phys. **27**, 1225 (1986).

[22]D. David, D. Levi, and P. Winternitz, Phys. Lett. A **118**, 390 (1986).

[23]B. Champagne and P. Winternitz, J. Math. Phys. **29**, 1 (1988).

[24]R. A. Leo, L. Martina, and G. Soliani, J. Math. Phys. **27**, 2623 (1986).

[25]B. Dorizzi, B. Grammaticos, A. Ramani, and P. Winternitz, J.

Math. Phys. **27**, 2848 (1986).
[26]L. Gagnon and P. Winternitz, J. Phys. A **21**, 1493 (1988).
[27]B. Champagne and P. Winternitz (unpublished).
[28]J. M. Lévy-Leblond, *Galilei Group and Galilean Invariance,* Vol. II of *Group Theory and Its Applications,* edited by E. Loebl (Academic, New York, 1974).
[29]J. Voisin, J. Math. Phys. **6**, 1519 (1965); **6**, 1822 (1965).
[30]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974).
[31]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[32]P. Painlevé, Acta Math. **25**, 1 (1902); B. Gambier, *ibid.* **33**, 1 (1910).
[33]M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21**, 715 (1980).
[34]D. Rand and P. Winternitz, Comput. Phys. Commun. **42**, 359 (1986).
[35]P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer, Berlin, 1971).